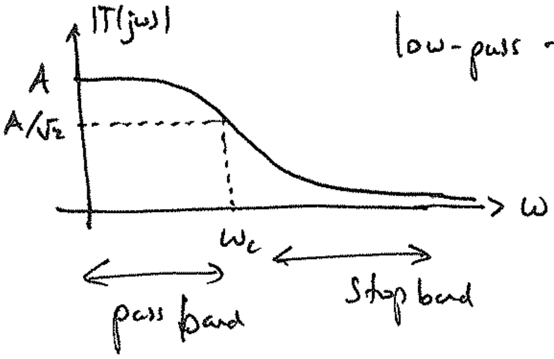
Chapter 12 : Frequency Response

Recall from last lecture:

$$x(t) = X_A \cos(\omega t) \longrightarrow \boxed{\text{Stable circuit}} \longrightarrow y_{ss}(t) = X_A |T(j\omega)| \cos(\omega t + \angle T(j\omega))$$

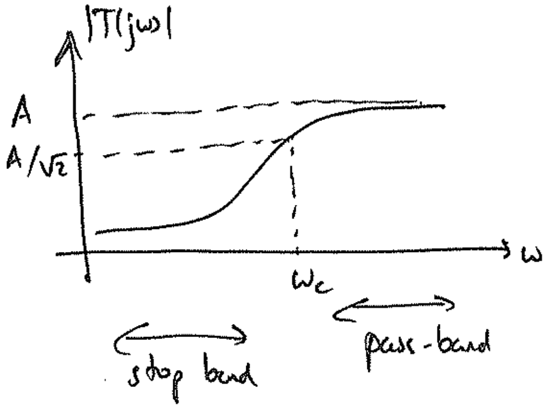
We would like to see the function $|T(j\omega)|$ as a function of ω (same for $\angle T(j\omega)$).

<u>Examples</u>:

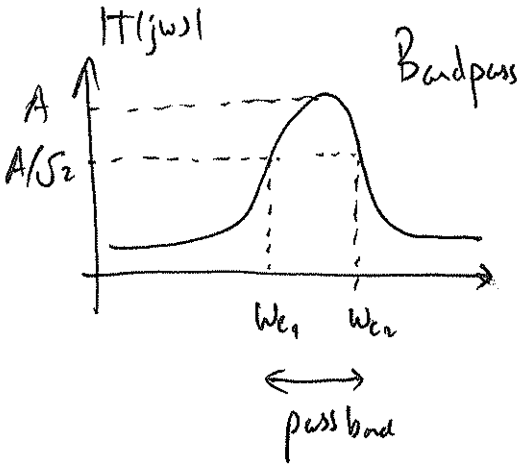


low-pass filter

W_c : cut-off frequency

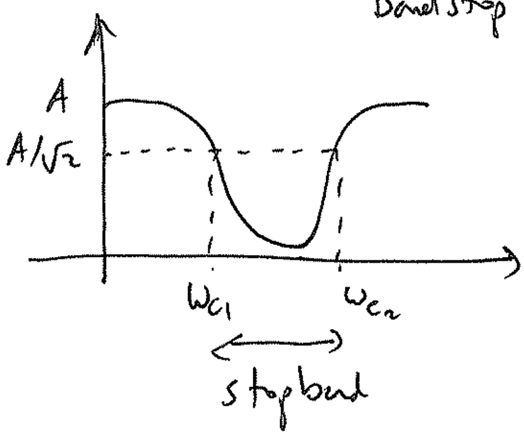


High-pass filter



Bandpass filter

Bandstop filter



Bode Plots:

graph of $20\log_{10}|T(j\omega)|$ and $\angle T(j\omega)$ in log-scale of ω

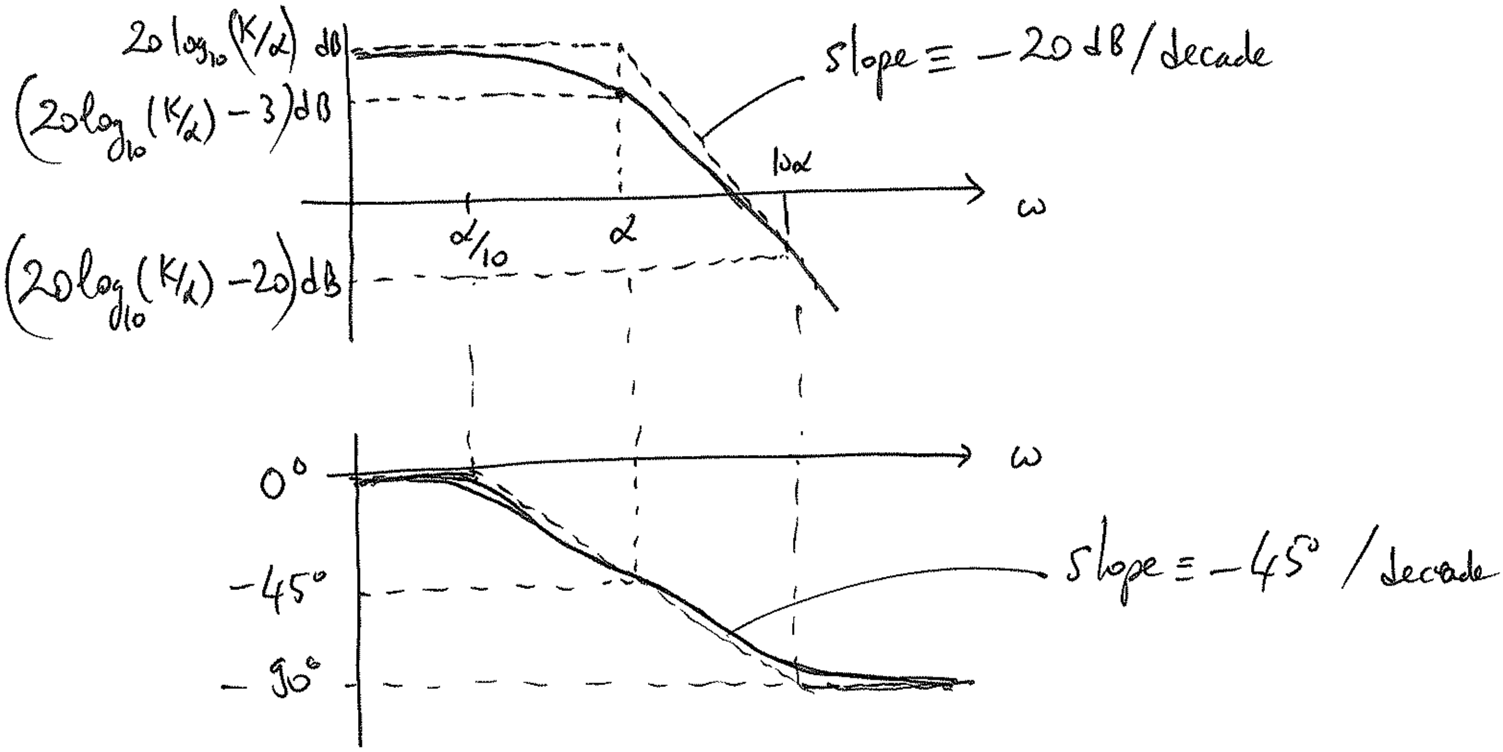
$$20\log_{10}\frac{A}{\sqrt{2}} = 20\log_{10}A - 20\log_{10}\sqrt{2} = 20\log_{10}A - 3\,dB$$

