A Thermodynamic Theory of Broadband Networks with Application to Dynamic Routing

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Abstract—We propose a thermodynamic theory for broadband networks relating quantities such as grade of service (GoS), bandwidth assignment, buffer assignment, and bandwidth demand. We propose a scalability postulate for these four quantities. Useful thermodynamic type relationships are then derived. The scalability postulate and thermodynamic type relations are then reexamined via statistical methods using the moment generating function. Large deviations theory is applied, and in the process, notions such as effective bandwidth are defined. We apply this theory to networks which allow dynamic routing of different call types. The probability of rare events expressed in conjunctive forms is characterized using large deviations theory. Based on this theory, a new dynamic routing method called effective bandwidth network routing (EBNR) is proposed.

I. INTRODUCTION

NETWORKS are large dynamical systems. The interactions of various components are complex, and the analytical computation of detailed phenomena very often is intractable, inaccurate, and sometimes irrelevant. Analytical approaches for network performance so far focus primarily on detailed and microscopic modeling and derivations of node and network dynamics. There is a need for a macroscopic theory which is more concerned with large quantities and qualitative behavior on a larger scale. There is also a need for a fundamental understanding concerning how these quantities scale with respect to each other. Given the scale of the network and the large capacity of network components such as memory and capacity, we could be more interested in the asymptotes and exponents of aggregated quantities. Qualitative behavior such as conditions for system equilibrium may generate more insight concerning resource allocation and system control than a detailed calculation based on often unverifiable data and models. After all chaotic phenomena, so prevalent in complex systems, soon render such detailed calculations irrelevant. Instead, appropriately defined aggregated quantities, which could be measured via empirical methods and related using a macroscopic theory, could render a better understanding and engineering of complex broadband networks.

While we are still far from attaining such an understanding, this paper is an attempt to define interesting quantities and relationships among these quantities. Section II examines a number of system parameters such as the grade of service (GoS) G, the bandwidth in the network C, the buffer capacity in the network B, and the bandwidth demand N. These quantities can be vector quantities denoting multiple commodities throughout the whole network. Basic relations are given among these extensive quantities and their corresponding intensive marginals. A fundamental postulate, called the scalability postulate, is used to derive a myriad of fundamental relationships. The approach taken is similar to that of the classical theory of Thermodynamics. Section III reexamines these quantities using large deviations theory and moment generating functions, focusing on the asymptotics and exponential behavior of such systems. The approach taken is similar to that of Statistical Mechanics, allowing description and computation of system parameters by modeling the microscopic dynamics. Section IV applies large deviations theory to analyzing events of a conjunctive form, with an application to network routing using alternative paths of multiple links. Section V concludes the paper and proposes an application of the above theory called effective bandwidth network routing (EBNR).

A model of traffic in interconnection networks using thermodynamics is presented by Benes in [4]. In this model the entropy of the system is defined as the Shannon entropy of the random variable denoting the state of the network. This entropy functional is not a measurable quantity on a real network contrary to the entropy defined in this paper. The statistical description of the traffic in Benes’ model is characterized only through average values and blocked calls are omitted which is an important performance measure of the network. Furthermore, this model can not be extended to multirate traffic.

Large deviations techniques have been used to obtain the asymptotics for small probabilities in network applications. The relationship between capacity, loss probability and offered traffic for multirate traffic on a blocking link is obtained by using transform domain methods [17]. The effect of changes in the parameters such as capacity and offered traffic on the performance of the network for routing and capacity allocation problems is discussed in [20]. Hui [15], [16] used the Chernoff bound to compute the tail of the probability distribution for Poisson offered traffic and showed that, for unbuffered networks, there is a notion of engineered bandwidth for each source, which depends on the statistics of the source
as well as the characteristics of the channel. Kelly [18] extended the notion of effective bandwidth to GI/G/1 queues using Kingman’s bound. The similarity between the effective bandwidths for unbuffered and buffered sources is also noted.

For queues driven by point arrival processes, the fluid-flow approximation is valid when the mean buffer length and the buffer size are large. The motivation for this approximation lies in the fact that the magnitudes of the discontinuous jumps are small as compared to the average queue length. Thus, the original discrete queue can be represented by a continuous fluid flow. The tail probabilities for multiple on-off sources sharing a buffer is obtained with a fluid-flow model [1]. Using large deviations techniques it is shown that the fluid-flow approximation corresponds to the most probable path for continuous time Markov processes [22]. The effective bandwidth for multiple Markov modulated fluid-flow sources sharing a statistical multiplexer with a large buffer is obtained [11], [14].

Large deviations techniques have also been used to bound traffic parameters to obtain exponential upper bounds on the queue length and delay in queuing networks [6], [8]. The application of Chernoff bound for the waiting time of a single queue using martingales is given in [13].

A limiting regime similar to the one used in Section IV is discussed in [7], [19] for Poisson arrival process and fixed routing in a circuit-switched network. When the capacity of links and offered traffic are increased together the blocking probability of a path is given by a product form, as if links are blocked independently. The analysis is also generalized to networks with alternative routing using Erlang fixed point approximation. An independence assumption has been used in networks to evaluate the probability of joint events (Kleinrock’s assumption for queuing networks and Lee’s assumption for alternative routing). The dynamic routing method EBNR proposed in Section V is closely related to the real-time network routing (RTNR) policy [2], in which routing decisions are based on the current number of idle circuits in each of the links throughout the network.

II. A THERMODYNAMICAL THEORY OF NETWORKS

The physical theory of thermodynamics is a fundamental study of the relationship among macroscopic quantities such as volume \( V \), energy \( E \), entropy \( S \), and the number of particles \( N \). A typical relation is expressed in the functional form \( E(S, N, V) \) among many other possible forms. These four quantities are termed extensive quantities in the sense that a functional quantity scales in the same manner as its arguments, i.e., \( E(\alpha S, \alpha N, \alpha V) = \alpha E(S, N, V) \). Certain physical laws, such as the conservation of energy (the first law) and the nondecreasing entropy of closed systems (the second law) further constrain the evolution of these quantities in time.

A modest parallel in networking similarly deals with the functional form \( C(G, N, B) \), in which \( C \) represents capacity, \( G \) represents GoS, \( N \) represents the number of connections, and \( B \) represents buffer resources. These could be vector quantities for entities throughout the network. We call this functional form the bandwidth demand (BD) equation, which gives the required bandwidth for carrying a traffic intensity \( N \) with a buffer size \( B \), while meeting the GoS requirement \( G \). The bandwidth demand is the minimum bandwidth required for satisfactory GoS, while an actual network may have more or less bandwidth than required. An alternative functional form, which is often more useful, is the GoS equation \( G(C, N, B) \) which gives the GoS for given network resources and traffic.

The quantities \( C, G, N, B \) bear a certain remarkable parallel, respectively, with the thermodynamic quantities \( E, S, N, V \) as shall be pointed out later. This resemblance is due largely to the mathematical structures of scalable systems rather than physical resemblance, and hence one should not overextend the interpretation of the parallels. Nevertheless, Table I lists the analogous quantities between Thermodynamics and our
network theory. The analogies made by Benes are stated in the third column. It is noteworthy that though there is a certain mathematical resemblance between our theory and that of Benes, the parameters and their interpretations are quite different between the two theories. Their applications are also different: Benes' work is largely concerned with enumerating equally likely states in a switching network, while we are concerned with computing GoS and BD using large deviations theory.

