Introduction to Wavelet Transform

A. Enis Çetin
Visiting Professor
Ryerson University

Overview of Wavelet Course

- Sampling theorem and multirate signal processing
- Wavelets form an orthonormal basis of $L^2(R)$
- Time-frequency properties of wavelets and scaling functions
- Perfect reconstruction filterbanks for multirate signal processing and wavelets
- Lifting filterbanks
- Adaptive and nonlinear filterbanks in a lifting structure
- Frames, Matching Pursuit, Curvelets, EMD, ...
- Applications

Wavelets form an orthonormal basis of L^2 :

Let $x(t) \in L^2$

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 2^{k/2} w_{k,l} \psi(2^k t - l)$$

where

- $\psi(\, . \,)$: wavelet basis function

- Wavelet (transform) coefficients: $w_{k,l}$
- Countable set of coefficients: *k,l* are integers
- There are many wavelets satisfying the above equation

Wavelet coefficients

$$w_{k,l} = \langle x(t), 2^{k/2}\psi(2^k t - l) \rangle = \int_{-\infty}^{\infty} 2^{k/2} x(t)\psi(2^k t - l)dt$$

- Mother wavelet $\psi(t)$ may have a compact support, i.e., it may be finite-extent => wavelet coefficients have temporal information
- The basis functions are constructed from the mother wavelet by translation and dilation
- Countable basis functions:

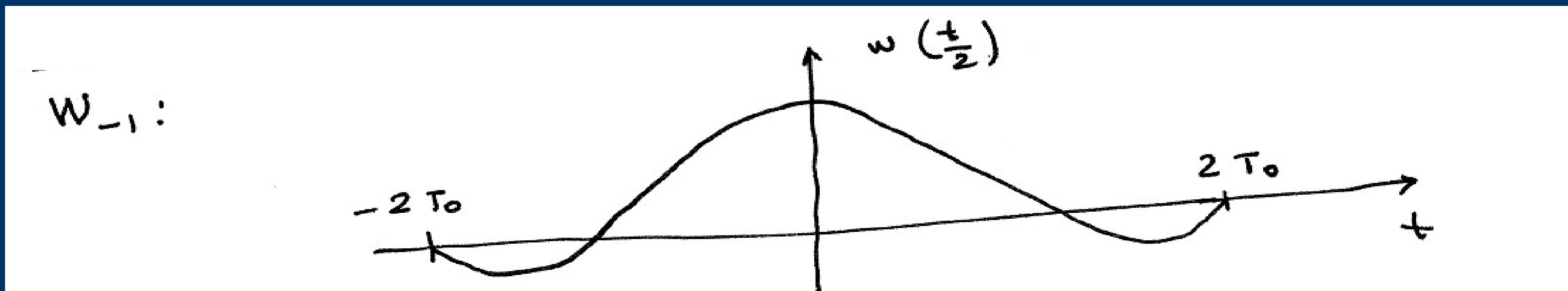
$$2^{k/2}\psi(2^k t - l),\ k, l \ are \ integers$$

- Wavelets are orthonormal to each other
- Wavelet is a "bandpass" function
- In practice, we don't compute the above integral!

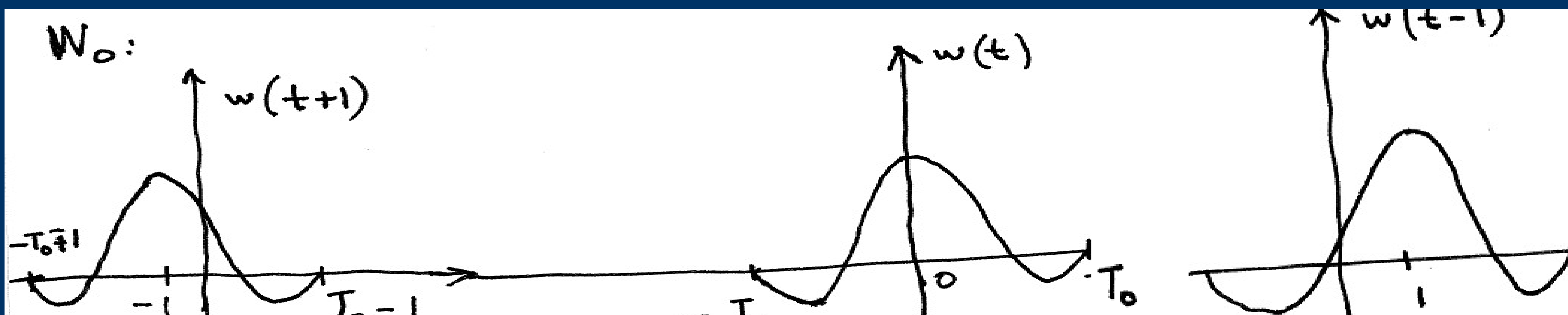
Multiresolution Framework

Let w(t) be a mother wavelet:

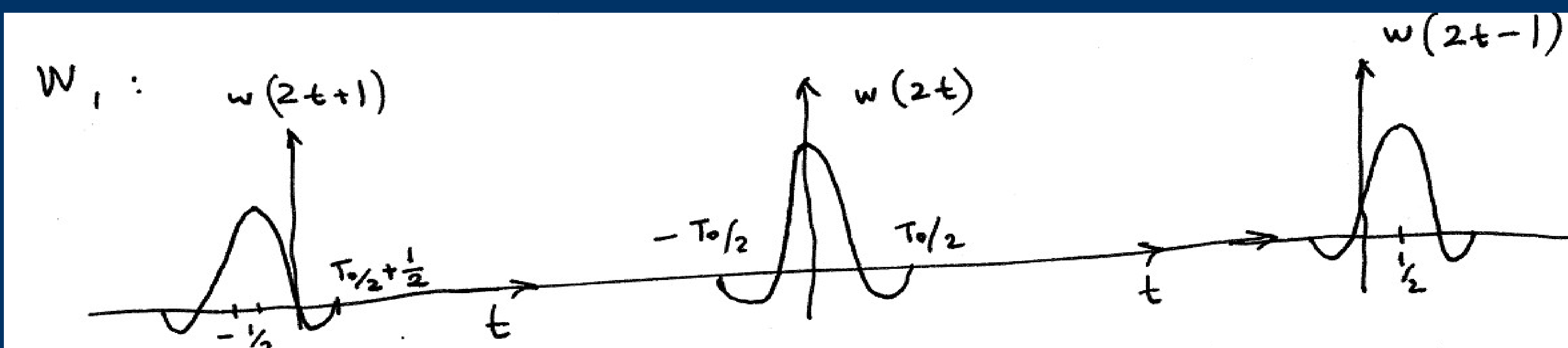
k= -1



k=0



k=1



Fourier Transform (FT)

- Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

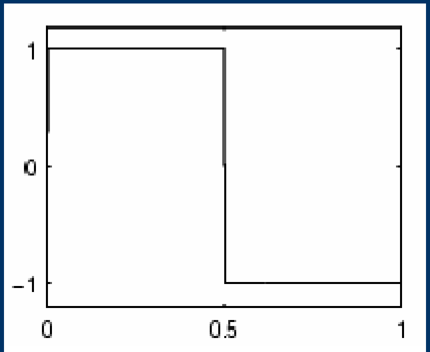
- $e^{j\omega t}$ does not have a compact support, i.e., it is of infinite extent : - ∞ < t < ∞ => no temporal info
- $e^{j\omega t}$ is also a bandpass function => delta at *ω*
- *F(ω)* is a continuous function (uncountable) of *ω*

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

- Uncountablity => integral in FT instead of summation in WT

Example: Haar Wavelet

$$\psi(t) = \begin{cases} \frac{1}{2}, & \textbf{if} \quad 0 \leq t \leq \frac{1}{2} \\ -\frac{1}{2}, & \textbf{if} \quad \frac{1}{2} < t < 1 \\ 0, & \textbf{otherwise} \end{cases}$$



Corresponding scaling function:

$$\phi(t) = \begin{cases} 1, \textbf{if} \quad 0 \leq t < 1 \\ 0, \textbf{otherwise} \end{cases}$$

- Haar wavelet is the only orthonormal wavelet with an analytic form
- It is not a good wavelet !

Wavelet and Scaling Function Pairs

- It is possible to have "zillions" of ortogonal mother wavelet functions
- It is possible to define a corresponding scaling function $\phi(t)$ for each wavelet
- Scaling function is a low-pass filter and it is orthogonal to the mother wavelet

$$\psi(t) \perp \phi(t)$$

- Scaling coefficients (low-pass filtered signal samples):

$$c_{l,k} = \int_{-\infty}^{\infty} x(t)\phi(2^k t - l)dt$$

Wavelet and Scaling Function Properties-II

- Scaling function $\varphi(t)$ is not orthogonal to $\varphi(kt)$
- Wavelet $\psi(t)$ is orthogonal to $\psi(kt)$, for all integer k
- Haar wavelet:

$$\varphi(t) = 1\varphi(2t) + 1\varphi(2t-1)$$
$$\psi(t) = 1\varphi(2t) - 1\varphi(2t-1)$$

