LIFTING FACTORIZATION OF WAVELET MULTIRESOLUTION ANALYSIS

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ABSTRACT

Decomposing the wavelet transform in lifting steps allows a simpler implementation of the transform filters and provides the flexibility necessary to satisfy other requirements, e.g., generating non-linear integer-to-integer wavelet transforms. The paper presents a flow-graph approach to the lifting factorization that gives a better insight to the main features of single-phase and two-phase wavelet transform representations. On this basis, truly loss-less signal compression algorithms using integer wavelet transform can be devised..

1. INTRODUCTION

Lossless compression schemes with multi-functionality support play a key role in medical image storage, retrieval and transmission, when fast interactive handling of large data sets over networks with limited and/or variable bandwidth is required, and when the option of decompression without distortion has to be retained. The analysis of the basic compression methods reveals that the embedded coding techniques based on the discrete wavelet transform fulfil the most important requirements for medical image compression. Moreover, the definition of a "medical" profile for JPEG2000 and the decision of DICOM [1] to join the efforts of the JPEG committee [2, 3]are turning wavelet based compression techniques into good candidates for a widely accepted standard within the medical community. Several coding techniques [4] featuring multi-functionality support, able to produce a compressed lossless embedded data stream that allows the progressive refinement of the decompressed image or of a certain set of user-defined regions of interest, have recently been developed based on the integer-to-integer wavelet transforms [6-8]. In medical imaging the lost of any information when processing or transmitting an image is not acceptable because minute image details might be essential to signal a pathological state. Standard wavelet compression techniques [6] cannot reconstruct the lossless version of the original image, even when loss-less in principle and when retaining all the coefficients of the wavelet transform, because these coefficients are generated as real (floating point) numbers. Coding rounds up or down the wavelet coefficients to integers, so that losses result. The alternative is the use of the lifting scheme to generate truly loss-less non-linear integer-to-integer wavelet transforms [7-9].

By using a signal flow-graph approach, the paper presents the lifting scheme as a simple and flexible tool for constructing pairs of forward and inverse wavelet transforms that can be adjusted to acquire various desired features while retaining the basic property to perfecty reconstruct the initial signal. In particular, the lifting scheme largely simplifies obtaining integer-to-integer true wavelet transform.

2. PERFECT RECONSTRUCTION CONDITION

Figure 1 shows the single-phase and the equivalent two-phase analysis step of a wavelet multiresolution representation of a signal [6, 8, 13-16].

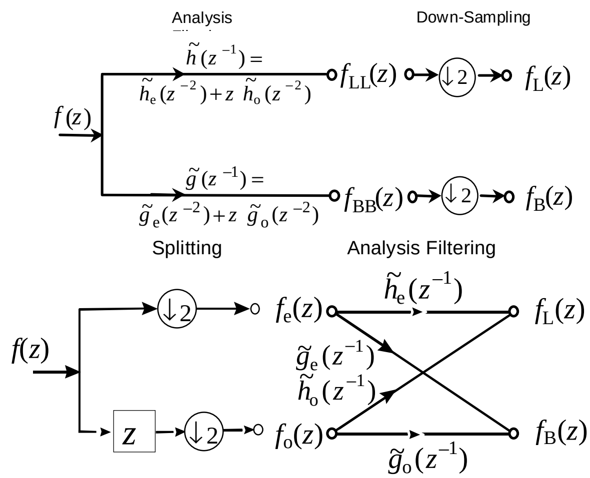

Figure 1. Equivalent single- and two-phase wavelet analysis steps

In the two-phase wavelet analysis, the low resolution-- $f_L(z)$ and detail -- $f_B(z)$ components of the signal are expressed in terms of the even -- $f_e(z)$ and odd -- $f_o(z)$ components, by the relation:

$$\begin{bmatrix} f_L(z) \\ f_B(z) \end{bmatrix} = \tilde{\mathbf{P}}^T(z^{-1}) \begin{bmatrix} f_e(z) \\ f_o(z) \end{bmatrix} \qquad (1)$$