$$|T(j\omega)|_{dB} = 20\log|T(j\omega)| \quad \text{in dB}$$

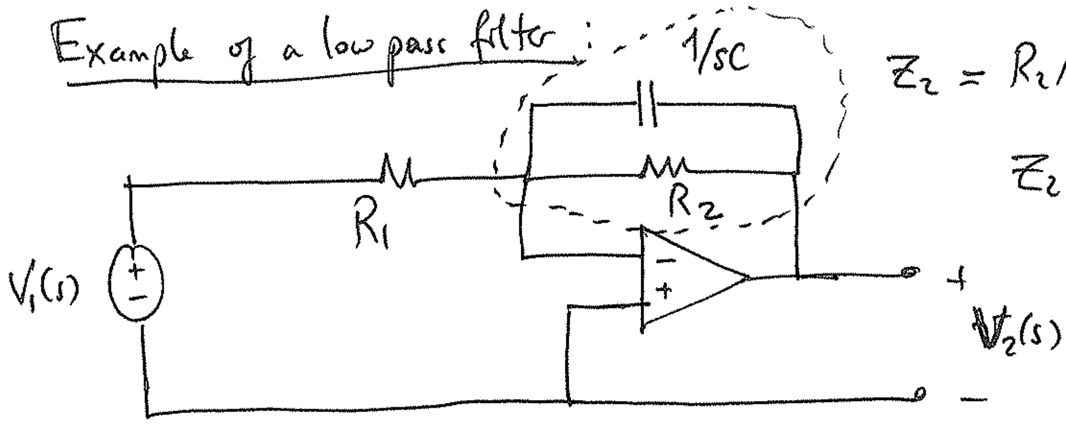
Bode Plots of First order low-pass and high pass filters:

$$T(s) = \frac{K}{s+\alpha} = \frac{K/\alpha}{s/\alpha + 1} \qquad T(j\omega) = \frac{K/\alpha}{1 + j\,\omega/\alpha} \qquad K > 0$$

$$|T(j\omega)| = \frac{K/\alpha}{\sqrt{1+\omega^2/\alpha^2}} \qquad \angle T(j\omega) = -\tan^{-1}\frac{\omega}{\alpha}$$



Example of a low pass filter:



$$Z_2 = R_2 // \frac{1}{sC}$$

$$Z_2 = \frac{1}{sC + 1/R_2} = \frac{R_2}{R_2 C s + 1}$$

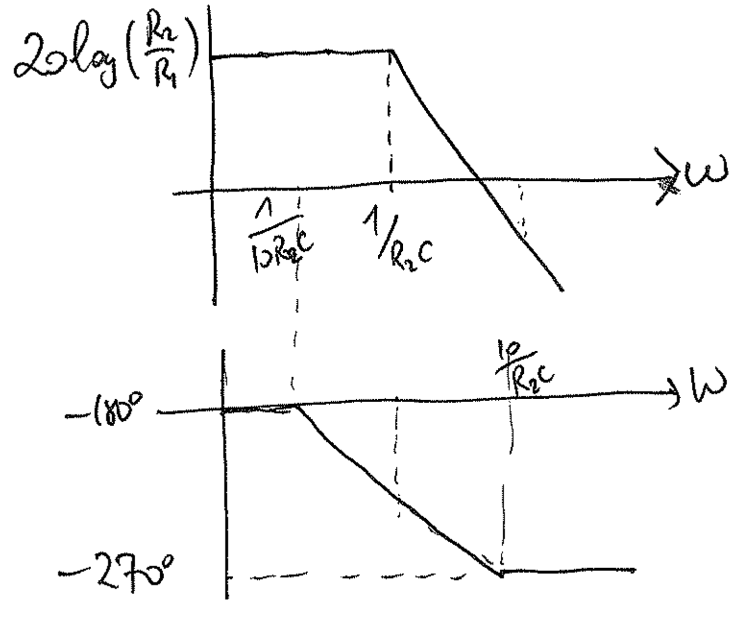
$$V_2(s) = -\frac{Z_2(s)}{R_1} = -\frac{R_2/R_1}{R_2 C s + 1} = -\frac{1/CR_1}{s + 1/R_2C}$$

