# Circuit Theory

## Basic Circuit Analysis

• Skip to Sec. 2.2 → **Connection Constraints** → **KCL, KCVL**

• **Kirchhoff's Current Law** (**KCL**) based on **Nodes**

• **Kirchhoff's Current Law :** Algebraic sum of currents **entering** any node is zero.

• Some books use **leaving** convention. Its essentially the same.

• **Physical Reason :** conservation of charge. → rate of change of total charge at any node is zero → KCL

• **Example on p. 21-22, Fig. 2-11**

• $$\mathbf{A:}\ -i_1 - i_2 = 0,\ \mathbf{B:}\ i_1 - i_3 - i_4 + i_5 = 0,\ \mathbf{C:}\ i_2 + i_3 + i_4 - i_5 = 0,$$

• **Linear dependence :** A set of equations are linearly dependent if a linear combination of them is exactly zero. → some equations can be derived from the others, hence unnecessary.

• **Example : L1** : $x_1 + x_2 = 1$, **L2** : $2x_1 + 2x_2 = 2$ → 2**L1** - **L2**$= 0$ → **L1** and **L2** are linearly dependent. → One equation is redundant.

• In the above example, we have **A**+ **B** + **C** $= 0$.

• **Conclusion :** All node equations are **linearly dependent**

- **Question :** Assume that there are $n$ nodes. How many node $KCL$ equations are linearly independent ?

- Any $n-1$ of KCL node equations are **linearly independent**.

- **Exercise 2.1 on p. 23, Fig. 2-12**

- **A:** $-i_1 - i_2 = 0$, **B:** $i_2 - i_3 - i_4 = 0$, **C:** $i_4 - i_5 - i_6 = 0$, **D:** $i_1 + i_3 + i_5 + i_6 = 0$

- Check that **A**+ **B** + **C** + **D** $= 0$.

- Given $i_1 = -1\ mA,\ i_3 = 0.5\ mA,\ i_6 = 0.2\ mA \rightarrow i_2 = 1\ mA$ (from **A**), $i_4 = 0.5\ mA$ (from **B**), $i_5 = 0.3\ mA$ (from **C**).

- An alternative view of KCL : Take any spherical volume, place it inside the circuit. Algebraic sum of currents **entering** to the volume is zero.

- **Physical Reason :** conservation of charge in any volume.

- Place the volume around any node $\rightarrow$ KCL node equation.

- **Matrix Formulation** : KCL equations can be written in matrix form as $Ai = 0$. Here, $A$ is called **incidence** matrix, $i$ is a vector containing all currents.

- $$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{pmatrix} = 0$$

- Due to redundancy, we may delete any row of matrix $A$.

- **Kirchhoff's Voltage Law** (**KVL**) based on **Loops**

- **Loop :** A closed path formed by tracing through an ordered sequence of nodes without passing any node more than once.

- **Closed path :** starts and end at the same node.

- **Closed node sequence :** Same as loop, without the "passing any node more than once" requirement $\rightarrow$ actually union of loops.

- **Kirchhoff's Voltage Law :** Algebraic sum of voltages around any loop (or any closed node sequence) is zero.

- **Physical Reason :** Due to conservative field, the potential difference between point $A$ nd $A$ is zero...The work done by moving a unit charge from point $A$ to $A$ is zero...

- Let $A - B - C - ... - A$ be the nodes in the loop....$w_{A-A} = 0 \rightarrow w_{A-A} = w_{A-B} + w_{B-C} + ... \rightarrow dw_{A-A}/dq = 0 \rightarrow dw_{A-B}/dq + dw_{B-C}/dq + .... = 0 \rightarrow v_{A-B} + v_{B-C} + .... = 0$

• **Example on p. 24, Fig. 2-13**

• Loop 1 : $-v_1 + v_2 + v_3 = 0$, Loop 2 : $-v_3 + v_4 + v_5 = 0$, Loop 3 : $-v_1 + v_2 + v_4 + v_5 = 0$

• **Example 2-5 on p. 24, Fig. 2-13**

• $v_1 = 5\ V,\ v_2 = -3\ V,\ v_4 = 10\ V \rightarrow$

• Loop 1 $\rightarrow v_3 = 8\ V$, Loop 2 $\rightarrow v_5 = -2\ V$.

• **Question :** Do we need all of these loop equations ? $\rightarrow$ Are they linearly independent?