So far, the definition of $G$ is deliberately abstract, similar to the notion of entropy for Thermodynamics before the days of Statistical Mechanics. A fundamental question is whether these four network parameters are extensive, that is, how they scale with respect to the other parameters. In this section, we propose the following scalability postulate

$$G(\alpha C, \alpha N, \alpha B) = \alpha G(C, N, B).$$

(1)

This postulate, which seems implausible at first sight is justifiable a priori for the resulting elegant structures; and aposteriori for simple systems such as the M/M/1/B queue and the M/M/C/C+B queue (see Appendix A), and for systems with no buffers ($B = 0$). The scalability property for the fluid flow model is asymptotically shown in [12], [22] for large $C$, $B$ and $N$. In each of these cases, the GoS is defined as the negative logarithm of the loss probability. The following theorem proves the linear scalability property for slotted time queueing systems as $C$, $B$ and $N$ go to infinity, such that bandwidth per source $c = C/N$ and buffer per source $b = B/N$ are kept constant.

**Theorem 2.1:** For a slotted time queueing system, $G$, defined as the negative logarithm of the loss probability, satisfies

$$\lim_{N \to \infty} g(\alpha N c, \alpha N, \alpha N b) = \alpha \lim_{N \to \infty} g(N c, N, N b) \triangleq g_N(c, b)$$

(2)

for any $\alpha > 0$.

Theorem 2.1 is proved by using Gärtner-Ellis Theorem and the proof is given in Appendix B. To demonstrate how fast linear scaling is achieved in practical systems, we consider a numerical example of multiplexing fluid flow on-off sources [1]. We plot various aspects of the function $G(C, N, B)$ and show that there is a strong linearity even for small values. The fluid flow model has $N$ identical on-off sources, with on and off periods of mean durations of 1 and 2 unit times, respectively, and each source require 1 unit of bandwidth when on. We define $\rho = Nb/C$, where $b$ is the average bandwidth demand per source. We also define $\gamma = B/C$. Figs. 1 and 2 plot GoS versus $C$ and $B$ for different values of $\rho$ (buffer size $B$ has units of 1 unit bandwidth times 1 unit time). In Figs. 3 and 4, $G$ versus $C$ is plotted for different values of $\rho$ and $\gamma$. We observe that the linear scalability holds for even small values of $B$ and $C$. In Figs. 5 and 6, $G$ versus $B$ is plotted while keeping $C$ fixed for different values of $\rho$ and $\gamma$. We observe that $G$ becomes a linear function of $B$ as $B$ increases.

We now explore further the implications of linear scaling. The following equation expresses the marginal change of the effective bandwidth in terms of the differentials of the parameters $G$, $N$, and $B$

$$dC = \left(\frac{\partial C}{\partial G}\right)_{N,B} dG + \left(\frac{\partial C}{\partial N}\right)_{G,B} dN + \left(\frac{\partial C}{\partial B}\right)_{G,N} dB.$$ 

(3)

These partial derivatives can be viewed as the shadow prices of capacity relative to GoS, traffic, and buffering, respectively. The computation of these quantities depends on the underlying dynamics. In parallel for Thermodynamics, the corresponding equation is

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial N} dN + \frac{\partial E}{\partial V} dV = T dS + \mu dN - P dV$$

in which $T$ represents the absolute temperature, $\mu$ represents the chemical potential of the particles, and $P$ represents the pressure exerted by the volume. These quantities are intensive in the sense that scaling the system does not change these quantities.

We summarize the implications of the scalability property (1) in the following theorem.

1In a network $C$, $G$, $N$, $B$ represent vectors, whereas their derivatives become Jacobian matrices. For example, the matrix $dC/dG = [dC_i/dG_j]$ corresponds to the matrix of shadow prices for capacities with respect to GoS’s of all nodes, whereas $dC_i/dG_J = [dC_i/dG_J]$ is the gradient vector representing the shadow prices.
Theorem 2.2: (i) The quantities $G$, $C$, $N$, $B$ are simultaneously extensive, e.g., $C(aG, aN, aB) = aC(G, N, B)$ given the scaling equation for $G$ in (1). (ii) The partial derivatives of $G$ with respect to $C$, $N$, $B$ are intensive, e.g., $C(aC, aN, aB)/\partial(aC)|_{N,B} = \partial G(C, N, B)/\partial C|_{N,B}$. (iii) The Euler form holds for the GoS equation, which is given by

$$G = \left( \frac{\partial G}{\partial C} \right)_{N,B} C + \left( \frac{\partial G}{\partial N} \right)_{C,B} N + \left( \frac{\partial G}{\partial B} \right)_{N,B} B \triangleq \theta C - \mu N + \zeta B. \quad (4)$$

For multiple call types with $N_i$, denoting the number of connections of type $i$, Euler form for the GoS is given by

$$G = \theta C - \sum_i \mu_i N_i + \zeta B \text{ where } \mu_i = \left( \frac{\partial G}{\partial N_i} \right)_{C,B,(N_j)_{j \neq i}}.$$

(iv) $G$ is linear in $C$ (or $B$, $N$) for given $\rho = N/C$ (assume $\delta = 1$), and $\gamma = B/C$, i.e., $G = g_C(\rho, \gamma)C$ (for given $c = C/N$ and $b = B/N$, $G = g_N(c, b)N$).

Proof: (i) $C_1 = C(G, N, B)$ is the solution to the GoS equation $G = G(C_1, N, B)$. Using the scalability property (1), we obtain $\alpha G = \alpha G(C_1, N, B) = G(\alpha C_1, \alpha N, \alpha B)$, which implies that $\alpha C_1$ is the solution to the bandwidth equation $\alpha C_1 = C(\alpha G, \alpha N, \alpha B)$. (ii) The proof follows by differentiating both sides of (1) with respect to $C$ while keeping $N$ and $B$ fixed. (iii) Differentiating both sides of (1) with respect to $\alpha$, we have

$$\frac{\partial G(\alpha C, \alpha N, \alpha B)}{\partial (\alpha C)} C + \frac{\partial G(\alpha C, \alpha N, \alpha B)}{\partial (\alpha N)} N + \frac{\partial G(\alpha C, \alpha N, \alpha B)}{\partial (\alpha B)} B = G(C, N, B). \quad (5)$$

Setting $\alpha = 1$ in (5), we obtain the Euler form (4).

(iv) For given $\rho$ and $\gamma$, we have $(C, N, B)$ being the scaling of $(1, \rho, \gamma)$ by $C$, and hence by the scalability property, there exists a $g_{\rho}(\rho, \gamma)$ such that $G = g_{\rho}(\rho, \gamma)C$.

In Fig. 7, the function $g_{\rho}(\rho, \gamma)$ is plotted versus $\rho$ for different values of $\gamma$ for the fluid flow model described previously. The simplicity of these curves allows for further parametric reduction.
An immediate application of linear scaling is that the performance of a queueing system can be obtained by scaling down the parameters of the system, computing the GoS for the scaled system through simulation and then rescaling the resulting GoS to obtain the GoS for the original system. This approach provides computational savings over the direct study of the given system since the loss probability is much larger in the scaled down system. Since effective bandwidth function is also linearly scalable, the effective bandwidth for a system can be computed by scaling down the system.

The Euler form of (4) for the GoS equation allows us to express $G$ in terms of its marginals corresponding to intensive variables (in thermodynamics we have correspondingly the well-known energy equation $E = TS + \mu N - PV$). Equation (4) allows us to eliminate one variable and express the functional dependence more succinctly as stated in part (iv) of Theorem 2.2, which states that $G$ is linear in $C$ for $g$ (given $\rho$ and $\gamma$).