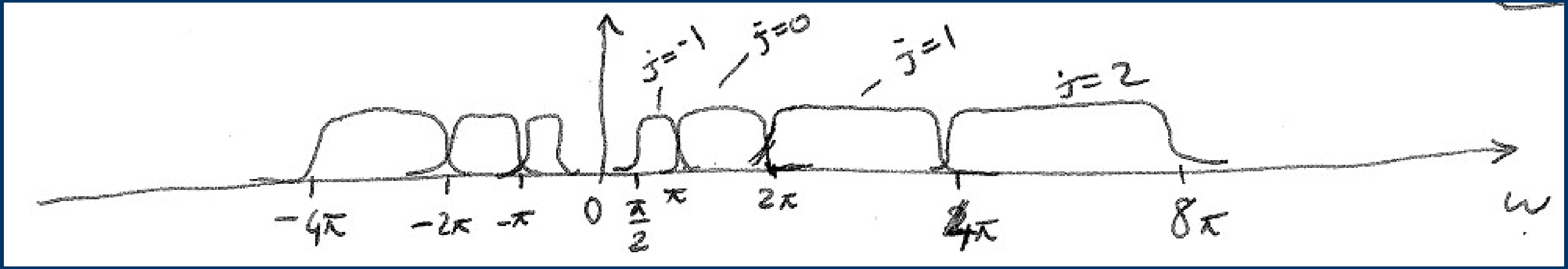
- Haar transform matrix: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- Daubechies 4th order wavelet:

$$\psi(t) = [(1-\sqrt{3})\varphi(2t) -(3-\sqrt{3})\varphi(2t-1)+(3+\sqrt{3})\varphi(2t-2) -(1+\sqrt{3})\varphi(2t-3)]/4\sqrt{2}$$
$$\varphi(t) =[(1+\sqrt{3})\varphi(2t)+(3+\sqrt{3})\varphi(2t-1)+(3-\sqrt{3})\varphi(2t-2)+(1-\sqrt{3})\varphi(2t-3)]/4\sqrt{2}$$

Wavelet family (... ψ(t/2), ψ(t), ψ(2t), ψ(4t),..) covers the entire freq. band

- Ideal passband of $\psi(t)$: $[\pi, 2\pi]$
- Ideal passband of $\psi(2t)$: $[2\pi, 4\pi]$
- Almost no overlaps in frequency domain:

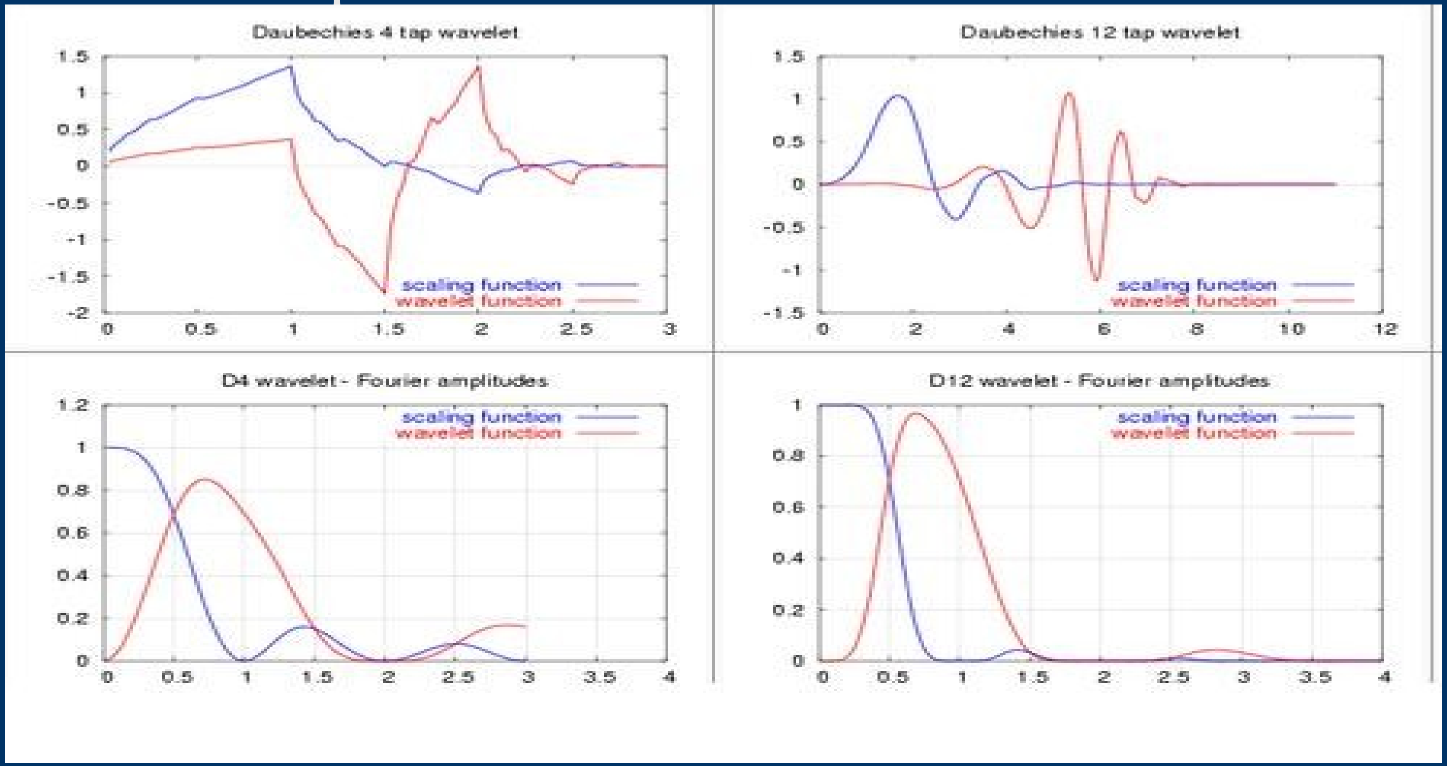


- Scaling function is a low-pass function:
- Ideal passband of $\varphi(t)$: $[0, \pi]$
- Ideal passband of $\varphi(2t)$: $[0, 2\pi]$
- Scaling coefficients: low-pass filtered signal samples of x(t):

$$c_{l,k} = \int_{-\infty}^{\infty} x(t)\phi(2^k t - l)dt$$

Daubechies 4 (D4) wavelet and the corresponding scaling function

- D4 and D12 plots:



- Wavelets and scaling functions get smoother as the number of filter coefficients increase
- D2 is Haar wavelet

Multiresolution Subspaces of $L^2(R)$

$$\vdots$$

$$V_{-1} = span\ \{\phi(t/2 - l), l\ integer\}$$

$$V_o = span\ \{\phi(t - l),\ l\ integer\}$$

$$V_1 = span\ \{\phi(2t - l), l\ integer\}$$

$$\vdots$$

$$A\ scale\ of\ subspaces:$$

$$\{0\} \subset ...V_{-1} \subset V_o \subset V_1 \subset ... \subset L^2(R)$$

- An ordinary analog signal may have components in all of the above subspaces: $c_{l,k} = \int_{-\infty}^{\infty} x(t)\phi(2^k t - l)dt$ $\neq 0$ for all k
-
- A band limited signal will have $c_{l,k} = 0$ for k > K

Properties of multiresolution subspaces V_j

Multiresolution Decomposition of

The subspaces V_j satisy:

1) $V_j \subset V_{j+1}$ and $\bigcap_j V_j = \{0\}$ and $\overline{\bigcup_j V_j} = L^2$

2) Scale invariance: $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$

3) Shift " : $f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0$

Wavelet subspaces

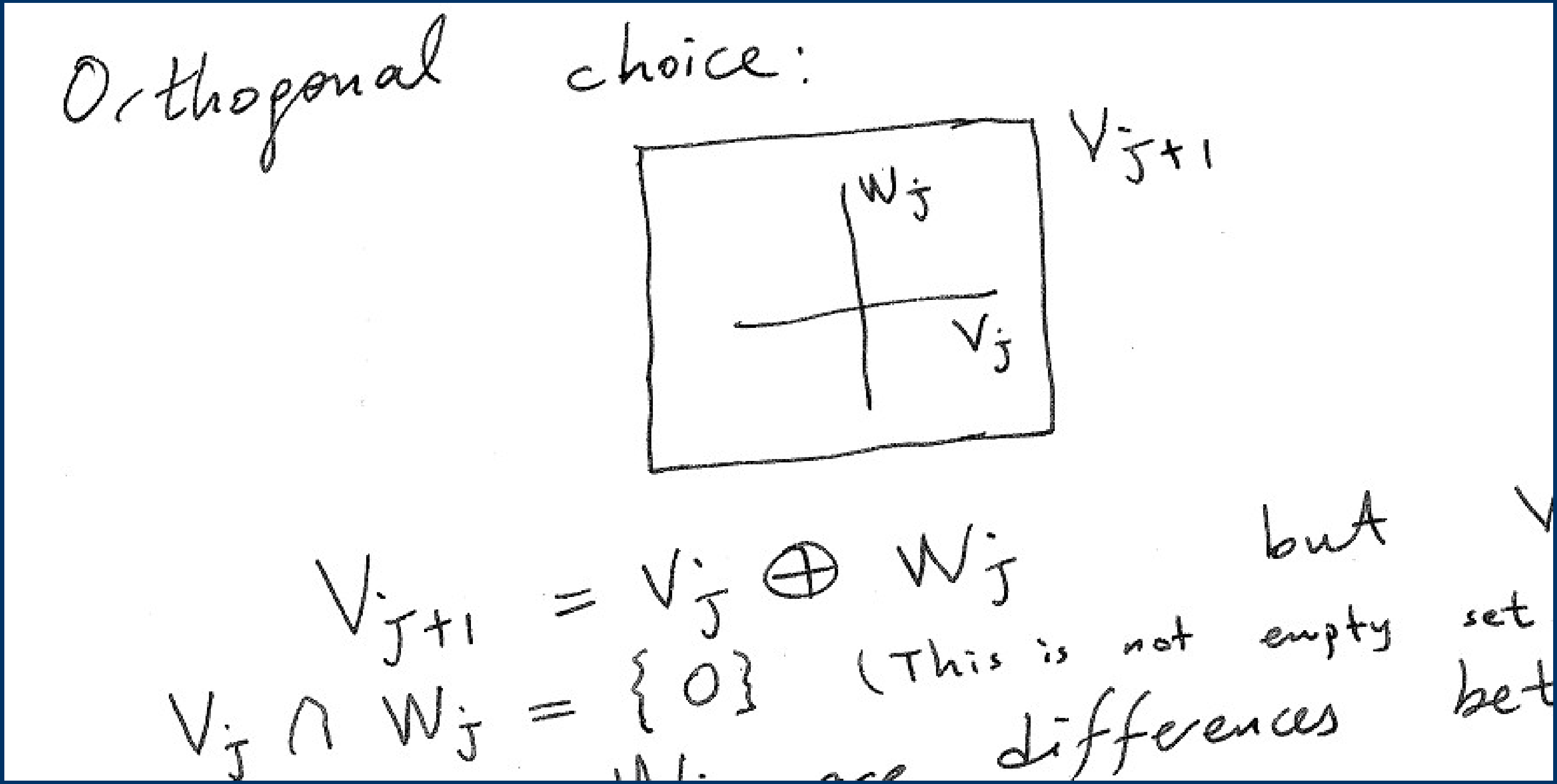
- $W_o = span\{\psi(t-l), integer\ l\}, \ldots$