$$K = \left(-1/CR_1\right) \qquad |K| = \frac{1}{R_1 C} \qquad \angle K = -180^\circ$$

$$\alpha = 1/R_2C \qquad (K/\alpha) = -R_2/R_1$$

$$|T(j\omega)| = \frac{R_2/R_1}{\sqrt{1 + \omega^2 (R_2C)^2}} \qquad \text{cut-off freq } \omega_c = 1/R_2C$$

$$\angle T(j\omega) = -180^\circ - \mathrm{Tan}^{-1}\, \omega R_2 C$$



Example select R_1, R_2, C
s.t. passband gain = 4
cut off freq = 100

$$R_2/R_1 = 4$$

$$\frac{1}{R_2 C} = 100$$

$$C = \frac{1}{100 R_2} \qquad R_1 = \frac{R_2}{4}$$

Let $R_2 = 1\,k\Omega$ $R_1 = 250\,\Omega$

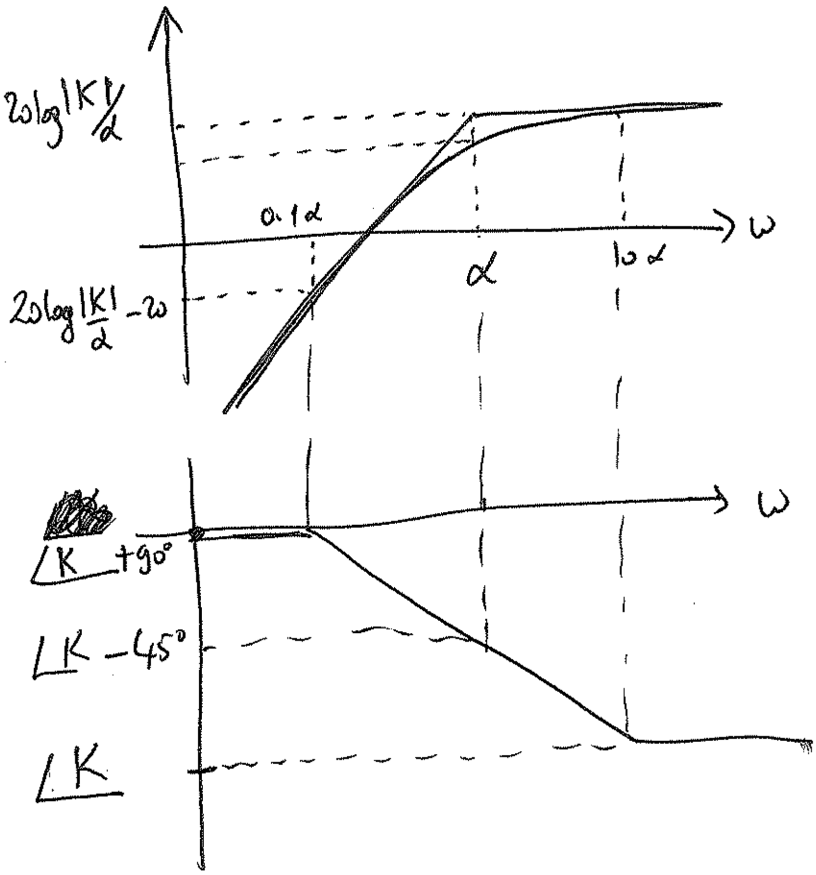
$C = 10\,\mu F$

First-Order Highpass Filter

$$T(s) = \frac{K\,s}{s+\alpha} \qquad |T(j\omega)| = \frac{\frac{|K|}{\alpha}\,\omega}{\sqrt{1 + j\,\omega/\alpha}}$$

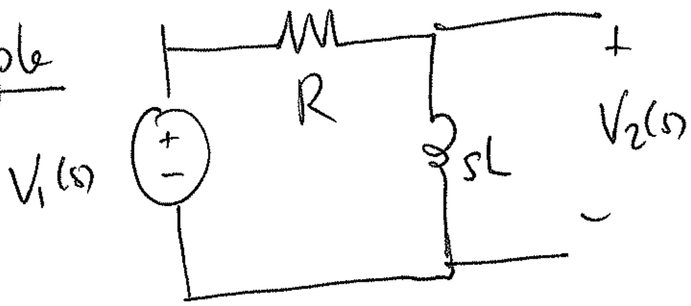
$\alpha > 0$

$$\angle T(j\omega) = \angle K + 90^\circ - \mathrm{Tan}^{-1}\frac{\omega}{\alpha}$$