The intensive variables $\theta$, $\zeta$, and $\mu_i$ can be used to explore the differential relations between extensive variables as listed in Table II. The relations in the second row of Table II can be obtained from the definitions of $\theta$, $\zeta$, and $\mu_i$ by using the identity

$$
\left( \frac{\partial X}{\partial \gamma} \right)_z = - \left( \frac{\partial Z}{\partial \gamma} \right)_x \left( \frac{\partial Z}{\partial X} \right)_y.
$$

For example, $\partial G/\partial C$ and $\partial G/\partial B$ represent the shadow prices of GoS with respect to capacity and buffer, whereas $\partial C/\partial B$ is the shadow price for capacity with respect to buffer and hence explains the trade-off between transmission and buffering. $\partial G/\partial N_i$ is the shadow price for GoS when an additional call of type $i$ joins the system. $\partial C/\partial N_i$ and $\partial B/\partial N_i$ are the effective bandwidth and effective buffer for a connection of type $i$, which correspond to shadow prices for capacity and buffer relative to traffic demand. Also, $\partial N_j/\partial N_i = \mu_j/\mu_i$ represents the relative price of a connection of type $j$ with respect to a connection of type $i$. These partial derivatives, which are given as ratios among the intensive parameters $\mu_i$, $\theta$, and $\zeta$ are useful for resource allocation and admission control. They will be used later for dynamic routing.

Many interesting partial differential relationships can also be obtained by using Legendre transformation: Given that we may measure the intensive parameters instead of the extensive parameters, different independent parameters could be chosen as the basis for further calculations. There are a total of 8 such bases, since we have a choice of either the extensive parameter or the associated intensive parameter for each independent variable of the GoS equation $G(C, N, B)$. Let $G_{P_1, P_2, P_3}$ be the function obtained by choosing $(P_1, P_2, P_3)$ as the set of independent variables and let $(Q_1, Q_2, Q_3)$ be the set of dependent variables with $\partial G_{P_1, P_2, P_3}/\partial P_i = Q_i$ (this follows from the properties of Legendre transformation). By using the property that mixed partial derivatives are equal independent of the order of differentiation, we write

$$
\frac{\partial^2 G_{P_1, P_2, P_3}}{\partial P_1 \partial P_2} = \frac{\partial^2 G_{P_1, P_2, P_3}}{\partial P_2 \partial P_1} \quad \text{or} \quad \frac{\partial Q_1}{\partial P_2} = \frac{\partial Q_2}{\partial P_1}.
$$

As mentioned, we have eight different choices of independent variables $(P_1, P_2, P_3)$ among the intensive and extensive quantities. Also, we have three choices among $(P_1, P_2, P_3)$ for taking second order partial differentiation. Therefore, there are a total of 24 equations of the type (6), which are called Maxwell relations. These relations are useful for calculating partial derivatives from other measured partial derivatives. All Maxwell relations are listed in Table III. We note that the derivation of Maxwell relations does not require the scalability property. The use of these relations is subject to further research.

So far this discussion has been general. Though the equations present useful relationships between the quantities, no practical means is provided to compute their values. A particular method of computation is given by using large deviations theory. The suggested analogy between the parameters of a network versus the parameters of thermodynamics will then be clarified.

III. LARGE DEVIATIONS THEORY APPLIED TO NETWORK

The scalability of GoS is largely a consequence of considering the asymptotic behavior of the exponent for GoS figures of interest, such as the loss probability $p$. Typically, such GoS is plotted on a logarithmic scale on the y-axis versus a linear scale for other parameters of interests, such as buffer size, traffic load, or capacity on the x-axis. It is often observed [11], [14] that $\log p/B$ becomes a constant for large $B$, and hence
<table>
<thead>
<tr>
<th>Function</th>
<th>Ind. vars.</th>
<th>Base</th>
<th>Maxwell equations</th>
</tr>
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<tbody>
<tr>
<td>(dG_{CNB} = \theta dC - \mu dN + \zeta dB)</td>
<td>C, N</td>
<td>(\frac{\partial}{\partial C} B_N)</td>
<td>(= \frac{\partial}{\partial N} B_C) (T3.1)</td>
</tr>
<tr>
<td>(dG_{BN} = -C d\theta - \mu dN + \zeta dB)</td>
<td>(\theta)</td>
<td>N, B</td>
<td>(\frac{\partial C}{\partial N} B_N) (T3.2)</td>
</tr>
<tr>
<td>(\mu)</td>
<td>C, B</td>
<td>(\frac{\partial C}{\partial B} B_B) (T3.3)</td>
<td></td>
</tr>
<tr>
<td>(dG_{\mu B} = \theta dC + N d\mu + \zeta dB)</td>
<td>C, (\mu)</td>
<td>B, (\mu)</td>
<td>(\frac{\partial N}{\partial C} B_B) (T3.4)</td>
</tr>
<tr>
<td>(dG_{CN})</td>
<td>C, N</td>
<td>(\frac{\partial C}{\partial N} B_C) (T3.5)</td>
<td></td>
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<tr>
<td>(dG_{\zeta B} = -C d\theta - \mu dN + \zeta dB)</td>
<td>(\theta)</td>
<td>N, B</td>
<td>(\frac{\partial C}{\partial N} B_N) (T3.6)</td>
</tr>
<tr>
<td>(\mu)</td>
<td>C, B</td>
<td>(\frac{\partial C}{\partial B} B_B) (T3.7)</td>
<td></td>
</tr>
<tr>
<td>(dG_{NC} = \theta dC - \mu dN - B d\zeta)</td>
<td>C, (\zeta)</td>
<td>C, (\zeta)</td>
<td>(\frac{\partial C}{\partial C} B_C) (T3.8)</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>N, (\zeta)</td>
<td>(\frac{\partial N}{\partial C} B_C) (T3.9)</td>
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<tr>
<td>(dG_{\mu C} = \theta dC + N d\mu - B d\zeta)</td>
<td>(\theta)</td>
<td>B, (\mu)</td>
<td>(\frac{\partial N}{\partial C} B_B) (T3.10)</td>
</tr>
<tr>
<td>(\mu)</td>
<td>C, (\mu)</td>
<td>(\frac{\partial C}{\partial B} B_B) (T3.11)</td>
<td></td>
</tr>
<tr>
<td>(dG_{\zeta C} = -C d\theta - \mu dN - B d\zeta)</td>
<td>(\theta)</td>
<td>N, (\zeta)</td>
<td>(\frac{\partial N}{\partial C} B_C) (T3.12)</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>C, (\zeta)</td>
<td>(\frac{\partial N}{\partial C} B_C) (T3.13)</td>
<td></td>
</tr>
<tr>
<td>(dG_{CN})</td>
<td>C, (\zeta)</td>
<td>C, (\zeta)</td>
<td>(\frac{\partial C}{\partial C} B_C) (T3.14)</td>
</tr>
<tr>
<td>(dG_{\mu C} = \theta dC + N d\mu - B d\zeta)</td>
<td>(\theta)</td>
<td>B, (\mu)</td>
<td>(\frac{\partial N}{\partial C} B_B) (T3.15)</td>
</tr>
<tr>
<td>(\mu)</td>
<td>C, (\mu)</td>
<td>(\frac{\partial C}{\partial B} B_B) (T3.16)</td>
<td></td>
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<tr>
<td>(dG_{\zeta C} = -C d\theta - \mu dN - B d\zeta)</td>
<td>(\theta)</td>
<td>N, (\zeta)</td>
<td>(\frac{\partial N}{\partial C} B_C) (T3.17)</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>C, (\zeta)</td>
<td>(\frac{\partial N}{\partial C} B_C) (T3.18)</td>
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indicating an asymptotically linear scaling of GoS with respect to buffer size. On the other hand, \(\log p\) often resembles a “water fall” for increasing \(C\), if load is kept constant. However, the linear scaling of \(\log p\) versus \(C\) is often restored if we allow load to scale linearly as \(C\). In this case, the utilization level of the capacity is kept constant. Combining the scaling of GoS with respect to buffer size with the scaling of GoS with respect to load and capacity, we formulated the scalability postulate of GoS with respect to buffer size, traffic load, and capacity in the previous section.