$$W_j = span\ \{\psi(2^j t - l), l\ integer\}$$

- $V_j \oplus W_j = V_{j+1}$
- $\bigcup_{j=-\infty}^{\infty} W_j = L^2(R)$

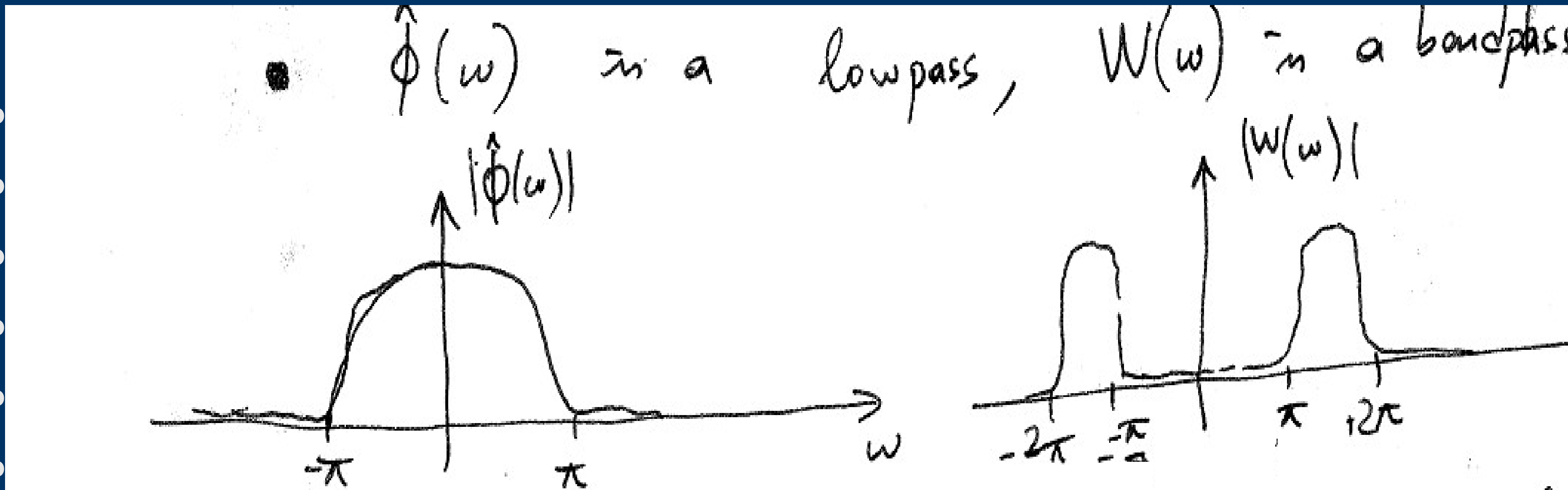
- W_j does not contain W_k, $j>k$ (*but* V_j *does contain* V_k)
- It is desirable to have V_j to be orthogonal to W_j

Geometric structure of subspaces



- W_{j+1} is the "z-axis", V_{j+2} is the 3-D space ...

Ideal frequency contents of wavelet and scaling subspaces:



- Subspace V_0 contains signals with freq. content $[0,\pi]$
- Subspace W_0 contains signals with freq. content $[\pi,2\pi]$
- Subspace V_1 contains signals with freq. content $[0,2\pi]$
- Subspace W_1 contains signals with freq. content $[2\pi,4\pi]$
- Subspace V_2 contains signals with freq. content $[0,4\pi]$

Structure of subspaces:

$$V_1 = V_0 \oplus W_0$$
$$V_2 = V_1 \oplus W_1 \quad \ldots \text{ etc.}$$
$$V_2 = V_0 \oplus W_0 \oplus W_1$$
$$V_3 = V_0 \oplus W_0 \oplus W_1 \oplus W_2 .$$
$$\vdots$$
$$\mathcal{L}^2(\mathbb{R}) = V_0 \oplus \sum_{\oplus}^{\infty} W_j$$

- Geometric analogy: each wavelet subspace adds another dimension

Projection of a signal onto a subspace V_0

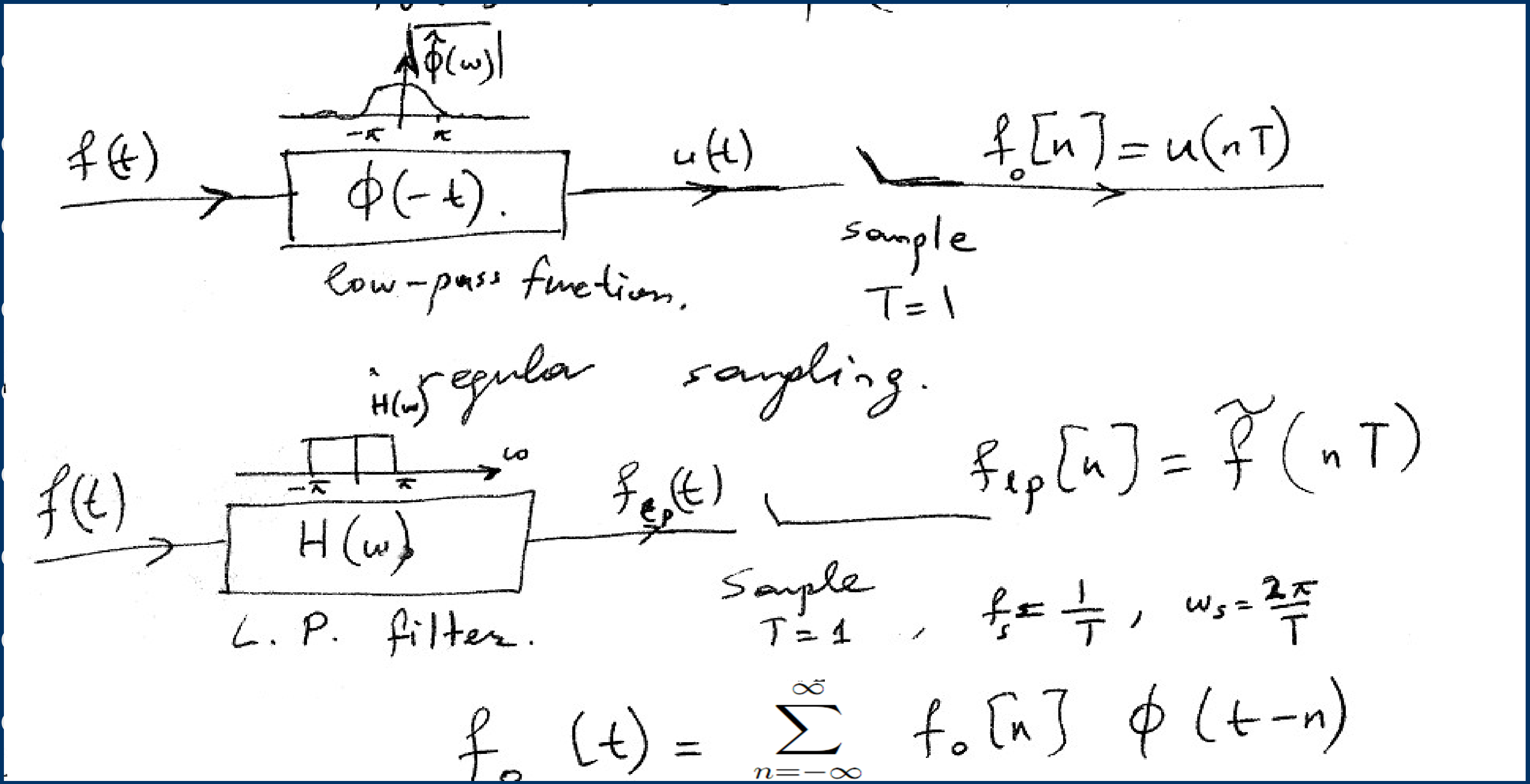
- Projection $x_o(t)$ of a signal $x(t)$ onto a subspace V_o means: 1st compute:

$$c_{n,o} = \int_{-\infty}^{\infty} x(\tau)\phi(\tau - n)d\tau$$ for all integer n

and form $x_o(t) = \sum_n c_{n,o}\, \varphi(t-n)$ which is a smooth approximation of the original signal $x(t)$