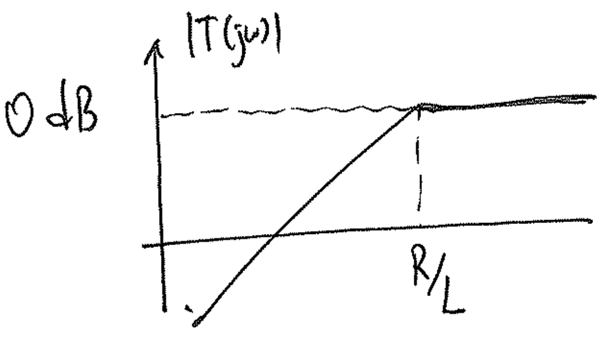


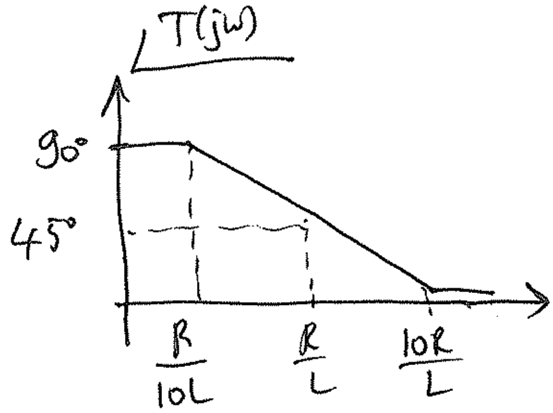
$$V_2(s) = \frac{sL}{R+sL}\,V_1(s)$$

Example



$$T(s) = \left(\frac{L/R\; s}{1 + s\,L/R}\right)$$

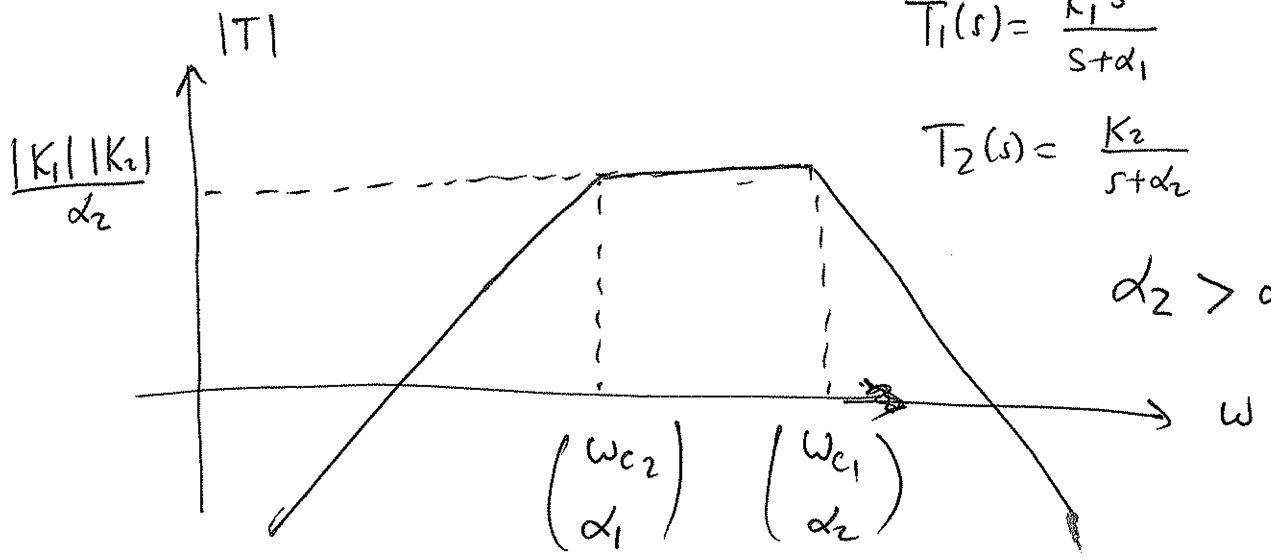




Bandpass-filter

$T(s) = T_1(s)\, T_2(s)$ (no loading)   Cascade stages

$$20\log|T(j\omega)| = \underbrace{20\log|T_1(j\omega)|}_{\text{High Pass}} + \underbrace{20\log|T_2(j\omega)|}_{\text{low - Pass}}$$



$$T_1(s) = \frac{K_1 s}{s+\alpha_1}$$

$$T_2(s) = \frac{K_2}{s+\alpha_2}$$

$$\alpha_2 > \alpha_1$$

low freq: $\quad |T(j\omega)| \rightarrow \dfrac{|K_1|\omega}{\alpha_1} \cdot \dfrac{|K_2|}{\alpha_2} = \dfrac{|K_1||K_2|}{\alpha_1\,\alpha_2}\,\omega$

high freq: $\quad |T(j\omega)| \rightarrow \dfrac{|K_1|\cancel{\omega}}{\cancel{\omega}} \cdot \dfrac{|K_2|}{\omega} = \dfrac{|K_1||K_2|}{\omega}$

mid freq. $\quad |T(j\omega)| \rightarrow \dfrac{|K_1| \cdot |K_2|}{\alpha_2}$

Band-Stop Filter

Parallel stages $\quad T_1(s) = \frac{K_1 s}{s+\alpha_1} \qquad T_2(s) = \frac{K_2}{s+\alpha_2}$