• Loop 1 + Loop 2 − Loop 3 = 0.

• All loop equations are linearly dependent. $\rightarrow$ How many loop equations do we need ?

• $b$ : number of two terminal elements, $n$ : number of nodes $\rightarrow$ There are $b - n + 1$ linearly independent loop equations...$\rightarrow$ Meshes ....

• **Matrix Formulation** : KVL equations can be written in matrix form as $Bv = 0$. Here, $B$ is called **loop** matrix, $v$ is a vector containing all voltages.

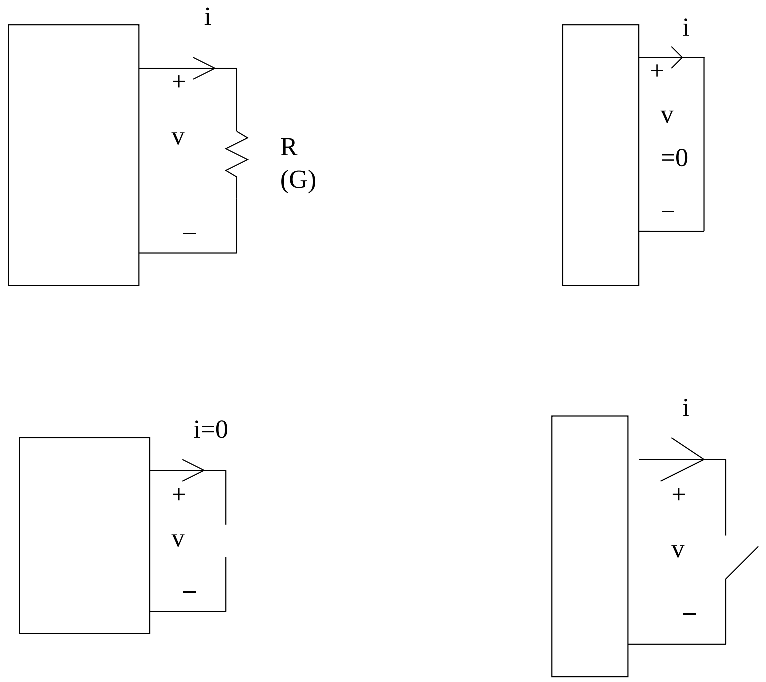
• $$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = 0$$

• Skip to Sec. 2.1 → **Element Constraints**

• Element constraints are algebraic or differential relations between terminal voltage(s) and current(s) of the device in question.

• **Resistors :** Defined by an **algebraic** relation between terminal $v$ and $i$.

• **Linear Resistor :**algebraic relation is **linear** → $av + bi = 0$




• $a \neq 0 \rightarrow v = Ri \rightarrow R$ : Resistance → unit : Ohm $\Omega$ → Ohm $= V/A$.

• $b \neq 0 \rightarrow i = Gi \rightarrow G$ : Conductance → unit : Siemens S → Siemens $= A/V$.

• if $G \neq 0 \rightarrow R = 1/G$, if $R \neq 0 \rightarrow G = 1/R$,

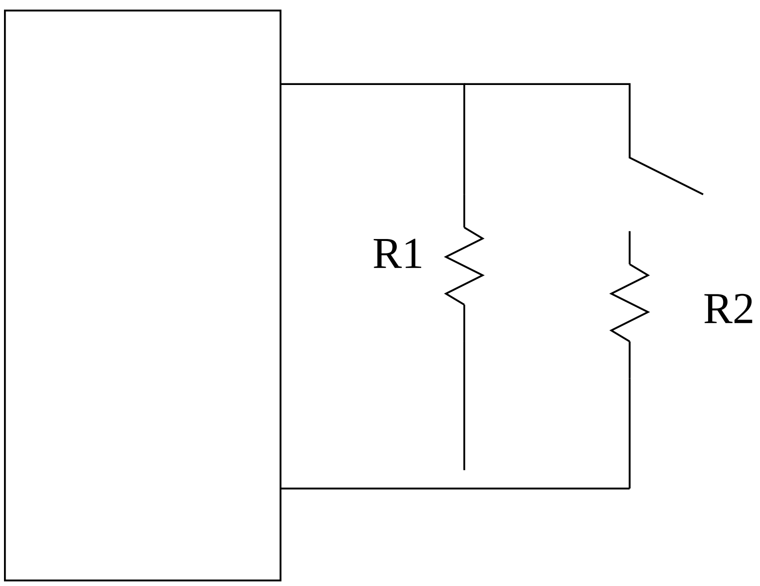
• $a = 0,\ b \neq 0 \rightarrow i = 0$ ( $v$ is arbitrary) $\rightarrow$ **Open Circuit**.