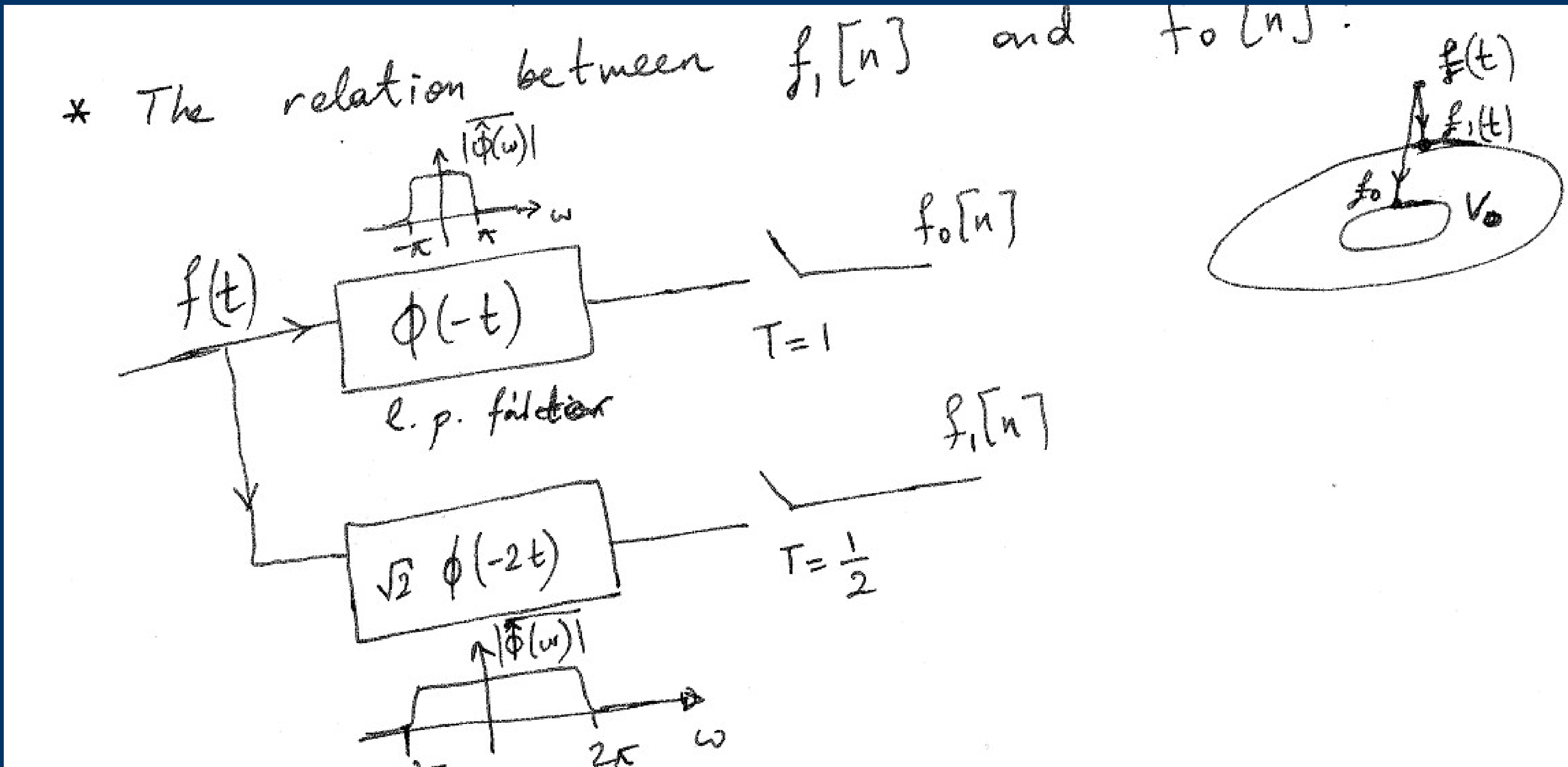
- This is equivalent to low-pass filtering $x(t)$ with a filter with passband $[0,\pi]$ and sample output with $T=1$
- As a result we don't compute the above integrals in practice: $x_o(t)=\sum_n x_o[n]\, \varphi(t-n)$

Sampling ≈ Projection onto V subspaces



Regular sampling: $f_{lp}(t) = \sum f_{lp}[n]\, sinc(t-n)$

Sampling-II

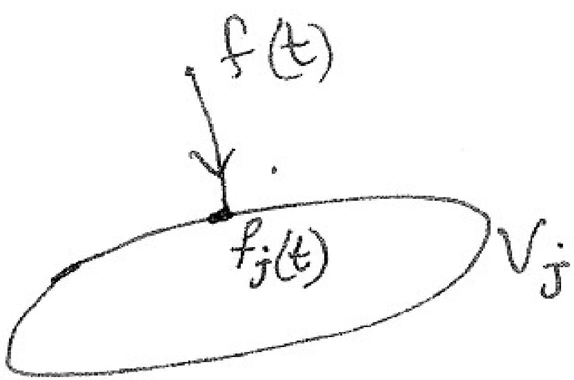


$f_1(t)=\sum_n f_1[n]\,\varphi(2t-n)$ *is a better approximation than* $f_o(t)$

Projection onto the subspace Vj (freq. content: [0, $2^j\pi$])

Given $f(t) \in \mathcal{L}^2(R)$:

$$f_j(t) = P_j\, f(t)$$



$$V_j = \text{span}\{ 2^{j/2}\, \phi(2^j t - n),\ n \in Z \}$$

$$f_j(t) = \sum_{n=-\infty}^{\infty} 2^{j/2} f_j[n]\, \phi(2^j t - n)$$

where $f_j[n] = \int_{-\infty}^{\infty} f(t)\, 2^{j/2}\, \phi(2^j t - n)\, dt =$

This is almost equivalent to Shannon sampling with $T=1/2^j$

Wavelet Equation (Mallat)

- $W_0 \subset V_1 \quad =>$

$$\psi(t)=\sqrt{2}\sum_k d[k]\varphi(2t-k)$$

- $d[k]=\sqrt{2} < \psi(t), \varphi(2t-k) >, \quad \psi(t)=2\sum_k g[k]\varphi(2t-k)$
- $g[k]= \sqrt{2}\, d[k]$ is a discrete-time half-band high-pass filter
- Example: Haar wavelet

  $\psi(t) = \varphi(2t) - \varphi(2t-1) => d[0]=\sqrt{2}/2 , d[1]= -\sqrt{2}/2$
- g and d are simple discrete-time high-pass filters

Scaling Equation

- Subspace V_0 is a subset of V_1 =>

$$\varphi(t)=2\sum_k h[k]\varphi(2t-k)$$

where $h[k]=\sqrt{2}\,\langle \varphi(t), \varphi(2t-k) \rangle$

- h[k]= $\sqrt{2}$ c[k] is a half-band discrete-time low-pass filter with passband: $[0,\pi/2]$
- In wavelet equation g[k] is a high-pass filtre with passband $[\pi/2,\pi]$

Fourier transforms of wavelet and scaling equations

$$\hat{\phi}(w) = \int_{-\infty}^{\infty} \phi(t) e^{-iwt} dt, \quad W(w) = \int_{-\infty}^{\infty} \psi(t) e^{-iwt} dt$$

$$\phi(t) = 2\sum h[k]\phi(2t-k) \Rightarrow \hat{\phi}(w) = H(e^{iw/2})\hat{\phi}(w/2)$$

Similarly, $W(\omega) = G(e^{i\frac{\omega}{2}})\,\hat{\phi}\left(\frac{\omega}{2}\right)$.

$$\hat{\phi}(\omega) = H(e^{i\frac{\omega}{2}})\, H(e^{i\frac{\omega}{4}})\, H(e^{i\frac{\omega}{8}}) \cdots\cdots$$

Orthogonality Condition:

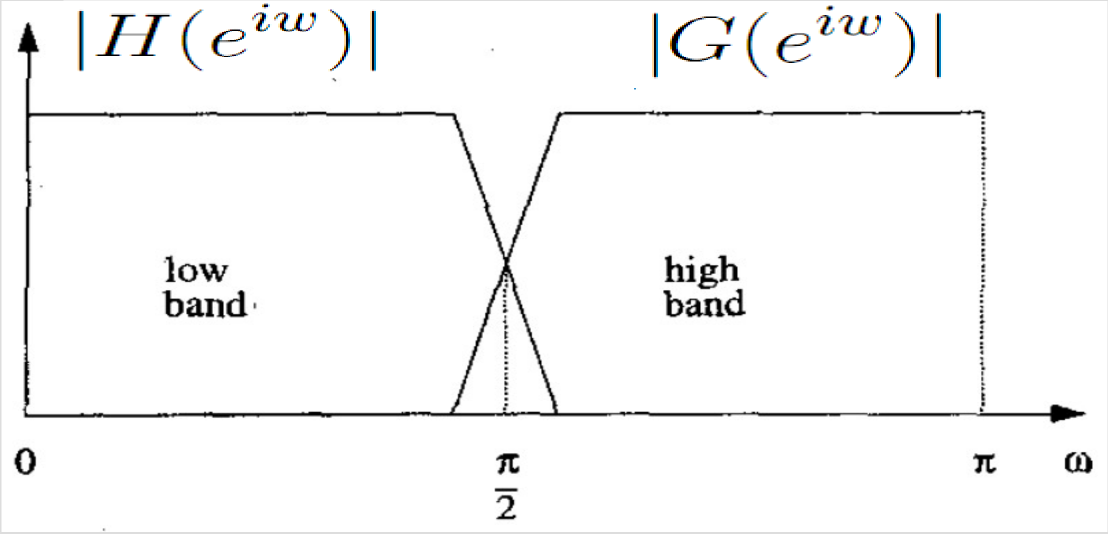
$$|H(e^{i\omega})|^2 + |H(e^{i(\omega+\pi)})|^2 = 1.$$

$H(\pi) = 0$ (zero at $\omega = \pi$ or $z = -1$)

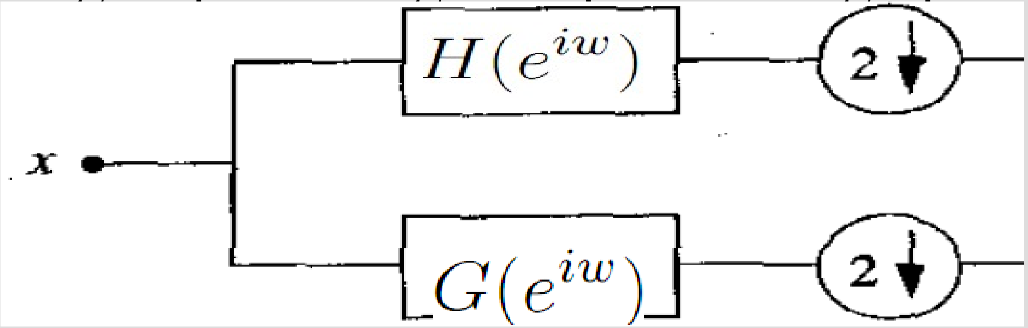
$H(e^{iw})$, $G(e^{iw})$ are the discrete-time Fourier transforms of h[k] & g[k], respectively.