$T(s) = T_1(s) + T_2(s) \qquad \alpha_2 < \alpha_1$

<u>low freq:</u> $\quad \omega \ll \alpha_2$

$$|T(j\omega)| \cong \frac{|K_2|}{\alpha_2}$$

<u>high freq:</u> $\quad \omega \gg \alpha_1 \qquad |T(j\omega)| \cong |K_1|$

Choose $|K_1| = \frac{|K_2|}{\alpha_2}$

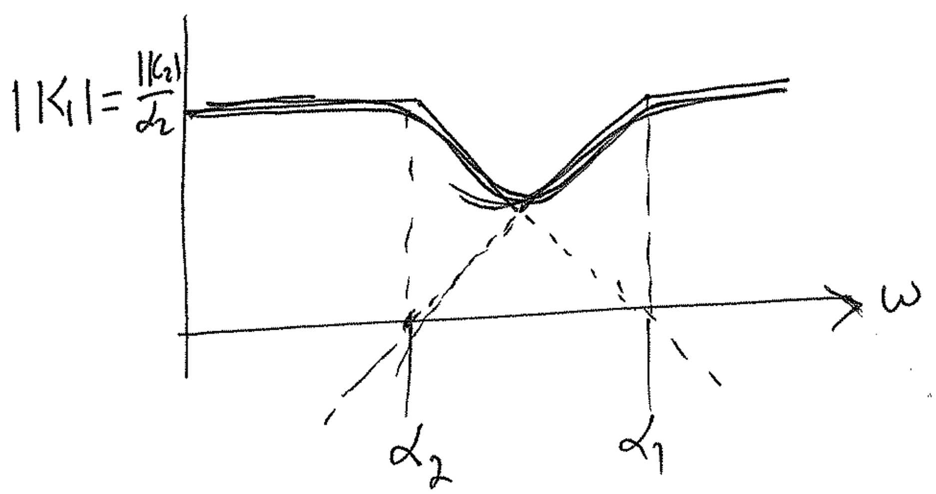
$\alpha_2 = \frac{|K_2|}{|K_1|}$

$\Downarrow$

<u>mid freq:</u> $\quad \alpha_2 \ll \omega \ll \alpha_1$

$$|T(j\omega)| = \frac{|K_1|\,\omega}{\alpha_1} + \frac{|K_2|}{\omega}$$

intersection frequency $\quad \frac{|K_1|\,\omega_0}{\alpha_1} = \frac{|K_2|}{\omega_0} \quad \Rightarrow \quad \omega_0 = \sqrt{\alpha_1 \alpha_2}$



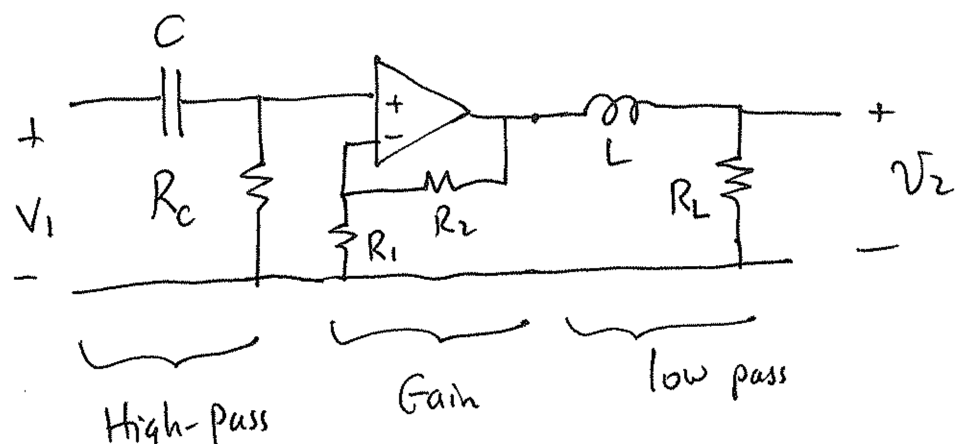
Example: Design a bandpass circuit with a passband gain of 10 and cut-off frequencies at 20Hz and 20kHz

20Hz $\longrightarrow$ $\omega_1 = 2\pi \times 20 = 40\pi$ rad/sec $= \alpha_1$

20 kHz $\longrightarrow$ $\omega_2 = 40000\pi$ rad/sec $= \alpha_2$

$$\frac{|K_1 K_2|}{\alpha_2} = 10$$

$$T(j\omega) = \frac{K_1 s}{s+\alpha_1} \cdot \frac{K_2}{s+\alpha_2} = \left(\frac{K_1 K_2}{\alpha_2}\right)\left(\frac{s}{s+\alpha_1}\right)\left(\frac{\alpha_2}{s+\alpha_2}\right)$$



$$T_1(s) = \frac{R_c}{R_c + \frac{1}{sC}} = \frac{R_c C s}{R_c C s + 1} = \left(\frac{s}{s + 1/R_c C}\right)$$

$$\alpha_1 = 1/R_c C = 40\pi$$

Gain: $\frac{R_2 + R_1}{R_1} > 1$ choose $\frac{R_2 + R_1}{R_1} = 10$

$$T_2(s) = \frac{R_L}{sL + R_L} = \frac{R_L/L}{s + R_L/L}$$

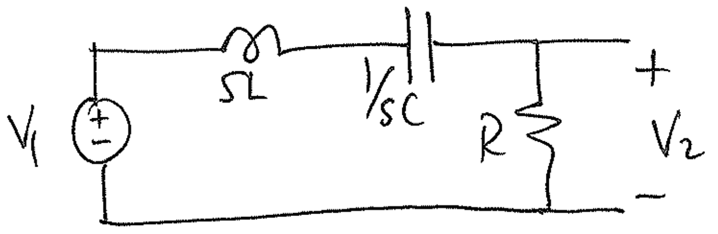
$$\alpha_2 = R_L/L = 40000\pi$$

RLC circuits Bode Plots

Series RLC band-pass circuit

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

$$\frac{2\xi}{\omega_0} = RC \qquad \boxed{\xi = \frac{R}{2}\sqrt{\frac{C}{L}}}$$

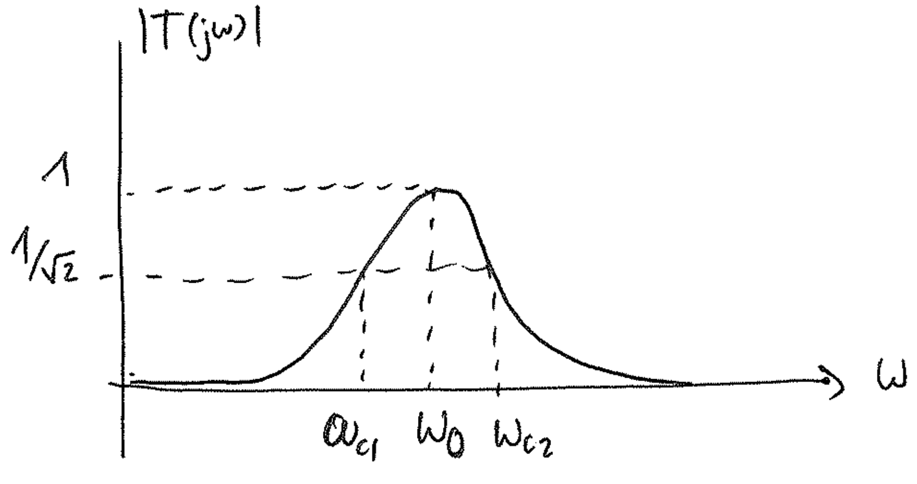


$$T(s) = \frac{R}{sL + \frac{1}{sC} + R} = \frac{RCs}{LCs^2 + RCs + 1} = \frac{2\xi\, s/\omega_0}{s^2/\omega_0^2 + 2\xi\frac{s}{\omega_0} + 1}$$