Large deviations theory has been used to obtain asymptotic results that apply to communication networks. The application of large deviations theory to statistical mechanics can be found in [10]. A better known technique of this theory is the Chernoff bound, used to bound the tail of random variables with known moment generating functions [13]. Chernoff bounds will be used to provide the link between thermodynamic theory of networks and large deviations theory.

Consider a node where \(X\) and \(Y\) denote random variables corresponding to traffic rate and buffer occupancy, respectively. The Chernoff bound is derived as follows

\[-\log P\{X > C, Y > B\} = -\log \int_B^{\infty} \int_C^{\infty} f_{XY}(x, y) dxdy\]

\[\geq -\log \int_B^{\infty} \int_C^{\infty} e^{\theta(x-C)+\zeta(y-B)} f_{XY}(x, y) dxdy\]

\[\geq -\log \left(e^{-\theta C} e^{-\zeta B} \int_0^{\infty} \int_0^{\infty} e^{\theta x} e^{\zeta y} f_{XY}(x, y) dxdy\right)\]

\[= \theta C + \zeta B - \psi(\theta, \zeta)\] (7)
where $f_{XY}(x, y)$ and $\psi(\theta, \zeta)$ denote, respectively, the joint probability density and logarithmic moment generating functions of $X$ and $Y$. We define the GoS function $G$ as given by the right-hand side of (7), namely

$$G \triangleq \theta C + \zeta B - \psi(\theta, \zeta). \quad (8)$$

The Laplace parameters $\theta$ and $\zeta$ are chosen to maximize the value of $G$. Since $\psi(\theta, \zeta)$ is convex, $\theta$ and $\zeta$, which maximize the right-hand side of (7), are the solutions to the equations

$$\frac{\partial \psi(\theta, \zeta)}{\partial \theta} = C, \quad \frac{\partial \psi(\theta, \zeta)}{\partial \zeta} = B, \quad (9)$$

where both partial derivatives are taken with $C, N$ and $B$ are kept fixed.

There is a remarkable similarity between equation (8), i.e.,

$$G = \theta C + \zeta B - \psi(\theta, \zeta),$$

and the Euler equation (4) resulting from the scalability postulate, i.e., $G = \theta C - \mu N + \zeta B$. In (4), $\theta = \partial G/\partial C$ and $\zeta = \partial G/\partial B$ by definition, which we now derive from (8) and (9) for the GoS function resulting from large deviations theory.

Using (8), the partial derivative of $G$ with respect to $C$, keeping $B$ and $N$ fixed can be written as

$$\frac{\partial G}{\partial B} = \theta + C \frac{\partial \theta}{\partial C} + B \frac{\partial \zeta}{\partial C} - \frac{\partial \psi(\theta, \zeta)}{\partial \theta} \frac{\partial \theta}{\partial C} - \frac{\partial \psi(\theta, \zeta)}{\partial \zeta} \frac{\partial \zeta}{\partial C} = \theta, \quad (10)$$

where the last equality follows from (9). Similarly, we obtain

$$\frac{\partial G}{\partial N} \bigg|_{C,N} = \zeta. \quad (11)$$

Furthermore from (8), the partial derivative of $G$ with respect to $N_i$, where $N_i$ denotes the number of traffic sources of type $i$, is given by

$$\frac{\partial G}{\partial N_i} \bigg|_{C,B} = C \frac{\partial \theta}{\partial N_i} + B \frac{\partial \zeta}{\partial N_i} - \frac{\partial \psi(\theta, \zeta)}{\partial \theta} \frac{\partial \theta}{\partial N_i} - \frac{\partial \psi(\theta, \zeta)}{\partial \zeta} \frac{\partial \zeta}{\partial N_i} = \left( \frac{\partial \psi(\theta, \zeta)}{\partial N_i} \right)_{C,B,N_i \theta}. \quad (12)$$

where the last equality follows from (9). The intensive parameter $\mu_i = -(\partial G/\partial N_i)_{C,B}$ defined in Section II corresponds to the large deviation parameter $\mu_i = \partial \psi(\theta, \zeta)/\partial N_i$. If the GoS function $G$ defined by (8) is linearly scalable in $C, B$ and $N_i$, then by using (10), (11) and (12) we can write $\psi(\theta, \zeta)$ as

$$\psi(\theta, \zeta) = \sum_i N_i \frac{\partial \psi(\theta, \zeta)}{\partial N_i}. \quad (13)$$

Table IV lists the first-order relations between extensive and intensive quantities obtained by using Chernoff bound that are not listed in Table II. The last equation can be obtained from (9) by using the identity

$$\left( \frac{\partial X}{\partial Y} \right)_2 = \left( \frac{\partial W}{\partial Y} \right)_2 \left( \frac{\partial W}{\partial X} \right)_2.$$

A. Notions of Effective Bandwidth with Small and Large Buffers

In our scalability postulate, we scale up $C, N$ and $B$ simultaneously. Previous works in the literature for effective bandwidth focus primarily on either the small buffer case [15], [16] or the large buffer case [11], [14]. In this subsection, we seek to harmonize these two views. Assuming linear scalability we summarize the results of the earlier part of this section by

$$G = \theta C + \zeta B - \sum_i N_i \mu_i$$

where $\mu_i = \left( \frac{\partial G}{\partial N_i} \right)_{C,B} = \left( \frac{\partial \psi(\theta, \zeta)}{\partial N_i} \right)_{C,B,N_i \theta}. \quad (14)$$

For the small buffer case, the aggregate effective bandwidth $\hat{C}$ is given by

$$\hat{C} = G + \sum_i N_i \frac{\mu_i}{\theta} \quad (15)$$

and it has been proved that the aggregate effective bandwidth, $\hat{C}$, is additive, i.e.

$$\hat{C} = \lim_{B \rightarrow \infty, G/B \rightarrow \zeta} C(G, N, B) = N \hat{c}. \quad (16)$$

For the large buffer case [11], [14], the aggregate effective bandwidth is defined as minimum bandwidth required such that $G = \zeta B$ for large $B$. We observe from (14) that the effective bandwidth $\hat{C}$ satisfies

$$\hat{C} = \sum_i N_i \frac{\mu_i}{\theta} \quad (17)$$

The value of $\theta$ in (18) can be determined by using (9), and $\theta$ is given by the solution of the equation

$$\psi(\theta, \zeta) = \frac{\partial \psi(\theta, \zeta)}{\partial \theta}. \quad (19)$$

Since $\psi(\theta, \zeta)$ is convex and $\psi(0, \zeta) > 0$ for $\zeta > 0$, the solution to (19) uniquely exists. The solution has also a geometric interpretation as the value of $\theta$ for which the tangent line to $\psi(\theta, \zeta)$ passes through the origin.

To unify the two notions of effective bandwidth for the small and large buffer cases, it should be noted that the effective bandwidth for a connection of type $i$ is given by

$$\hat{c}_i = \frac{\mu_i}{\theta} = \left( \frac{\partial G}{\partial N_i} \right)_{C,B} \quad (20)$$
which is the additional bandwidth required for the connection if \( G \) and \( B \) are to be kept constant. The interpretation in (20) can be used for resource trade-off.