• $a \neq 0,\ b = 0 \rightarrow v = 0$ ( $i$ is arbitrary) $\rightarrow$ **Short Circuit**.

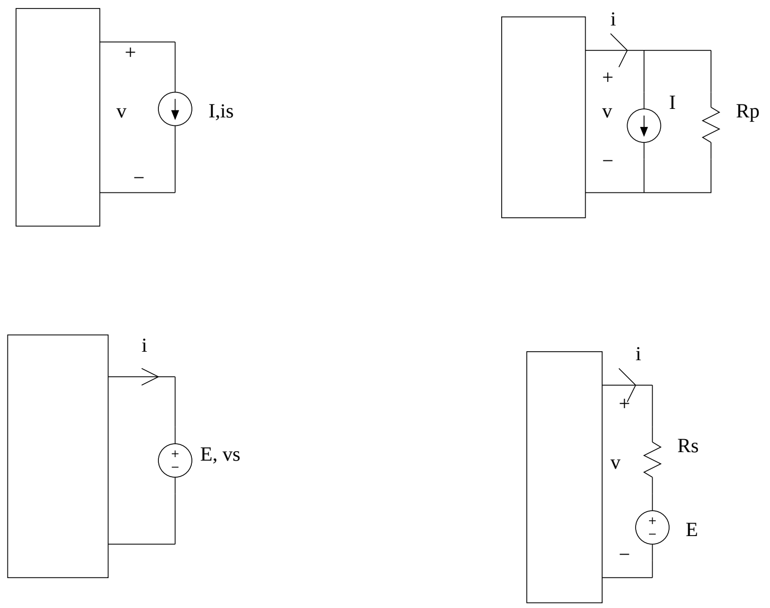
• **Ideal Switch** A switch which changes between open and short circuits...

• **Practical Switch** has a small resistance when short and large resistance when open...

• $$R_S = \begin{cases} R_s & \text{short} \\ R_o & \text{open} \end{cases}$$



• **Ideal Sources :** Generates power for the circuit...




• **Ideal Current Sources :** $i = I$ (DC source), or $i = i_s(t)$ ( a given time function).

• **Ideal Voltage Sources :** $v = E$ (DC source), or $v = v_s(t)$ ( a given time function).

• **Practical Current Sources :** An ideal current source parallel to a linear resistor $i = v/R_p + i_s$

• **Practical Voltage Sources :** An ideal voltage source series to a linear resistor $v = R_s i + v_s$

- Skip to Sec. 2.3 → **Combined Constraints**

- When we write KCL + KVL + Element constraints → Combined constraints.

- Assume that we have $n$ nodes and $b$ elements. Each element has its $v$ and $i$ as **unknowns** → $2b$ unknowns. Need that many **linearly independent** equations.

- **How many equations do we have ?**

- KCL equations → $n - 1$

- KVL equations → $b - n + 1$

- Element Constraints : $b$

- Total Equations : $2b$.

- If all of these equations are linearly independent → circuit has a unique solution.

- **Example 2.8 on p. 29-30, Fig. 2-21**

- unknowns → $v_A,\ i_A, v_1\ , i_1, v_2,\ i_2$ → 6.

- KCL equations → **A :** $-i_A - i_1 = 0$, **B :** $i_1 - i_2 = 0$

- KVL equations → **Loop 1** $-v_A + v_1 + v_2 = 0$

- Element Constraints : **D1 :** $v_A = V_0$, **D2 :** $v_1 = R_1 i_1$, **D3 :** $v_2 = R_2 i_2$,

- $V_0 =\ 10\ V$, $R_1 = 2\ K\Omega$, $R_2 = 3\ K\Omega$

- KCL equations $\rightarrow -i_A = i_1 = i_2 = i$

- KVL + Element Constraints $\rightarrow$
- $V_0 = (R_1 + R_2)i \rightarrow i = V_0/(R_1 + R_2) = 2\ mA$

- Can find the rest of the variables by using $i$.