$$|T(j\omega)| = \frac{2\xi\frac{\omega}{\omega_0}}{\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2\right]^{1/2}} = \frac{1}{\left[1 + \left(\frac{1-\omega^2/\omega_0^2}{2\xi\,\omega/\omega_0}\right)^2\right]^{1/2}}$$

when $\omega \ll \omega_0$ $\qquad |T(j\omega)| \longrightarrow 2\xi\frac{\omega}{\omega_0} = RC\omega$

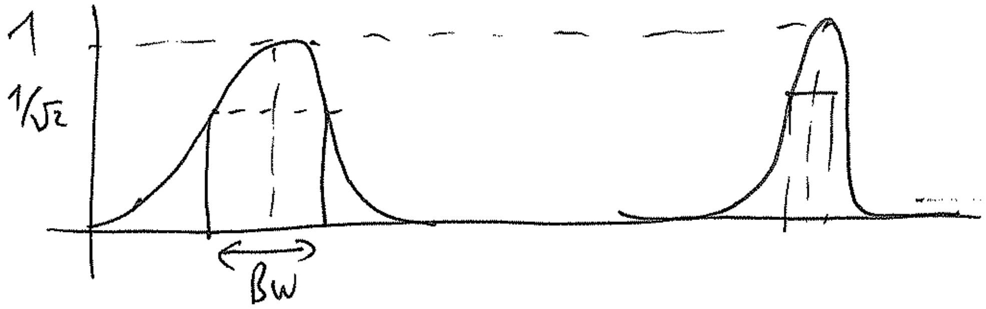
When $\omega \gg \omega_0$ $\qquad |T(j\omega)| \longrightarrow \frac{RC\omega}{\omega^2 LC} \to \frac{R}{L}\frac{1}{\omega}$



$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad \omega_{c2} = +\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_{c1}\,\omega_{c2} = \frac{1}{LC} = \omega_0$$

Quality factor $= Q = \dfrac{\omega_0}{BW}$



In our case $\quad BW = \omega_{c2} - \omega_{c1} = \dfrac{R}{L} \qquad \omega_0 = \dfrac{1}{\sqrt{LC}}$

$$Q = \frac{1/\sqrt{LC}}{R/L} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{2\xi}$$

Bode Plots with real poles and zeros

$$T(s) = \frac{K s\,(s+\alpha_1)}{(s+\alpha_2)(s+\alpha_3)}$$

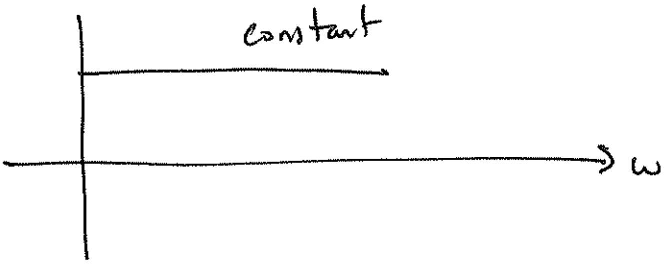
zeros at $s=0$, $s=-\alpha_1$
poles at $s=-\alpha_2$, $s=-\alpha_3$

$$T(j\omega) = \frac{(K\alpha_1)}{(\alpha_2\alpha_3)} \; \frac{s\,(s/\alpha_1+1)}{(s/\alpha_2+1)(s/\alpha_3+1)}$$

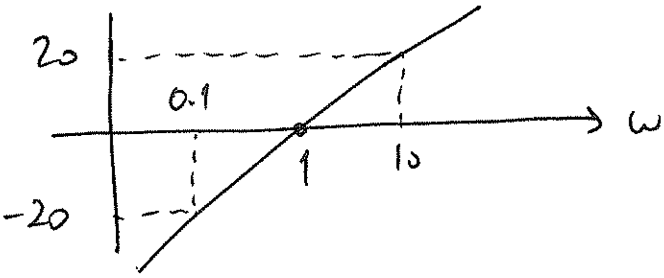
$$|T(j\omega)| = \left|\frac{K\alpha_1}{\alpha_2\alpha_3}\right| \; \frac{|j\omega|\cdot|j\omega/\alpha_1+1|}{|j\omega/\alpha_2+1|\cdot|j\frac{\omega}{\alpha_3}+1|} = \underbrace{\left|\frac{K\alpha_1}{\alpha_2\alpha_3}\right|}_{K_0} \; \frac{\omega\overbrace{\sqrt{1+(\omega/\alpha_1)^2}}^{M_1}}{\underbrace{\sqrt{(1+(\omega/\alpha_2)^2}}_{M_2}\;\underbrace{\sqrt{1+(\omega/\alpha_3)^2}}_{M_3}}$$

$$20\log|T(j\omega)| = 20\log|K_0| + 20\log\omega + \underset{20\log}{\wedge}M_1 - \underset{20\log}{\wedge}M_2 - \underset{20\log}{\wedge}M_3$$