where

$$\tilde{\mathbf{P}}^{\mathsf{T}}(z^{-1}) = \begin{bmatrix} \tilde{h}_{\mathrm{e}}(z^{-1}) & \tilde{h}_{\mathrm{o}}(z^{-1}) \\ \tilde{g}_{\mathrm{e}}(z^{-1}) & \tilde{g}_{\mathrm{o}}(z^{-1}) \end{bmatrix} \tag{2}$$

is the transposed of the analysis two-phase matrix [13-15].

Similarly, for the synthesis step, the even and odd components of the signal are restored in terms of the low resolution and detail components with the relation:

$$\begin{bmatrix} f_{\mathrm{e}}(z) \\ f_{\mathrm{o}}(z) \end{bmatrix} = \mathbf{P}(z) \begin{bmatrix} f_{\mathrm{L}}(z) \\ f_{\mathrm{B}}(z) \end{bmatrix}, \tag{3}$$

where

$$\mathbf{P}(z) = \begin{bmatrix} h_{\mathrm{e}}(z) & h_{\mathrm{o}}(z) \\ g_{\mathrm{e}}(z) & g_{\mathrm{o}}(z) \end{bmatrix} \tag{4}$$

is the transposed of the synthesis two-phase matrix, dual to $\tilde{\mathbf{P}}^{\mathsf{T}}(z^{-1})$.

Figure 2 shows the equivalence of the single- and two-phase synthesis steps.

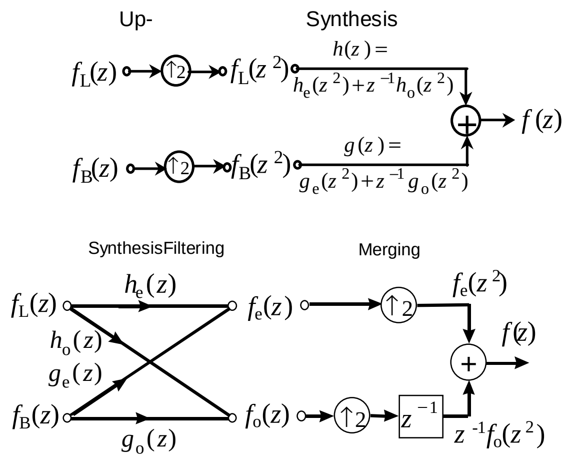

Figure 2. Equivalent single- and two-phase wavelet synthesis steps

The perfect reconstruction condition results in a natural form for the two-phase scheme:

$$\mathbf{P}(z)\tilde{\mathbf{P}}^{\mathsf{T}}(z^{-1}) = \mathbf{I} \Leftrightarrow \tilde{\mathbf{P}}(z) = \left(\mathbf{P}^{-1}(z^{-1})\right)^{\mathsf{T}}. \tag{5}$$

To obtain FIR analysis and synthesis filters, $\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$ must contain only Laurent polynomials, so that $\det \mathbf{P}(z) = C z^{k}$. By dividing $g_{\mathrm{e}}(z)$ and $g_{\mathrm{o}}(z)$ with $\det \mathbf{P}(z)$, it can be arranged that $\det \mathbf{P}(z) = 1$, or

$$h_{\mathrm{e}}(z) g_{\mathrm{o}}(z) - h_{\mathrm{o}}(z) g_{\mathrm{e}}(z) = 1. \tag{6}$$

Consequently, the analysis filters can be expressed in terms of the synthesis filters by the relations:

$$\begin{aligned} \tilde{h}_{\mathrm{e}}(z) &= g_{\mathrm{o}}(z^{-1}), & \tilde{h}_{\mathrm{o}}(z) &= -g_{\mathrm{e}}(z^{-1}), \\ \tilde{g}_{\mathrm{e}}(z) &= -h_{\mathrm{o}}(z^{-1}), & \tilde{g}_{\mathrm{o}}(z) &= h_{\mathrm{e}}(z^{-1}). \end{aligned} \tag{7}$$