Two-channel subband decomposition filter banks (Esteban&Galant 1975)

$$|H(e^{iw})|^2 + |H(e^{i(w+\pi)})|^2 = 1 \quad or \quad |H(e^{iw})|^2 + |G(e^{iw})|^2 = 1$$



Filterbanks in multirate signal processing: low-pass and high-pass filter the input discrete signal x[n] and downsample outputs by a factor of 2:



low resolution subsignal

detail subsignal

It is possible to reconstruct the original signal from subsignals using the synthesis filterbank

Wavelet construction for Multiresolution analysis

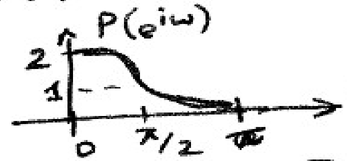
- Start with a perfect reconstruction filter bank:

1) Filter bank $h[k]$ and $g[k]$
2) $\hat{\phi}(\omega) = \prod_{l=1}^{\infty} H(e^{i\frac{\omega}{2^l}})$ (Convergence problems may occur!)
3) $\phi(t) = \mathcal{F}^{-1}_{CT}\{\hat{\phi}(\omega)\}$ and $w(t) = \mathcal{F}^{-1}_{CT}\{G(e^{i\frac{\omega}{2}})\,\hat{\phi}(\frac{\omega}{2})\}$

- <u>But we don't compute inner products with Ψ(t) and φ(t) in practice!</u>
- <u>We only use the discrete-time filterbanks!</u>

Filter Bank Design (Daubechies in 1988 but earliest examples in 1975)

Half-Band Filter: (



)

$$P(z) + P(-z) = 2 \quad \text{and} \quad P(z) = H_0(z)\,H_0(z^{-1})$$

$$P(e^{i\omega}) + P(e^{i(\omega+\pi)}) = 2 \quad \text{and} \quad P(e^{i\omega}) = H_0(e^{i\omega})\,H_0(e^{-i\omega})$$

$$|H_0(e^{i\omega})|^2 + |H_0(e^{i(\omega+\pi)})|^2 = 2 .$$

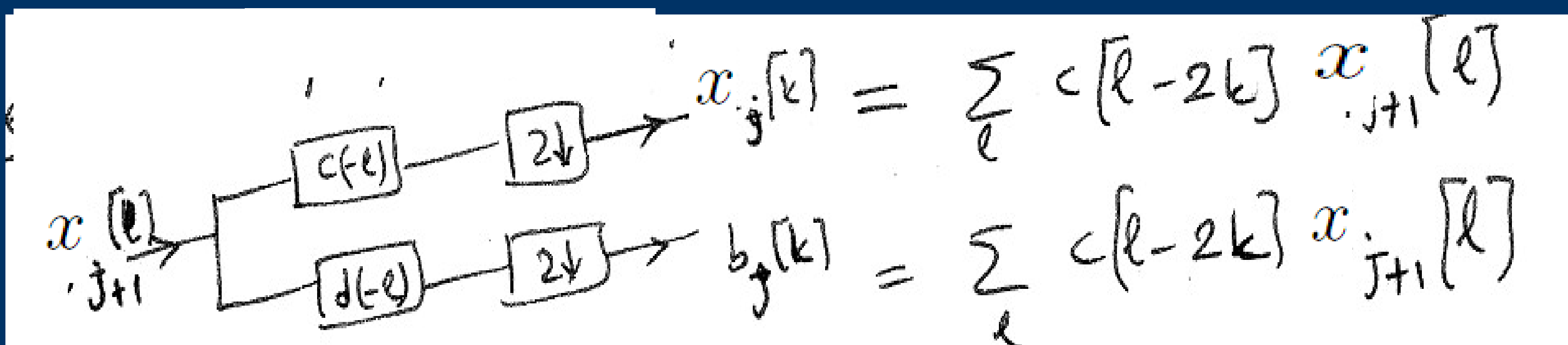
L.P. Filter $H_0(z)$ (filter order $N+1$)

H.P Filter $H_1(z) = -z^{-N} H_0(-z^{-1})$, N. odd

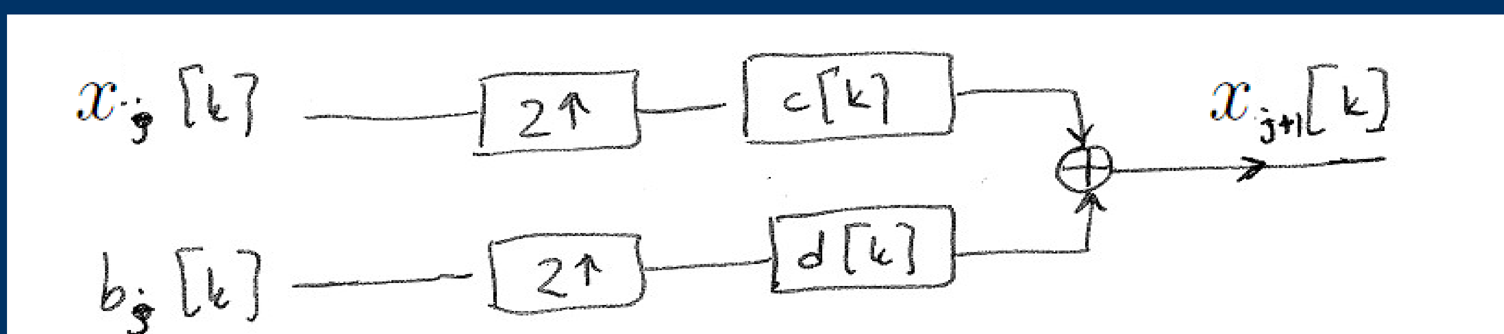
- Example half-band filters: Lagrange filters p[n]:
- p[n]= [½ 1 ½] , p[n] = 2*[-1/32 0 9/32 1 9/32 0 -1/32],...

Mallat's Algorithm (≡ Signal analysis with perfect reconstruction filter banks)

You can obtain lower order approximation and wavelet coefficients from higher order approximation coefficients:



Reconstruction:



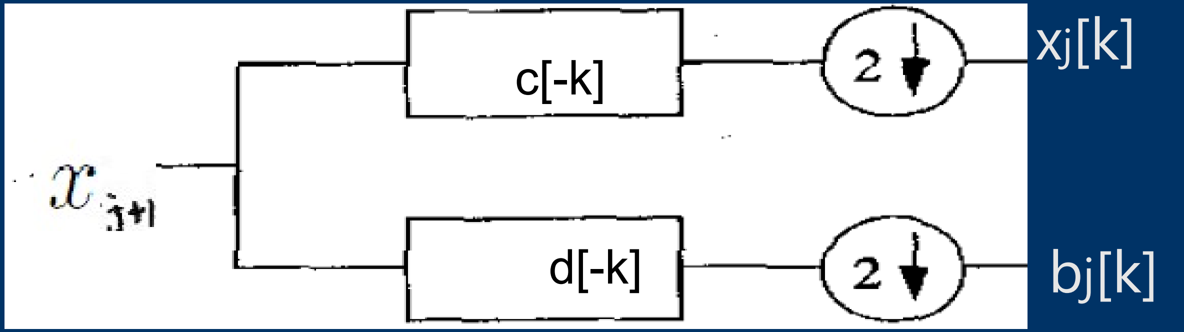
c[k]=h[k]/√2 and d[k]= g[k]/√2 are discrete-time low-pass and high-pass filters, respectively

Mallat's Algorithm (≡ Signal analysis with perfect reconstruction filter banks)

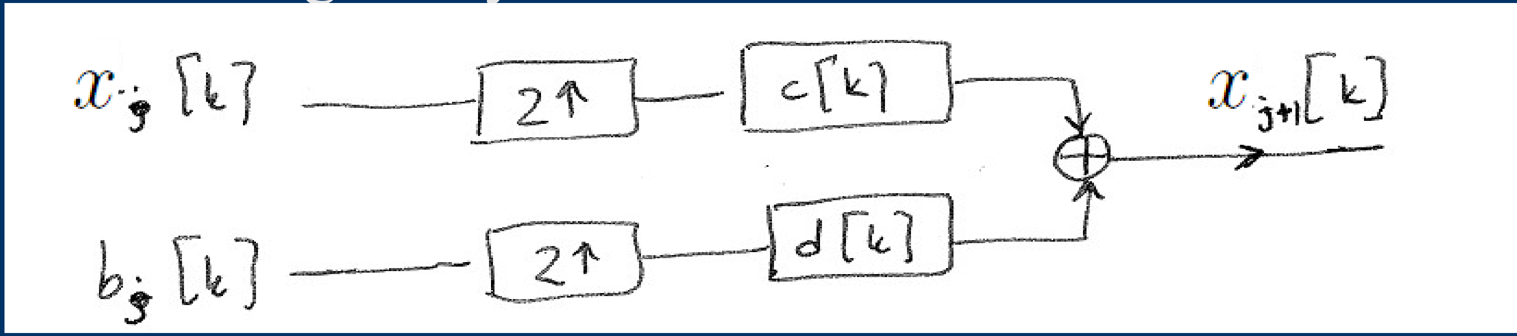
You can obtain lower order approximation and wavelet coefficients from higher order approximation coefficients:

$$x_j[k] = \sum_{\ell} c[\ell-2k]\, x_{j+1}[\ell]$$

$$b_j[k] = \sum_{\ell} d[\ell-2k]\, x_{j+1}[\ell]$$



Reconstruction using the synthesis filterbank:



c[k]=h[k]/√2 and d[k]= g[k]/√2 are discrete-time low-pass and high-pass filters, respectively

Mallat's algorithm (tree structure)