**B. Measurement of Intensive Variables**

To apply the above thermodynamical and statistical theories, it is necessary to develop either analytical or experimental methods to measure parameters such as \( \psi, \theta, \zeta, \) or \( \mu_s \). Among these parameters \( \psi \) is probably the most important and can be measured on line using the following procedure.

Denote the moment generating function as \( \phi(\theta) \), i.e., \( \psi(\theta) = \ln \phi(\theta) \). Consider in discrete time step \( k \), we obtain a measurement \( X_k \), which is the average value of parameters such as bit rate or buffer occupancy over the duration of the time step. Here, \( X_k \) could be a vector with \( \theta_k \) being the associated vector of marginal prices. For simplicity of notations, we treat \( X_k \) as a scalar instead. We update the moment generating function \( \phi(\theta) \) at each step as

\[
\phi_{k+1}(\theta) = (1 - \alpha)\phi_k(\theta) + \alpha e^{\theta X_k}.
\]

We use the Taylor series expansions of \( \phi \) and \( \psi \) with finite number of coefficients \( M \). Experimental results indicate that the choice of \( M = 5 \) is sufficient to obtain the estimates with a reasonable amount of error. Let \( a[m] \) and \( d[m], 0 \leq m \leq M - 1 \) denote the \( m \)th coefficients of Taylor series expansions of \( \phi \) and \( \psi \), respectively. In each iteration, we upgrade the following values.

1) Update the coefficients \( a[.] \) by using the Taylor series expansion of (21)

\[
a_{k+1}[m] = (1 - \alpha)a_k[m] + \alpha \frac{(X_k)^m}{m!}
\]

for \( m = 0, \ldots, M - 1 \), and update the coefficients \( d[.] \) recursively as (see [16])

\[
d_{k+1}[0] = 0, d_{k+1}[1] = a_{k+1}[1]
\]

and

\[
d_{k+1}[m] = a_{k+1}[m] - \sum_{j=1}^{m-1} j a_{k+1}[m-j] d_{k+1}[j]
\]

for \( m = 2, \ldots, M - 1 \).

2) Use Newton's method to find \( \theta_{k+1} \) given as the solution of the implicit equation given by (9)

\[
\sum_{m=1}^{M-1} m d_{k+1}[m](\theta_{k+1})^{m-1} = C.
\]

Currently, we are testing the stability and convergence properties of this algorithm for empirical measurements of the shadow prices.

**IV. PROBABILITY OF EVENTS IN HIGH-SPEED NETWORKS EXPRESSED IN CONJUNCTIVE FORM**

Earlier discussion was primarily focused on the single component systems even though the extensive and intensive parameters can be defined as vector quantities for multicomponent systems. In the rest of the paper, we shall be dealing with multicomponent systems using the Gärtnér-Ellis Theorem. An event in a multicomponent system can be written as a logical expression and the set of points corresponding to an event can be expressed in a conjunctive form, namely the intersection of union of subsets, where each subset represents the set of points corresponding to a subevent.

In this section, by using the theory of large deviations, we show that the asymptotic probability of a rare event expressed as a union of a set of subevents is governed by the subevent with the largest probability. As an illustration of the results, consider the graph in Fig. 8 corresponding to a network with alternative routing. \( P \) and \( M \) denote, respectively, the set of paths and the set of cutsets between nodes \( s \) and \( d \). Our results show that the probability of the event for a single path is given exclusively by the most critical link on the path. The probability of the event for all paths in \( P \) is given by the most critical cutset in \( M \), which can be obtained by using the logarithmic moment generating function. The results apply to networks with unbuffered or buffered resources. In the following analysis, the occurrence probability of a rare event is evaluated by using the logarithmic moment generating function which can be expressed analytically if possible or measured by monitoring the statistical quantities relevant for the computation of the rare event.

Let \( G = (N, E) \) be the graph corresponding to the network. \( P_{sd} = \{P_1, \cdots, P_m\} \) denotes the set of all possible paths between nodes \( s \) and \( d \), where \( P_i = (l_{i1}, \cdots, l_{in}) \), \( l_{ij} \in E \) is the \( j \)th link in \( P_i \), and \( q_y \) is the number of links in \( P_i \). We assume that a rare event \( A \) corresponding to nodes \( s \) and \( d \) can be written in a conjunctive form, which is intersection of union of events

\[
A = \bigcap_{i=1}^{m} \bigcup_{j=1}^{q_y} A_{ij}
\]

where \( A_{ij} \) is an event corresponding to link \( l_{ij} \), \( \cap \) and \( \cup \) denote the intersection and union of events, respectively. A typical event that can be expressed in the form (22) is the blocking between nodes \( s \) and \( d \) under alternative routing where all possible paths are allowed, with the event \( A_{ij} \) corresponding to blocking on link \( l_{ij} \).

**Definition 4.1:** A subset \( E' \subseteq E \) is called a \((s,d)\)-cutset if \( s \) and \( d \) are in different components in \( G' = (N, E - E') \).

Let \( M_{sd} = \{M_1, \cdots, M_k\} \) be the set of minimal \((s,d)\)-cutsets, where \( M_k = (k_{11}, \cdots, k_{1q}) \), with \( q_k \) being the number of links in \( M_k \). The following lemma gives the disjunctive form (union of intersection of events) corresponding to (22) which will be used in Theorem 4.4 to obtain the probability
of a rare event between \( s \) and \( d \). When \( A_{ij} \) corresponds to congestion on link \( l_{ij} \), Lemma 4.1 demonstrates the fact that all paths between \( s \) and \( d \) are congested if and only if there exists a congested cutset between \( s \) and \( d \).

**Lemma 4.1:**

\[
\bigcup_{i=1}^{m} \bigcup_{j=1}^{q_i} A_{ij} = \bigcap_{k=1}^{r} \bigcap_{l=1}^{t_k} A_{kl}.
\]

**Proof:** Let \( A \) be an event such that \( A \in \bigcap_{k=1}^{r} \bigcap_{l=1}^{t_k} A_{kl} \) for some \( k, 1 \leq k \leq r \). \( A \) is contained in \( A_{kd} \) for all links of cutset \( M_k \) and each path between \( s \) and \( d \) uses at least one link from \( M_k \). Therefore, for any path \( i \), \( A \) is contained in \( A_{ij} \) for some link \( l_{ij} \), \( 1 \leq j \leq q_i \). Hence

\[
A \in \bigcap_{i=1}^{m} \bigcup_{j=1}^{q_i} A_{ij}.
\]

Conversely, let \( A \in \bigcap_{i=1}^{m} \bigcup_{j=1}^{q_i} A_{ij} \). For each path \( P \), \( A \) is contained in \( A_{ij} \) for some link \( l_{ij} \), \( 1 \leq j \leq q_i \). \( \{l_{ij}\}_{i=1}^{M} \) is a \((s,d)\)-cutset since \( P_{sd} \) contains all possible \((s,d)\) paths in \( G \) and by deleting an edge from each path, \( s \) and \( d \) become disconnected in \((\bar{G},E-\{l_{ij}\})\). There exists a cutset \( M_k \) in \( M_{sd} \) which is a subset of \( \{l_{ij}\} \) since \( M_{sd} \) contains all minimal \((s,d)\)-cutsets. Hence

\[
A \in \bigcap_{k=1}^{r} \bigcup_{l=1}^{t_k} A_{kl}.
\]

This lemma is useful in two ways as we shall see later. First, it simplifies the calculation by expressing the events in terms of critical cutsets. Second, the disjunctive expression of the event is critical in the following proofs.

Consider a sequence of networks indexed by \( n \) (each having the same topology), where \( Z_n \) denotes a random vector representing a set of random variables each corresponding to a resource occupancy measure in the \( n \)th network. We assume that \( E[Z_n] = nE[Z_1] \), i.e., the average values of the random variables constituting \( Z_n \) increase linearly as \( n \) gets larger.