3. PRIMAL AND DUAL LIFTING STEPS

The lifting scheme [10-13] allows to change one of the analysis or synthesis filters, keeping unchanged its complementary filter and conserving the perfect reconstruction condition. In the following, a pair of filters (h, g) is called complementary if the corresponding two-phase matrix $\mathbf{P}(z)$ has the determinant one, i.e., $\det \mathbf{P}(z) = 1$. From (5) it results that, if the pair of filters (h, g) is complementary, so are the filters $(\tilde{h}, \tilde{g})$. Figure 3 shows the two phase representation of a ***primal lifting*** step which maintains the complementarity of the filters and the perfect reconstruction condition.

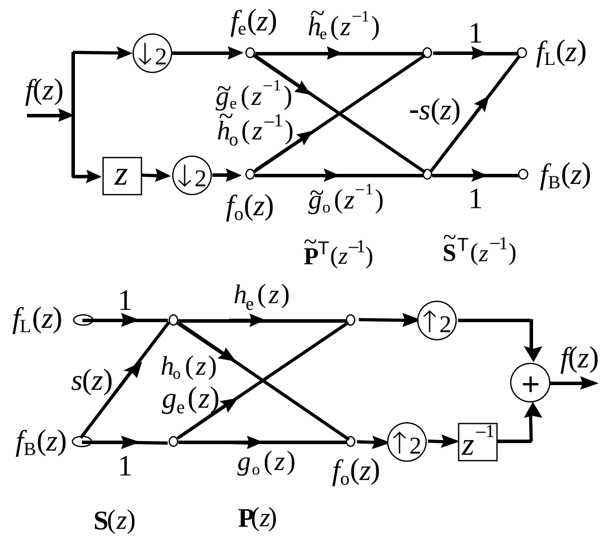

Figure 3. Primal lifting step in the two-phase wavelet multiresolution representation

The new synthesis and analysis two-phase matrices are:

$$\mathbf{P}'(z) = \mathbf{P}(z) \begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix} = \mathbf{P}(z)\, \mathbf{S}(z), \tag{8}$$

and

$$\tilde{\mathbf{P}}'(z) = \tilde{\mathbf{P}}(z) \begin{bmatrix} 1 & 0 \\ -s(z^{-1}) & 1 \end{bmatrix} = \tilde{\mathbf{P}}(z)\tilde{\mathbf{S}}(z), \tag{9}$$

so that, for any Laurent polynomial $s(z)$:

$\det \mathbf{P}'(z) = \det \mathbf{P}(z) = 1$ and $\det \tilde{\mathbf{P}}'(z) = \det \tilde{\mathbf{P}}(z) = 1$.

A ***dual lifting*** step is shown in Figure 4.

The modified two-phase matrices are given by:

$$\mathbf{P}'(z) = \begin{bmatrix} h_{\mathrm{e}}(z) & g_{\mathrm{e}}(z) \\ h_{\mathrm{o}}(z) & g_{\mathrm{o}}(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t(z) & 1 \end{bmatrix} = \mathbf{P}(z)\mathbf{T}(z), \tag{10}$$

and

$$\tilde{\mathbf{P}}'(z) = \tilde{\mathbf{P}}(z) \begin{bmatrix} 1 & -t(z^{-1}) \\ 0 & 1 \end{bmatrix} = \tilde{\mathbf{P}}(z)\,\tilde{\mathbf{T}}(z), \tag{11}$$