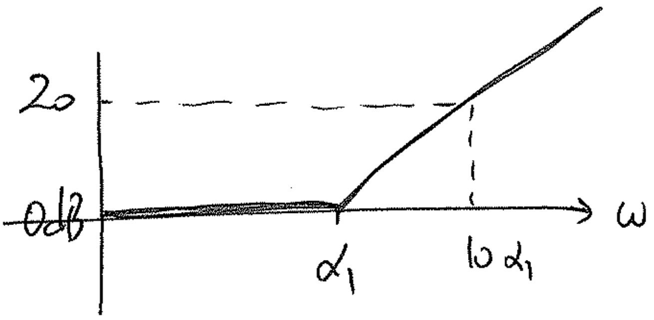
$20\log|K_0|$



$20\log\omega$



$20\log M_1$

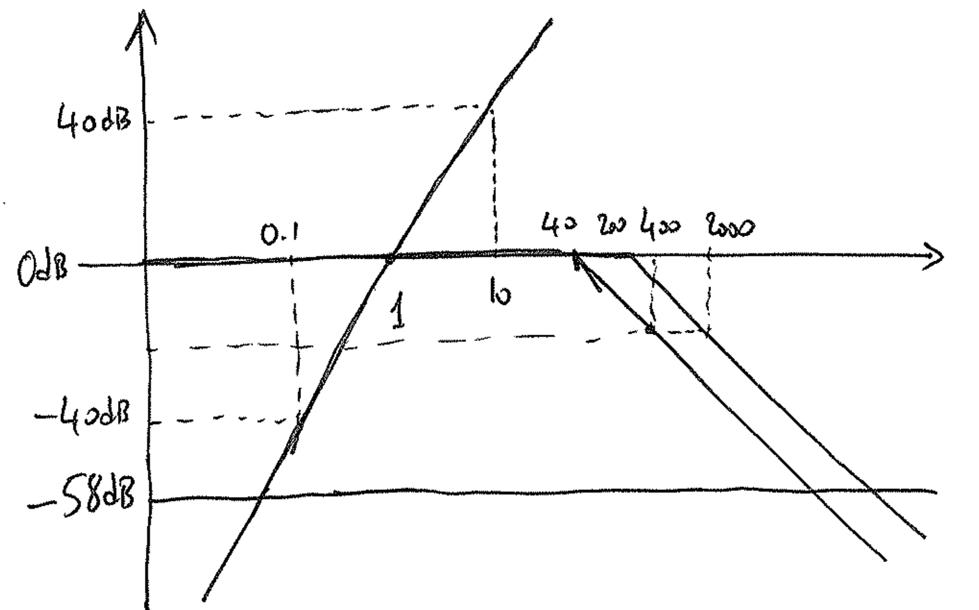


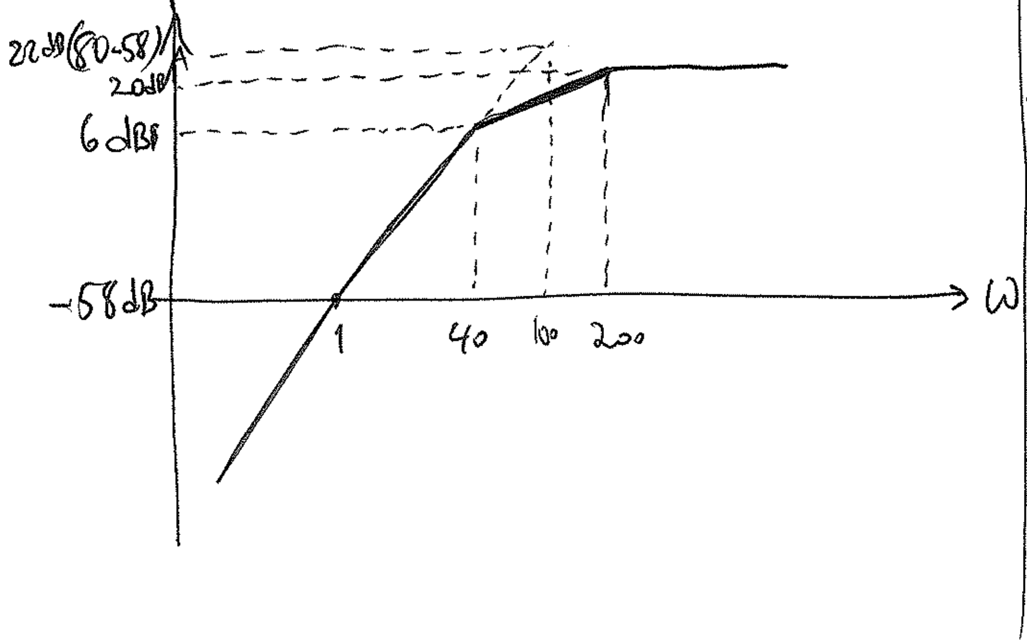
Example: $$T(s) = \frac{10\, s^2}{(s+40)(s+200)} = \left(\frac{10}{40 \times 200}\right) \frac{s^2}{(1+s/40)(1+s/200)}$$

$$T(j\omega) = \left(\frac{-1}{800}\right) \frac{\omega^2}{(1+j\,\omega/40)(1+j\,\omega/200)}$$

$$20 \log \frac{1}{800} = -20 \log 800 = -20\left(\log 8 + \log 100\right)$$

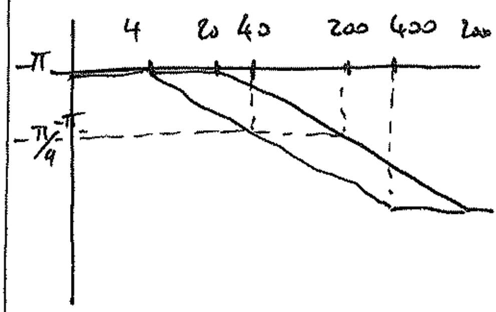
$$= \underbrace{-20 \log 2^3}_{-3\times 6} + \underbrace{(-20 \log 100)}_{-40\,dB} = -58\,dB$$