Let \( A_n \) be an event corresponding to congestion in the \( n \)th network, that can be written as a union of subevents such that

\[
A_n = \bigcup_{i} A_{ni}
\]

with \( \Pr(A_{ni}) = \Pr(Z_n \in nS_i) \) where \( S_i \subset \mathbb{R}^d \) is a convex set. Since \( A_n = \bigcup_{i} A_{ni} \), we have \( \Pr(A_n) = \Pr(Z_n \in nS) \), where \( S = \bigcup_{i} S_i \).

We show that the probability of \( A_n \) as \( n \to \infty \) is governed by the probability of the subevent with the largest probability. The motivation under this result lies in the fact that the probability of \( A_n \) can be upperbounded (by union bound) as a sum of exponentials with large negative exponents, which is dominated by the term with the exponent of smallest magnitude (Laplace’s principle).

This result can be applied to a queueing network, where \( A_n \) denotes the event that a packet will be lost at some buffer on its route. \( Z_n \) is the random vector corresponding to buffer occupancies and \( S \) is the set of points for which the occupancy of one of the buffers on the path exceeds the buffer size. The computation of this probability requires the analysis of the queue with largest loss probability along the path. Thus, using this analysis all methods applicable to a single buffer can be extended to a network of buffers. In the remaining part of this section, we analyze the congestion between two nodes in a network.

We evaluate the probability of events expressed in the disjunctive form derived earlier in this section in Lemma 4.1. We consider a network with \( C = (C_1, \ldots, C_M)^T \) denoting a threshold vector above which congestion occurs. This threshold vector may contain both transmission and buffering constraints. The network operates under alternative routing, where a connection request blocked on a path can try other paths. A connection request is rejected when all paths between origin and destination are blocking. The notion of a path here is more general than a physical path of links in a circuit-switched network. Paths could be considered as a constituent comprising an alternative and links could be considered as components of a path which may be a transmission or a buffering facility. \( \psi(\bar{\theta}) \) is the logarithmic moment generating function corresponding to a measure of congestion.

We show that when the threshold vector and amount of the traffic are increased together, asymptotically the loss probability on a path is given by the most congested link along the path. We then characterize the congestion between a node pair which corresponds to the congestion of all possible paths.

Consider a sequence of networks indexed by \( n \). Let the random vector \( Z_n \) denote a measure of utilization in the \( n \)th network. Let \( C_n \) denote the vector corresponding to congestion thresholds in the \( n \)th network. We set \( E[Z_n] = nE[Z_1] \) and \( C_n = nC \) so that the ratios of average utilizations to congestion thresholds are kept fixed. Assume that the logarithmic moment generating function \( \psi_n(\bar{\theta}) \) of \( Z_n \) satisfies the assumptions of Gärtner-Ellis Theorem given in Appendix C.

**Theorem 4.1 Gärtner-Ellis [5]:** If Assumptions C.1 and C.2 are true, then

\[
\lim_{n \to \infty} \sup_{\bar{\theta} \in \mathbb{R}^d} \frac{1}{n} \log \Pr\left( \frac{Z_n}{n} \in F \right) \leq - \inf_{\bar{\theta} \in \mathbb{R}^d} I(\bar{\theta})
\]

for any closed \( F \).

If Assumptions C.1, C.2 and C.3 are true, then

\[
\lim_{n \to \infty} \inf_{\bar{\theta} \in \mathbb{R}^d} \frac{1}{n} \log \Pr\left( \frac{Z_n}{n} \in G \right) \geq - \inf_{\bar{\theta} \in \mathbb{R}^d} I(\bar{\theta})
\]

for any open \( G \), where \( I(\bar{\theta}) \) is the Legendre transform of \( \psi(\bar{\theta}) \) defined by

\[
I(\bar{\theta}) = \sup_{\bar{\theta} \in \mathbb{R}^d} \left[ \theta \cdot \bar{\theta} - \psi(\bar{\theta}) \right]
\]

where

\[
\psi(\bar{\theta}) = \lim_{n \to \infty} \psi_n(\bar{\theta})/n \text{ and } \psi_n(\bar{\theta}) = \log E[e^{\bar{\theta} \cdot Z_n}].
\]

In the \( n \)th network, for a path \( P \in P_{sd} \)

\[
\Pr(P \text{ is congested}) = \Pr\left( \bigcup_{i=t_i \in P} \{ Z_{ij} > nC_j \} \right)
\]

where \( Z_{ij} \) denotes the utilization on link \( l_{ij} \).
Before discussing the applications of the Gärtner-Ellis Theorem for alternative routing in high-speed networks, we present the following min-max theorem which will be used later in the proofs of Theorems 4.3 and 4.4. The proof of Theorem 4.2 follows from the fact that \(< \theta, \bar{z} > \geq -\psi(\bar{\theta})\) is a saddle-function [21].

**Theorem 4.2:** For a closed convex set \( F \),
\[
\inf_{\bar{\theta} \in F} \sup_{\bar{z} \in G} \left( \langle \theta, \bar{z} \rangle - \psi(\bar{\theta}) \right) = \inf_{\bar{\theta} \in F} \sup_{\bar{z} \in G} \left( \langle \theta, \bar{z} \rangle - \psi(\bar{\theta}) \right).
\]

The following theorem states that the probability that \( P_i \) is congested is given by the most congested link along \( P_i \).

**Theorem 4.3:**
\[
\lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ \bigcup_{l_j \in P_i} \{ Z_{n_j} > nC_j \} \right\} \leq - \inf_{x \in \overline{G}_i} I(\bar{x})
\]

where \( \psi(\theta) = \psi(\bar{\theta}) \left|_{\theta_i = 0, \theta_j \neq j} \right. \) and \( \theta_j^* \) is given by \( \frac{\partial \psi(\theta_j^*)}{\partial \theta_j^*} = C_j \).

**Proof:**
\[
\lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ \bigcup_{l_j \in P_i} \{ Z_{n_j} > nC_j \} \right\} = \lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ Z_{n_j} \in G_i \right\}
\]

where \( G_i = \{ \bar{x} : \bar{x} \geq 0 \) and \( x_j > C_j \) for some \( j \) such that \( l_j \in P_i \) and \( G_i \) denotes the closure of \( G_i \). Define \( F_j = \{ \bar{x} : \bar{x} \geq 0 \) and \( x_j > C_j \} \). We have \( G_i = \bigcup_{l_j \in P_i} F_j \).

\( \{ F_j \} \) are closed and convex, hence we can apply min-max theorem
\[
\inf_{x \in \overline{G}_i} I(\bar{x}) = \inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x})
\]

we obtain
\[
\inf_{x \in \overline{G}_i} I(\bar{x}) = \inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x})
\]

where \( \psi(\theta_j^*) \) is the marginal logarithmic moment generating function for link \( j \), and the second equality follows from
\[
\frac{\partial \psi(\theta_j^*)}{\partial \theta_j^*} = \frac{E[Z_{n_j}e^{\langle \theta_j^*, \bar{z}_j \rangle}]}{E[e^{\langle \theta_j^*, \bar{z}_j \rangle}]} \geq 0 \quad \text{since} \quad Z_{n_j} \geq 0 \quad \text{all} \quad n
\]

and from the uniform convergence of \( \psi(\theta_j^*) \), i.e., \( \psi(\theta_j^*) \) is increasing in \( \theta_j \). The last equality is obtained by using the concavity of \( C_j \theta_j - \psi(\theta_j^*) \), with \( \theta_j^* \) defined in the statement of the theorem.