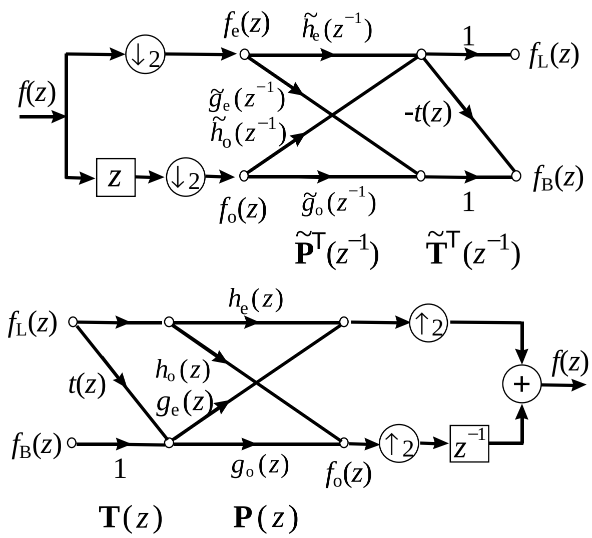

Figure 4. Dual lifting step in the two-phase wavelet multiresolution representation.

so that again, for any Laurent polynomial $t(z)$,.
$\det \mathbf{P}'(z) = \det \mathbf{P}(z) = 1$ and $\det \tilde{\mathbf{P}}'(z) = \det \tilde{\mathbf{P}}(z) = 1$.

4. LIFTING FACTORIZATION

Consider a pair of complementary filters (h, g). According to (6), $\det \mathbf{P}(z) = h_e(z) g_o(z) - h_o(z) g_e(z) = 1$,so that the Laurent polynomials $h_e(z)$ and $h_o(z)$ are relatively prime. Thus, the Euclid algorithm gives $\mathrm{GCD}\left(h_e(z), h_o(z)\right) = Kz^p$, i.e., a monomial. Using the non-uniqueness of the division on the ring of Laurent polynomials, the quotients can be chosen to reduce the GCD to a constant K, to obtain the factorization:

$$\begin{bmatrix} h_e(z) \\ h_o(z) \end{bmatrix} = \left(\prod_{i=1}^{n} \begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} K \\ 0 \end{bmatrix} \tag{12}$$

If $|h_o(z)| > |h_e(z)|$, the first quotient is zero: $q_1(z) = 0$. We will consider n = even; if n = odd we multiply $h(z)$ with z, and $g(z)$ with z^{-1}, so that n becomes even, without changing det $\mathbf{P}(z)$ = 1. Given the low-pass filter $h(z)$ and using the factorization (12), a complementary $g^{(0)}(z)$ can be obtained by taking:

$$\begin{aligned} \mathbf{P}^{(0)}(z) &= \begin{bmatrix} h_e(z) & g_e^{(0)}(z) \\ h_o(z) & g_o^{(0)}(z) \end{bmatrix} \\ &= \left(\prod_{i=1}^{n} \begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix} \end{aligned} \tag{13}$$

The last factor is taken to ensure $\det \mathbf{P}^{(0)}(z) = 1$ for n = even. A pair of successive factors can be rewritten:

$$\begin{bmatrix} q_{2i-1}(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{2i}(z) & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & q_{2i-1}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{2i}(z) & 1 \end{bmatrix} \tag{14}$$

so that relation (13) becomes:

$$\begin{aligned} \mathbf{P}^{(0)}(z) &= \left(\prod_{i=1}^{n/2} \begin{bmatrix} 1 & q_{2i-1}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{2i}(z) & 1 \end{bmatrix} \right) \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix} \\ &= \left(\prod_{i=1}^{n/2} \begin{bmatrix} 1 & s_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z) & 1 \end{bmatrix} \right) \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix} \end{aligned} \tag{15}$$

which factors $\mathbf{P}^{(0)}(z)$ in $n/2$ pairs of primal and dual lifting steps, followed by a scaling. An additional primal lifting step (8) can always be included to bring $g^{(0)}(z)$ to the original complementary filter $g(z)$:

$$\mathbf{P}(z) = \mathbf{P}^{(0)}(z) \begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}. \tag{16}$$