- Obtain $x_{j-1}[n]$ and wavelet coefficients $b_{j-1}[n]$ from $x_j[n]$
- Obtain $x_{j-2}[n]$ & wavelet coefficients $b_{j-2}[n]$ from $x_{j-1}[n]$
- Obtain $x_{j-3}[n]$ & wavelet coefficients $b_{j-3}[n]$ from $x_{j-2}[n]$

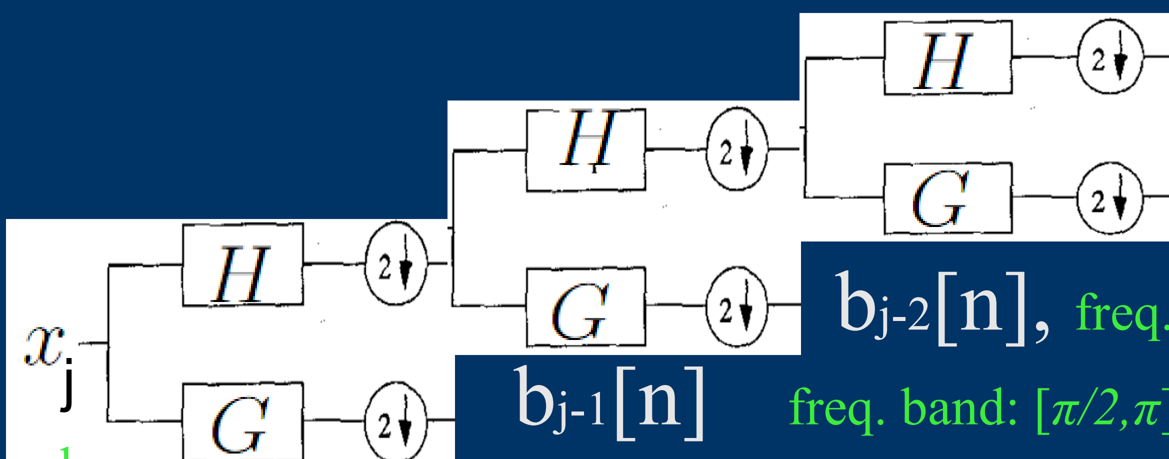
  :

- Wavelet tree representation of $x_j[n]$:

  $x_j[n] \equiv \{ b_{j-1}[n], b_{j-1}[n],...,b_{j-N}[n]; x_{j-N}[n] \}$
  where $b_{j-1}[n], b_{j-1}[n],...,b_{j-N}[n]$ are the wavelet coefficients at lower resolution levels
- Use a filterbank (e.g. Daubechies-4) to obtain the wavelet coefficients

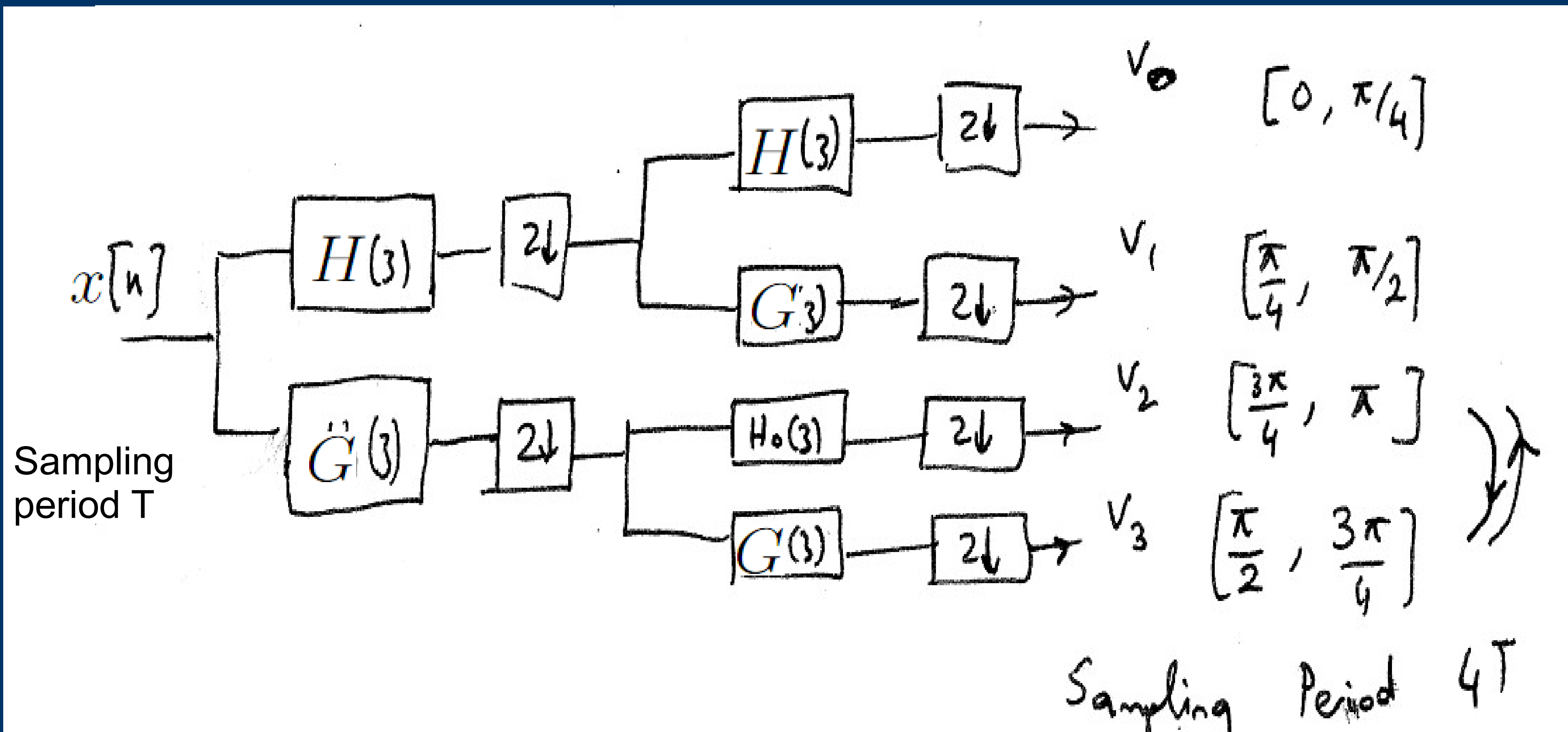
Discrete-time Wavelet Transform

- Discrete-time filter-bank implementation:
  H is the low-pass and G is the high-pass filter of the wavelet transform



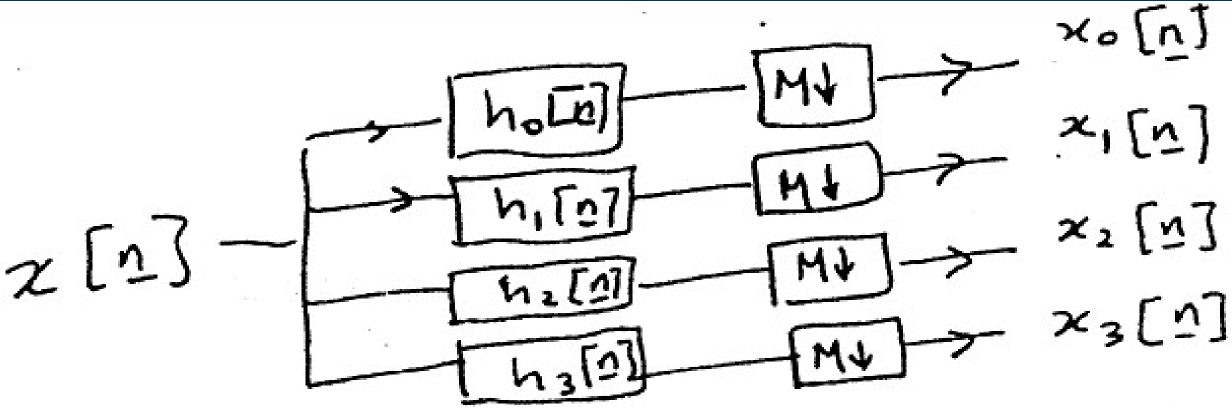
- Subband decomposition filterbank acts like a “butterfly” in FFT
- Perfect reconstruction of xj from subsignals, $x_{j-3}[n],..,b_{j-1}[n]$ is possible
- Both time and freq. information is available but Heisenberg's principle applies

Wavelet Packet Transform



Length of x[n] is N => Lengths of v_0, v_1, v_2, and v_3 are N/4

Two-dimensional filterbanks for image processing



filters:

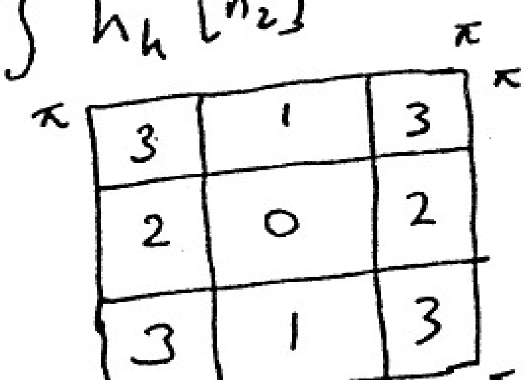
$$h_0[\underline{n}] = h_\ell[n_1]\, h_\ell[n_2]$$

$$h_1[n_1, n_2] = h_\ell[n_1]\, h_h[n_2]$$

$$h_2[\underline{n}] = h_h[n_1]\, h_\ell[n_2]$$

$$h_3[\underline{n}] = h_h[n_1]\, h_h[n_2]$$

Freq. domain picture



Example

- Cont. time signal x(t) = 1 for t<5 and 2 for t >5
- Sample this signal with T=1 ≡ Project it onto V_0 of Haar multiresolution decomposition using h={½ ½}, g={½ -½}:
- x[n] = (... 1 1 1 1 1 2 2 2 2 2 2 2 ....)
- Perform single level Haar wavelet transform:

Lowpass filtered signal: (... 1 1 1 1 1.5 2 2 2 2...)
downsample by 2
Low-resolution subsignal: (... 1 1 1.5 2 2...)
Highpass filtered signal: (... 0 0 0 0 0.5 0 0 0 0...)
downsample by 2
1st scale wavelet subsignal (... 0 0 0.5 0 0...)