Next, we look at the probability that all paths between \( s \) and \( d \) are congested. In proving the following result, we use the disjunctive form proved in Lemma 4.1, in order to be able to express the set corresponding to congestion as a union of convex sets. The theorem states that the probability of a rare event is dominated by the most critical cutset. It is also mentioned as a consequence that when the links in a cutset are independent, the upper bound can be written as a sum over links constituting the critical cutset, where each term corresponds to the exponent of the upper bound for individual links.

**Theorem 4.4:**
\[
\lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ \bigcup_{i=1}^m \bigcap_{j=1, j \in M_i} \{ Z_{n_j} > nC_j \} \right\} \leq \inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x})
\]

where \( \phi^* \) satisfies
\[
\frac{\partial \psi(\phi^*)}{\partial \phi^*} = C_j
\]

for \( j \in M_i \) and \( \theta_j^* = 0 \) otherwise.

**Proof:**
\[
\lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ \bigcup_{i=1}^m \bigcap_{j=1, j \in M_i} \{ Z_{n_j} > nC_j \} \right\} = \lim_{n \to \infty} \frac{1}{n} \log \Pr \left\{ Z_{n_j} \in S \right\} \leq - \inf_{x \in \overline{G}_i} I(\bar{x})
\]

since \( \bigcap_{l_j \in P_i} F_j \) is closed and convex. We have from the min-max theorem
\[
\inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \sup_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \sup_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x})
\]

We can write
\[
\inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \sup_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \sup_{x \in \bigcup_{l_j \in P_i} F_j} \left[ \sum_{j=1}^m C_j \theta_j + \psi(\theta_j^*) \right]
\]

Hence, we obtain
\[
\inf_{x \in \bigcup_{l_j \in P_i} F_j} I(\bar{x}) = \sup_{x \in \bigcup_{l_j \in P_i} F_j} \left[ \sum_{j=1}^m C_j \theta_j + \psi(\theta_j^*) \right]
\]

where \( \theta_j^* \) satisfies
\[
\frac{\partial \psi(\phi^*)}{\partial \phi^*} = C_j
\]

for \( j \in M_i \) and \( \theta_j^* = 0 \) otherwise. The last equality is obtained by using the fact that \( \psi(\phi) \) is increasing in \( \phi \) for all \( k \) and using the concavity of \( \langle \theta, \bar{z} \rangle - \psi(\bar{\theta}) \) with respect to \( \bar{\theta} \).
Remark: If the links belonging to the same cutset are blocked independently, $\psi_j(\theta^*_j)$ can be written as a sum of the individual logarithmic moment generating functions each corresponding to a link constituting cutset $M$. We have

$$\limsup_{n \to \infty} \frac{1}{n} \log \text{Pr} \left\{ \bigcup_{i=1}^{m} \bigcap_{j \in M_i} \{ Z_{nj} > nC_j \} \right\}$$

$$\leq \inf_{i} \left[ \sum_{j \in M_i} C_j \theta^*_j - \psi^j(\theta^*_j) \right]$$ (23)

where $\theta^*_j$ satisfies

$$\frac{\partial \psi^j(\theta^*_j)}{\partial \theta^*_j} = C_j$$

and $\psi^j(\theta_j)$ is the marginal logarithmic moment generating function for link $j$.

The loss probability between $s$ and $d$ with multiple paths is given asymptotically by the loss probability of the most congested $(s, d)$-cutset. The probability of congestion for a cutset can be computed by numerical differentiation for the solution of the corresponding Laplace parameter for each link in the cutset. It is noteworthy that for events consisting of subevents, the right-hand side of (23) is an upper bound, not just an asymptotic result.

If the function $I(\varepsilon)$ is continuous on the boundaries of the sets $\{ F_j \}_{j=1}^M$, then the infimum taken over the closure set $\overline{\mathcal{F}}_t$ will be same as the infimum taken over $F_t$. In this case, by using the lower bound in the Gärtner-Ellis Theorem, we conclude that the upper bound is actually reached as the limit.

V. EFFECTIVE BANDWIDTH NETWORK

ROUTING AND CONCLUSION

In this section we propose a dynamic network routing strategy for which decisions are based on the residual effective bandwidth on network components. This policy is similar to RTNR method [2] used for circuit-switched networks. The EBNR mechanism is applicable to any network containing both unbuffered and buffered resources. The routing decisions are done according to the following criteria: For a connection to be established we choose the path that maximizes the residual effective bandwidth. The residual effective bandwidth on a path is determined by the component of the path with the smallest residual effective bandwidth. The motivation for this rule is twofold. First, the effective bandwidth is a good measure of the required bandwidth to achieve a given GoS criteria on a component as a function of the traffic demand and buffer size. Second, as proven in Section IV, the probability of a congestion on a path is dominated by the most congested component.

The routing rule for EBNR is expressed as: For a connection request between $s$ and $d$ of traffic demand type $i$, first try the direct connection, say component $j$, if any. Use the direct connection if

$$C_j \geq C_j(G^*_j, N_j + e_i, B_j)$$

where $C_j$ is the capacity for component $j$ and $C_j(G^*_j, N_j, B_j)$ is the required bandwidth for component $j$ as a function of the required GoS $G^*_j$, current traffic demand $N_j$ and buffer size $B_j$, which is given by the BD equation. $e_i$ is the unit vector in the $i$th component with appropriate dimension. Otherwise, choose the alternative path $P_k \in P_{ad}$ that maximizes

$$R_k = \min_{l \in P_k} C_l - C_l(G^*_l, N_l, B_l)$$ (24)

and use $P_k$ if

$$R_k \geq C_l(G^*_l, N_l + e_i, B_i) - C_l(G^*_l, N_l, B_l)$$ (25)

where component $l$ on path $P_k$ is the one that achieves the minimum in (24). Otherwise, the connection request is rejected. The actual GoS $G_l$ may be larger or smaller than the required GoS $G^*_l$. When the number of traffic sources using component $l$ is large, the right-hand side of (25) can be approximated by the derivative of the aggregate effective bandwidth with respect to $N_l$ given in Table II. (25) is then equivalent to

$$R_k \geq \frac{\mu_i(\theta_l, \zeta_l)}{\theta_l}$$ (26)

where $\theta_l$ and $\zeta_l$ are the intrinsic variables corresponding to component $l$. The right-hand side of (26) corresponds the additional effective bandwidth required for routing a traffic demand of type $i$ through component $l$. The change in actual GoS for component $l$ after routing demand $i$ is given by $(\partial G_i/\partial N_l)_{C_j, B_j} = -\mu_i(\theta_l, \zeta_l)$, where $N_l$ is the number of traffic demands of type $i$ on component $l$.

The implementation of the routing algorithm requires the monitoring of extensive parameters. The effective bandwidth of a network component $l$ can be obtained from the measurements of the intensive parameters $\theta_l$, $\zeta_l$ and $\mu_i$ for component $l$ as described in Section III.

$$C_l(G^*_l, N_l, B_l) = \frac{G^*_l}{\theta_l} + \frac{1}{\theta_l} \sum_i N_i \mu_i - \frac{\zeta_l}{\theta_l} B_l.$$  

The algorithm also requires the exchange of the necessary information between components. The stability of the routing method need to be addressed. A bandwidth reservation technique similar to the trunk reservation method used in circuit-switched networks may be incorporated into the algorithm.