It results that a two-phase matrix corresponding to any pair of complementary filters (f, g) that define a wavelet multiresolution representation can always be factored into $m = n/2 + 1$ pairs of primal and dual lifting steps, followed by a scaling:

$$\mathbf{P}(z) = \left(\prod_{i=1}^{n/2} \begin{bmatrix} 1 & s_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z) & 1 \end{bmatrix} \right) \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix}, \tag{17}$$

where $s_m(z) = K^2\, s(z)$ and $t_m(z) = 0$. The Laurent polynomials $s_i(z)$, $t_i(z)$, for $i = 1, \ldots, m\text{-}1 = n/2$, result from applying Euclid algorithm to the pair of Laurent polynomials $h_e(z)$ and $h_o(z)$.

The decomposition of $\mathbf{P}(z)$ into lifting factors given by relation (17) generates the ladder structure of the two-phase flow-graph shown in Figure 5, which corresponds to a synthesis step of the wavelet multiresolution representation.

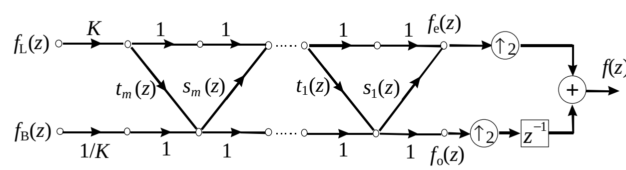

Figure 5. Lifting factor decomposition of a synthesis step of the wavelet multiresolution representation.

Similarly, the dual two-phase matrix is factored as:

$$\tilde{\mathbf{P}}(z) = \left(\prod_{i=1}^{n/2} \begin{bmatrix} 1 & 0 \\ s_i(z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} 1 & -t_i(z^{-1}) \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{K} & 0 \\ 0 & K \end{bmatrix}, \tag{18}$$

to which corresponds an analysis step of the wavelet multiresolution representation described by the two-phase flow-graph shown in Figure 6 that has a mirror ladder structure with respect to the flow-graph in Figure 5.

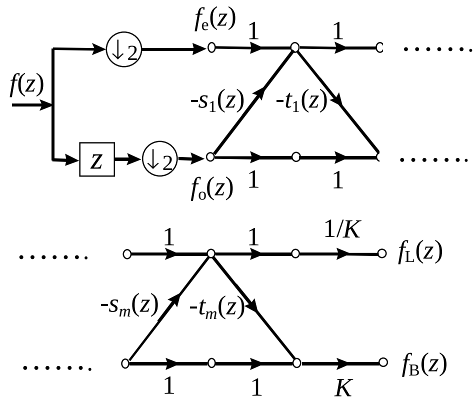

Figure 6. Lifting factor decomposition of an analysis step of a wavelet multiresolution representation.

5. CONCLUSIONS AND SUMMARY

The paper uses a signal flow-graph approach to present the factoring of wavelet transforms using the lifting scheme. The implementation advantages and the better insight offered by the two-phase analysis and synthesis steps are stressed. The lifting scheme is a simple and flexible tool for constructing pairs of forward and inverse wavelet transforms that can be adjusted to acquire various desired features while retaining the basic property to perfectly reconstruct the initial signal. Beside the increased flexibility, which makes possible to obtain reversible non-linear wavelet transforms, lifting allows: (1) *fast implementation* - by making optimal use of similarities between high and low pass filters, the necessary number of flops can be reduced to half, (2) *fully in-place calculation* - by gradually replacing the original image with its transform, the need for an auxiliary memory can be avoided and the hardware implementation can be simplified, (3) *simple inverse transform*, of the same computational complexity as the forward one - the inverse transform being composed of the inverse elementary operations of the forward one, taken in reversed order.

An important application of the lifting scheme is to generate non-linear integer-to-integer wavelet transform, which are essential for true lossless compression methods [10-16].

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