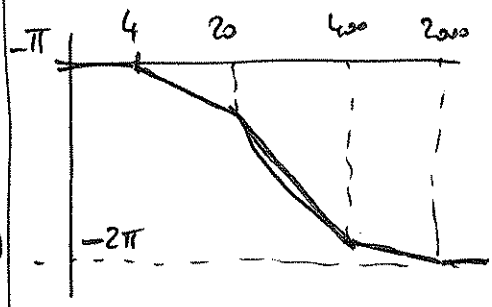




Phase

$$\angle T(j\omega) = -\pi - \tan^{-1}\left(\frac{\omega}{40}\right) - \tan^{-1}(\omega/200)$$

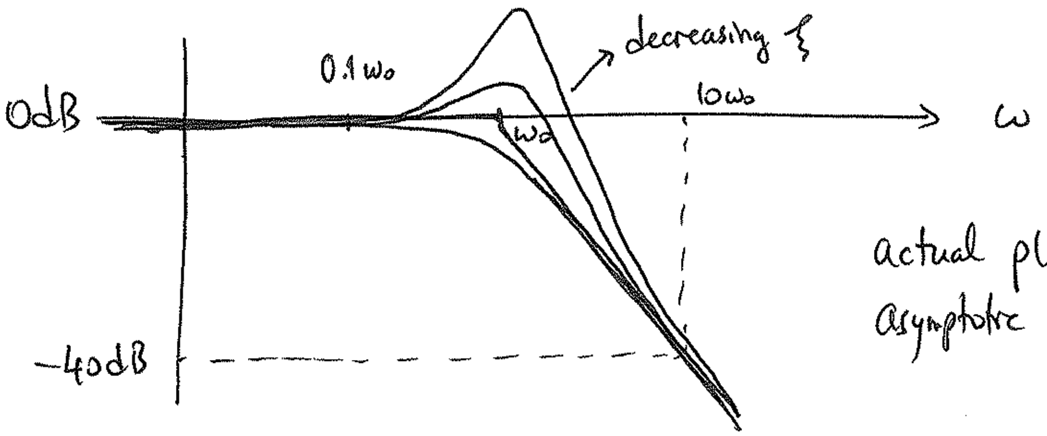




Bode Diagrams with Complex poles and zeros

$$T(s) = \frac{\omega_0^2}{\left(s^2 + 2\xi\omega_0 s + \omega_0^2\right)} \Rightarrow \frac{1}{\left(1-(\omega/\omega_0)^2\right) + j2\xi\left(\frac{\omega}{\omega_0}\right)} = T(j\omega)$$

$$|T(j\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_0}\right)^2}}$$



actual plot is below the asymptotic plot if $\xi > 0.7$ $\frac{1}{\sqrt{2}}$

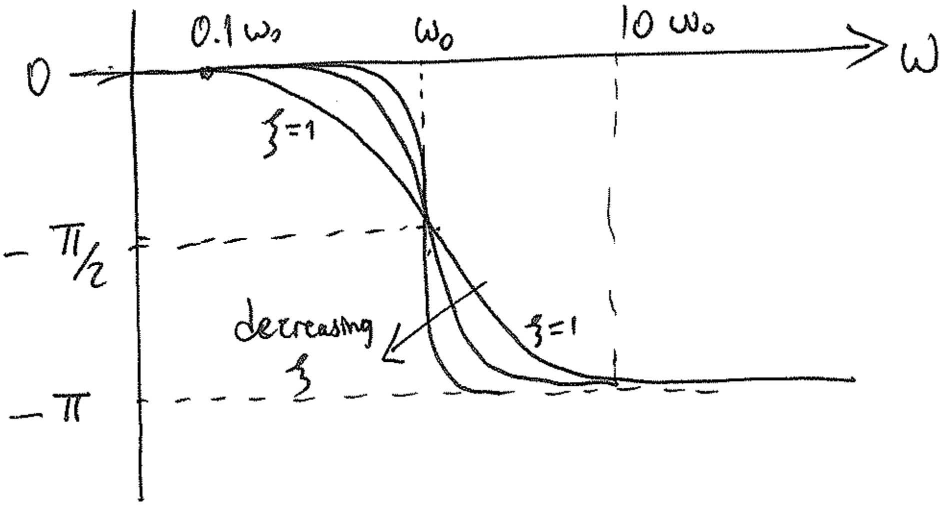
$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^4}} \quad \text{if} \quad \xi \approx \frac{1}{\sqrt{2}}$$

at $\omega = \omega_0$ $\quad |T(j\omega)| = \frac{1}{\sqrt{2}} \equiv -3\,dB$ $\quad$ if $\xi = \frac{1}{\sqrt{2}}$

$$\left.\begin{array}{ll} |T(j\omega)| \geq & \\ |T(j\omega)| = & \\ |T(j\omega)| \leq & \end{array}\right\{ \frac{1}{\sqrt{1+(\omega/\omega_0)^4}} \quad \begin{array}{ll} \text{if} & \xi < \frac{1}{\sqrt{2}} \\ \text{if} & \xi = \frac{1}{\sqrt{2}} \\ \text{if} & \xi > 1/\sqrt{2} \end{array}$$

when $\xi = 0$ $\quad |T(j\omega)| = \frac{1}{\left|1 - \left(\frac{\omega}{\omega_0}\right)^2\right|}$

$$\angle T(j\omega) = -\,\mathrm{Tan}^{-1}\left(\frac{2\xi\,(\omega/\omega_0)}{1-(\omega/\omega_0)^2}\right)$$



Maximum gain is attained at $\left(\text{for } \xi < \frac{1}{\sqrt{2}}\right)$

$$\left.\begin{aligned} \omega_m &= \omega_0\sqrt{1-2\xi^2} \\ |T(j\omega_m)| &= \frac{1}{2\xi\sqrt{1-\xi^2}} \end{aligned}\right\} \text{exercise: verify this}$$