In this paper, we have addressed the problem of identifying the fundamental relations between quantities such as GoS, transmission capacity, buffer resources and traffic demand in a network. The way these parameters scale with respect to each other defines the set of very useful intensive variables. In this context the problem of effective bandwidth for buffered resources is revisited. These parameters can be obtained either analytically or by measurements. Empirical methods to measure these quantities are discussed. The theory is applied to networks with dynamic routing and the probabilities of rare events expressed in conjunctive forms are characterized by using large deviations theory. Further research in this area should be directed at the scalability problem in general systems, measurement methods and stability issues.
APPENDIX A

In this part we justify the scalability property for M/M/1/B and M/M/C+C/B queues when C and B are large.

I. M/M/1/B:

The logarithm of the loss probability is given by

\[
\log p_B = \log(1 - \rho) - \log(1 - \rho^B) + B \log \rho
\]

where \( \rho = \lambda/\mu < 1 \) is the utilization factor with \( \lambda \) and \( \mu \) denoting the arrival and transmission rates, respectively. The logarithm of the loss probability is asymptotically given by

\[
\log p_B = B \log \rho
\]

(27)

since \( \log(1 - \rho) \) and \( \log(1 - \rho^B) \) become very small compared to \( B \log \rho \). When \( \lambda, \mu \) and \( B \) are all scaled by a large alpha (\( \alpha \) is kept constant), the GoS equation (27) shows that the logarithm of the loss probability is scaled by \( \alpha \).

II. M/M/C+C/B:

The logarithm of the loss probability is given by

\[
\log p_{C+B} = \log p_0 + C \log C! + (C + B) \log \rho
\]

where \( \rho = \lambda/\mu < 1 \) is the utilization factor with \( \lambda \) and \( \mu \) denoting the arrival rate and transmission rate per channel, respectively, and \( p_0 \) is given by

\[
p_0 = \left[ \sum_{n=0}^{C} \frac{(C \rho)^n}{n!} + \frac{C^C}{C!} \sum_{n=C+1}^{B+C} \rho^n \right]^{-1}.
\]

When \( C \) is large, using Stirling’s formula we obtain

\[
\log p_{C+B} = \log p_0 + (C + B) \log \rho + C.
\]

Using Stirling’s formula we have

\[
e^{C\rho} = \max_{0 \leq n \leq C} \frac{(C \rho)^n}{n!} < \sum_{n=0}^{C} \frac{(C \rho)^n}{n!} < \sum_{n=0}^{\infty} \frac{(C \rho)^n}{n!} = e^{C\rho}
\]

and

\[
\frac{CC}{C!} \sum_{n=C+1}^{B+C} \rho^n = \frac{(C \rho)^C}{C!} \sum_{n=1}^{B} \rho^n = (\epsilon \rho)^C \sum_{n=1}^{B} \rho^n.
\]

\[- \log p_0 = C \rho + \log(1 + (\epsilon \rho)^C \sum_{n=1}^{B} \rho^n) \approx C \rho
\]

since \( \epsilon \rho < \rho \) for \( \rho < 1 \) and the summation is upper bounded by \( \rho/(1 - \rho) \). The logarithm of the loss probability is asymptotically given by

\[
\log p_{C+B} = (C + B) \log \rho + C(1 - \rho).
\]

(28)

When \( \lambda, C \) and \( B \) are scaled together by a large \( \alpha \) (keeping \( \mu \) and thus \( \rho \) fixed), we observe from (28) that the logarithm of the loss probability is also scaled by \( \alpha \). Hence, the GoS equation for M/M/C+C/B queue is scalable.

In Fig. 9, we plot GoS versus capacity for \( \rho = 0.8 \) and different values of \( \gamma \). We observe that linear scalability holds even for small values of \( C \) and \( B \).

APPENDIX B

Proof of Theorem 2.1: Consider a slotted time queuing system with \( N \) statistically independent sources multiplexed into the system. The multiplexer is served with a fixed rate channel of capacity \( C = N c \) bits per time slot. Bits arriving when the bandwidth is fully utilized are placed in a buffer of size \( B = N b \) bits. Let \( X^i \) denote the number of bits arriving during time slot \( j \) generated by source \( n \). Assume the system is in steady state and let \( p(Nc, N, Nb) \) denote the loss probability. The random variable \( Y_i = \sum_{n=1}^{N} \sum_{j=1}^{N} X^i_{n,j} \) denotes total number of bits arriving at the system in the interval \([t_i, t_{i+1})\). Let \( A_i \) denote the event that the system overflows at time \( t_i \) and the buffer was last empty at time \( t_i \).

We have \( p(Nc, N, Nb) = \sup_{i \geq 1} \Pr\{A_i\} \). A necessary condition for \( A_i \) is that \( Y_i > NcB + Nb \). Hence

\[
\Pr\{A_i\} \leq \Pr\{Y_i > NcB + Nb\}.
\]

(29)

Since the right-hand side of (29) is a lower bound on \( p(Nc, N, Nb) \) for any \( i \), we have for large \( N \)

\[
p(Nc, N, Nb) \leq \sup_{i \geq 1} \Pr\{Y_i > NcB + Nb\}.
\]

(30)

We use Gärtner-Ellis Theorem to obtain

\[
\lim_{N \rightarrow \infty} \frac{1}{N} \log \Pr\left\{ \frac{1}{N} Y_i > c + \frac{b}{i} \right\} = -I(c + \frac{b}{i})
\]

(31)

where

\[
I(x) = \sup_{\theta \geq 0} \left\{ \theta x - \psi(x) \right\}
\]

\[
\psi(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \log E(e^{\theta Y_i}) = \frac{1}{i} \log E(e^{\theta W})
\]

where \( W \) is a random variable having the same distribution as \( \sum_{n=1}^{N} \sum_{j=1}^{N} X^i_{n,j} \). Hence, we have

\[
\lim_{N \rightarrow \infty} \frac{G(Nc, N, Nb)}{N} = \inf_{\theta, \phi \geq 0} \left\{ \phi + i(\theta - \psi(x)) \right\}.
\]

(32)

Equation (32) establishes the claim of Theorem 2.1. □
APPENDIX C

In this appendix, we present the assumptions under which Gartner-Elfss Theorem holds. Consider a sequence of random vectors $\tilde{Z}_n \in \mathbb{R}^d$ which has a logarithmic moment generating function

$$
\psi_n(\theta) = \log E[e^{i \theta \cdot \tilde{Z}_n}]
$$

where $\theta \in \mathbb{R}^d$ and $<,>$ denotes the inner product operation. It can proved that $\psi_n(\theta)$ is convex and lower semi-continuous [9].

Assumption C.1: For each $\theta \in \mathbb{R}^d$, the limit defined as

$$
\psi(\theta) = \lim_{n \to \infty} \frac{1}{n} \psi_n(\theta)
$$

exists as an extended real number. Furthermore, $\psi(\theta)$ is convex and lower semi-continuous.

Assumption C.2: The origin belongs to the interior of the domain $D_\psi$ of $\psi$ defined by $D_\psi = \{ \theta : \psi(\theta) < \infty \}$.

Definition C.1: A convex function $\psi: \mathbb{R}^d \to \mathbb{R}$, differentiable on the domain $D_\psi$, is steep if a sequence $\{ q_n \} \subset \mathbb{R}^d$ with $q_n \to \theta$ where $\theta$ is on the boundary of $D_\psi$ implies that

$$
\lim_{n \to \infty} \| \psi(q_n) \| = \infty.
$$

Assumption C.3: $\psi(\theta)$ is steep.

Remarks: Since $\{ \psi_n(\theta) \}$ are convex functions, $\psi(\theta)$ is convex if $D_\psi$ is a convex set. $\psi(\theta)$ is lower semi-continuous if $\psi_n/n \to \psi$ uniformly. Assumption C.2 is rather a mild condition and holds for almost all random vectors sequences.

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REFERENCES


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