- We can estimate the location of the jump from the nonzero value of the wavelet signal
- Haar is not a good wavelet transfrom because the wavelet signal of x[n-1] would be (...0 0 0 0 0...)

Toy Example: signal data compression

- Original x[n] = (1 1 1 1 2 2 2 2)
- 8 bits/sample => 8x8=64 bits
- Single level Haar wavelet transform:

  Low-resolution subsignal: (1 1 1.5 2 2)

  5*8 bits/pel =40 bits

  1st scale wavelet signal: (0 0 0.5 0 0)

  Only store the nonzero value (9 bits) and its location (3 bits)

  Total # of bits to store the wavelet signals= 52 bits
- Since 52bits < 64bits it is better to store the wavelet subsignals instead of the original signal

Denoising Example

- Original: $x[n] = (\ldots\ 1\ 1\ 1\ 1\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ \ldots.)$
- Corrupted: $x_c[n] = (\ldots\ \ 1\ 1.2\ 1\ 1\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ \ldots.)$
- Single level Haar wavelet transform of $x_c[n]$

  using $h=\{\sqrt{2}/2 \quad \sqrt{2}/2\}$, $g=\{\sqrt{2}/2 \quad -\sqrt{2}/2\}$:

  Low-resolution subsignal xl= (...1.49 1.59 1.51 2.828 2.828...)

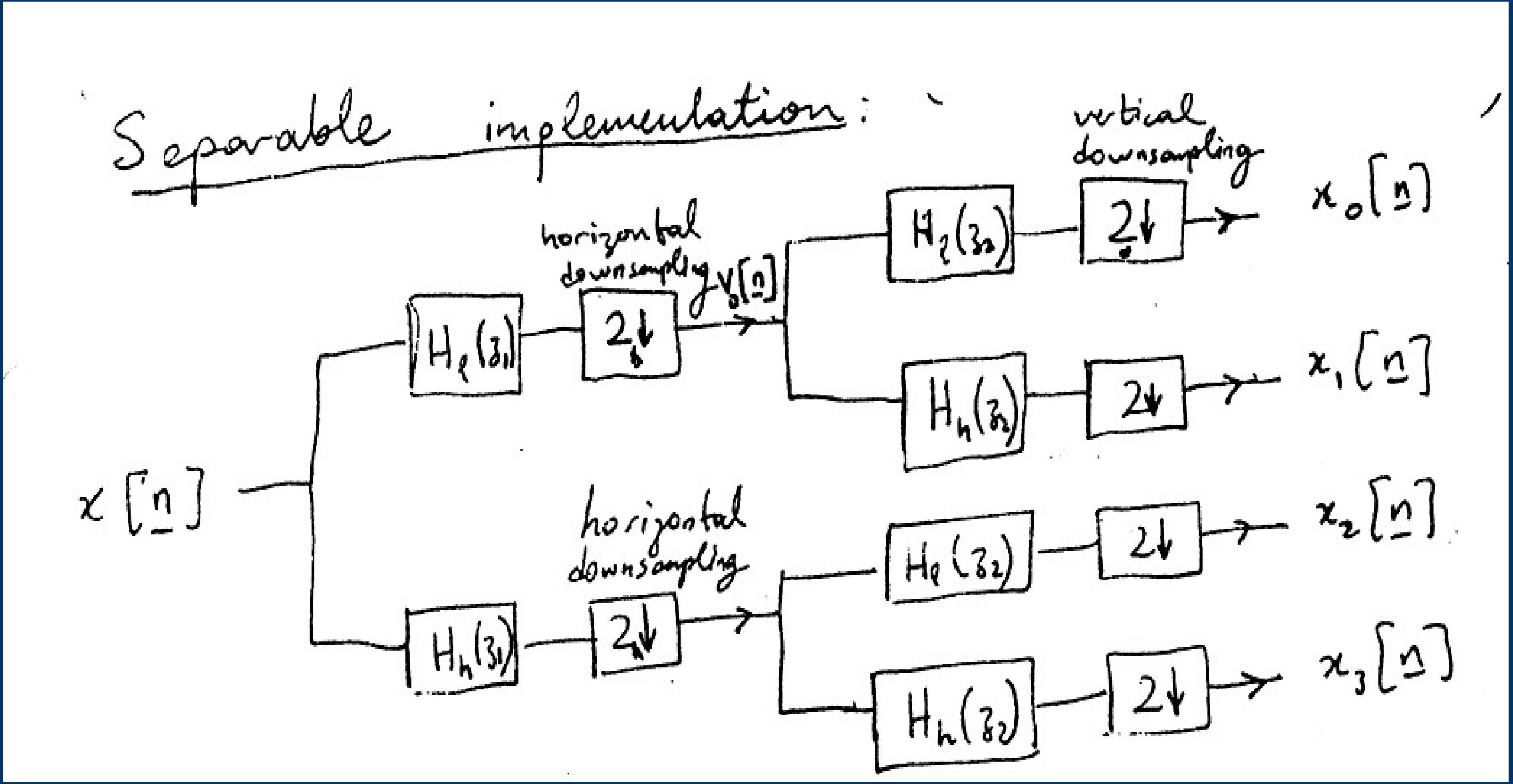
  1st scale wavelet signal: (... -.15 -.06 0.354 0 0...)

  Soft-thresholded wavelet signal: xs=(... 0 0 0.354 0 0...)

- Restored signal from xl and xs:

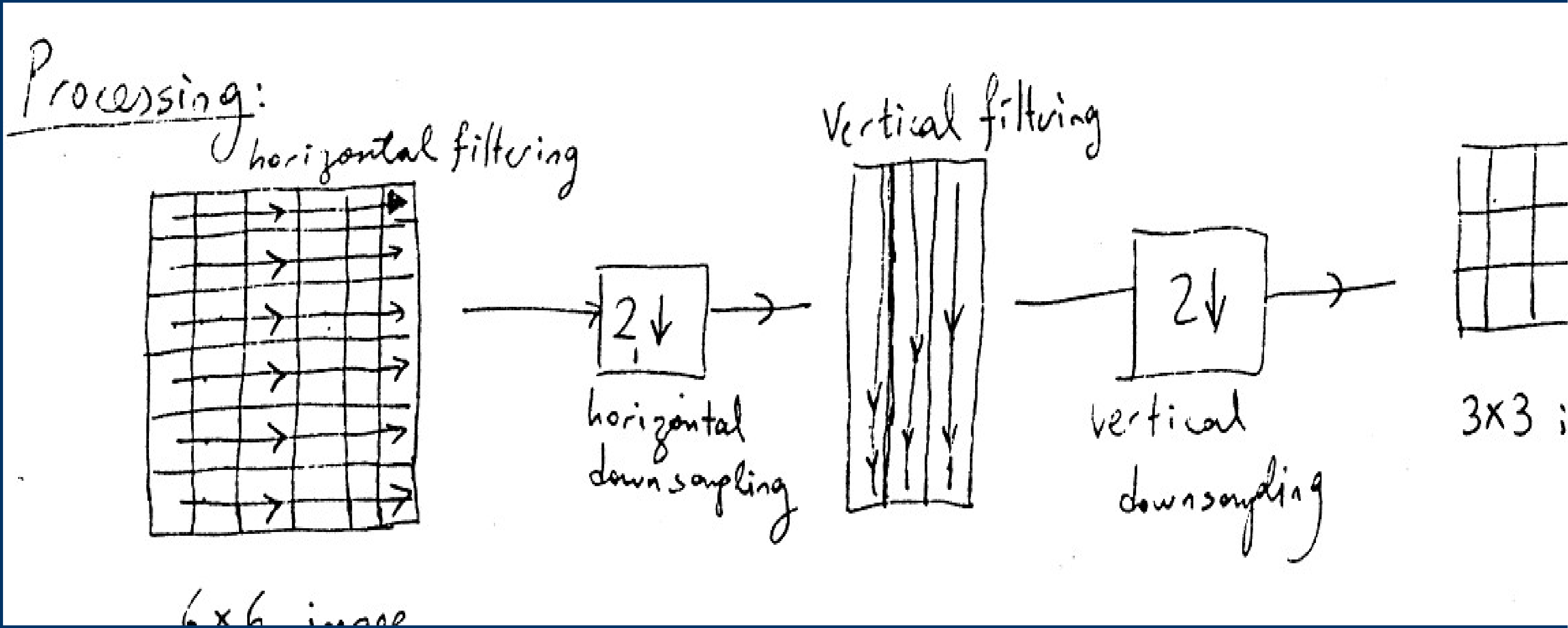
  $x_r[n] = (\ldots\ \ 1.1\ 1.13\ 1.04\ 0.98\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ \ldots.)$
- Better denoising results can be obtained with higher order wavelets using longer filters which provide better smoothing of the low-resolution signal

2-D image processing using a 1-D filterbank (separable filtering)



2-D image processing using a 1-D filter

Seperable processing in each channel of the 2-D filterbank:



2-D wavelet transform of an image

- Single scale decomposition:

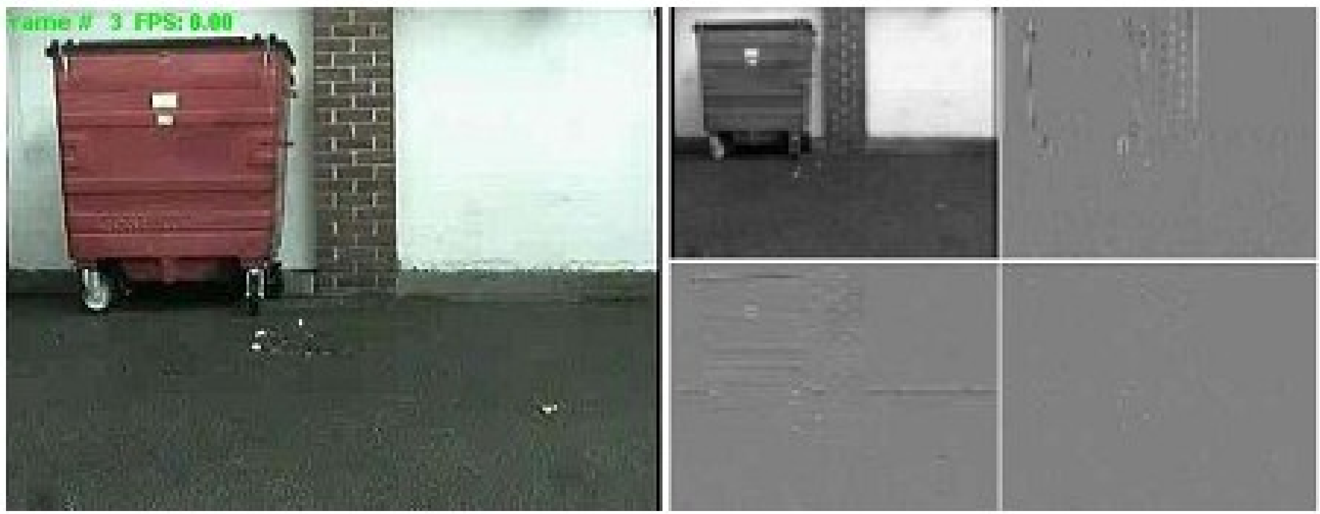


Figure 1: Original frame and its single level wavelet subimages.

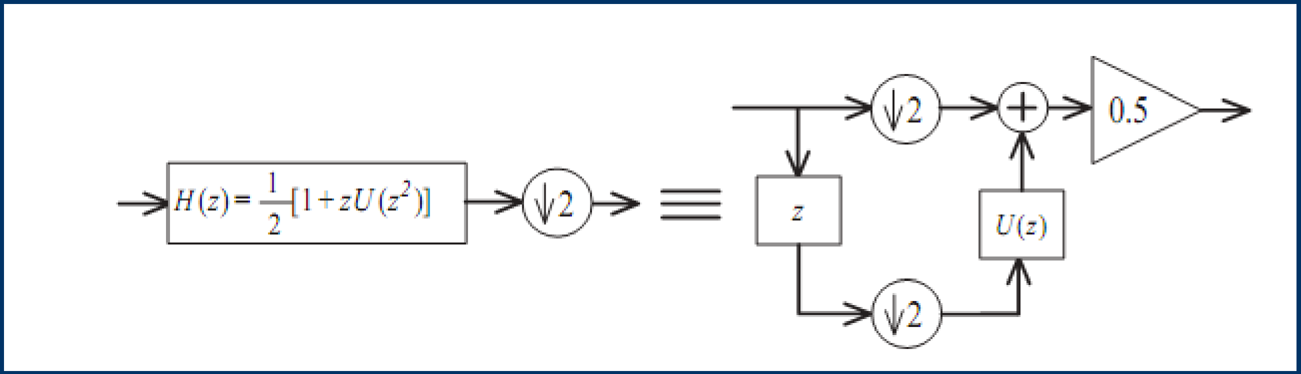
- "low-low" subimage can be further decomposed to subimages

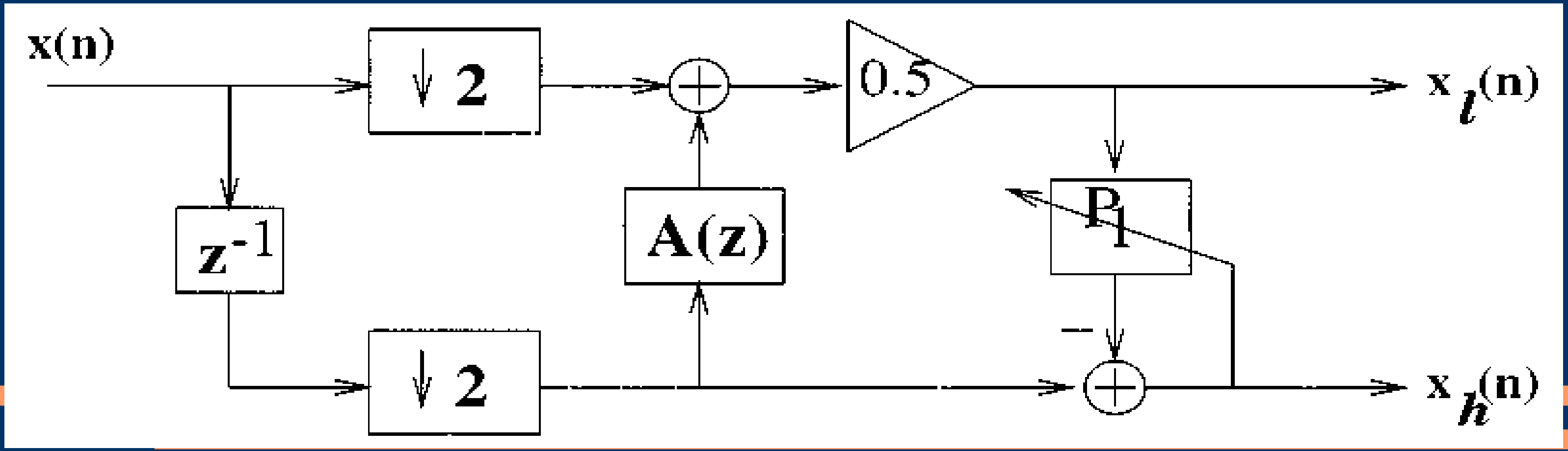
Image Compression

- JPEG-2000 (J2K) is based on wavelet transform
- Energy of the high-pass filtered subimages are much lower than the low-low subimage
- Most of the wavelet coefficients are close to zero except those corresponding to edges and texture
- Threshold low-valued wavelet coefficients to zero
- Take advantage of the correlation between wavelet coefficients at different resolutions
- JPEG and MPEG are still prefered because of local nature of DCT and Intellectual Property issues of J2K

Lifting (Sweldens)

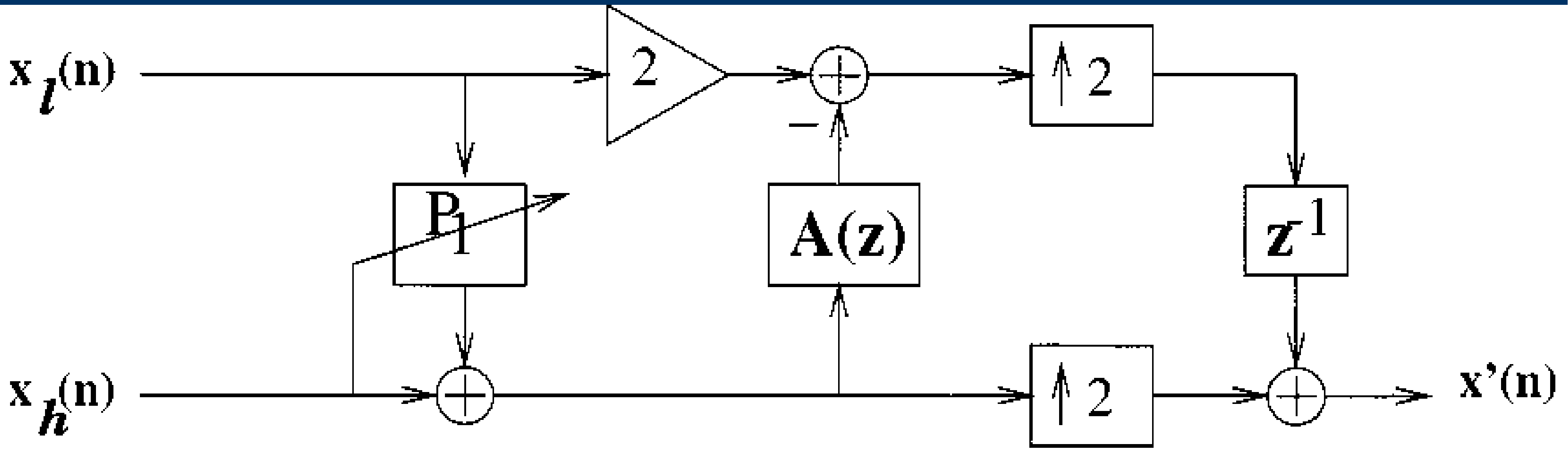
- Filtering after downsampling:



- It reduces computational complexity
- It allows the use of nonlinear (Pesquet), binary and adaptive filters (Cetin) as well
- 

Adaptive Lifting-II

- Reconstruction filterbank structure from Gerek and Cetin, 2000



Lifting

- The basic idea of lifting: If a pair of filters (h,g) is complementary, that is it allows for perfect reconstruction, then for every filter s the pair (h',g) with allows for perfect reconstruction, too.
- $H'(z)=H(z)+s(z^2)G(z)$ or
- $G'(z)=G(z)+s(z^2)H(z)$
- Of course, this is also true for every pair (h,g') of the form .
- The converse is also true: If the filterbanks (h,g) and (h',g) allow for perfect reconstruction, then there is a unique filter s with .

http://pagesperso-orange.fr/polyvalens/clemens/lifting/lifting.html

Equations

- x ~~ \sum _{n=-\infty}^{\infty} |H(e^{iw})|^2 + |H(e^{i(w+\pi)})|^2 =1 ~~ or ~~|H(e^{iw})|^2 + |G(e^{iw})|^2 =1\\
- \phi(t) = 2\sum h[k] \phi(2t-k)~ => ~\hat\phi(w) = H(e^{iw/2})\hat\phi(w/2)\\
- \hat\phi(w)=\int_{-\infty}^{\infty} \phi(t) e^{-iwt} dt, ~~~W(w)=\int_{-\infty}^{\infty} \psi(t) e^{-iwt} dt