- **Tableau Equations**

- Write KCL+ KVL + Element Constraints in **matrix** form...

- KCL $\rightarrow Ai = 0$

- KVL $\rightarrow Bv = 0$

- Element Constraints $\rightarrow Mv + Ni = u \rightarrow M$ and $N$ depends on resistors, $u$ depends on sources.

- $$\begin{pmatrix} B & 0 \\ 0 & A \\ M & N \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \qquad \rightarrow Tx = u_s$$

- If $T$ is invertible $\rightarrow x = T^{-1}u_s$.

- If $T$ is not invertible $\rightarrow$ circuit either has no solution, or have infinitely many solutions...

• Sec. 2.4 → **Equivalent circuits**

• Two circuits are said to be equivalent if they have identical $i - v$ relation between a specified pair of terminals.

• **Result :** The electrical behaviour inside the **rest of the circuit** will not change if we replace two identical circuits connected to it.

• **Equivalent Resistance : Series Combination :** (Fig. 2.25 and Fig 2.35).

• KCL : $i = i_1 = i_2 = i_3$, KVL : $v = v_1 + v_2 + v_3$,

• Elements : $v_j = R_j i_j,\ j = 1, 2, 3$

• $\rightarrow v = (R_1 + R_2 + R_3)i \rightarrow v = R_{eq}i, \quad R_{eq} = R_1 + R_2 + R_3$

• **Equivalent Resistance : Parallel Combination :** (Fig. 2.26 and Fig. 2.41).

• KCL : $i = i_1 + i_2 + i_3$, KVL : $v = v_1 = v_2 = v_3$,

• Elements : $v_j = R_j i_j, \ \rightarrow \ i_j = G_j v_j \ j = 1, 2, 3$

• $\rightarrow i = (G_1 + G_2 + G_3)v \rightarrow i = G_{eq}V, \quad G_{eq} = G_1 + G_2 + G_3$

• $\rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

- $j = 2 \rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$ (Fig. 2.26)

- **Voltage Sources in series :** (Fig. 2.34)

- KCL : $i = i_1 = i_2$, KVL : $v = v_1 + v_2$, Elements : $v_1 = V_1,\ v_2 = V_2$

- $\rightarrow v = V_{eq} = V_1 + V_2$

- **Current Sources in parallel :** (Fig. 2.34)

- KCL : $i = i_1 + i_2$, KVL : $v = v_1 = v_2$, Elements : $i_1 = I_1,\ i_2 = I_2$

- $\rightarrow i = I_{eq} = I_1 + I_2$

- **Equivalence of practical sources :** (Fig. 2.29)

- Practical Current Source :$i = v/R_p + i_s$, Practical Voltage Source : $v = R_s i + v_s$

- **Equivalence when** $R_p = R_s,\ v_s = -R_p i_s$

- $\rightarrow$ **Thevenin-Norton equivalent circuits**

- A Voltage source parallel to a resistor case : (Fig. 2.32)

- $v = v_s \rightarrow$ Resistance can be omitted !

- A Current source series to a resistor case : (Fig. 2.33)

• $i = i_s \rightarrow$ Resistance can be omitted !

• Sec. 2.5 $\rightarrow$ **Voltage and Current Division :**

• **Voltage Division :** (Fig. 2.35)

• $\rightarrow v = R_{eq}i, \quad R_{eq} = R_1 + R_2 + R_3,\ i = i_1 = i_2 = i_3, v_j = R_j i_j$

• $\rightarrow v_j = \frac{R_j}{R_1+R_2+R_3}v$ (Note that $v = v_s$).

• **Current Division :** (Fig. 2.41)

• $\rightarrow i = G_{eq}v, \quad G_{eq} = G_1 + G_2 + G_3,\ i = i_1 + i_2 + i_3, i_j = G_j v_j$

• $\rightarrow i_j = \frac{G_j}{G_1+G_2+G_3}i$ (Note that $i = i_s$).

• $\rightarrow$ For 2 resistors $\rightarrow$, $i_1 = \frac{G_1}{G_1+G_2}i = \frac{R_2}{R_1+R_2}i$ (Fig. 2.42)

• $\rightarrow$ For 2 resistors $\rightarrow$, $i_2 = \frac{G_2}{G_1+G_2}i = \frac{R_1}{R_1+R_2}i$

• Sec. 2.6 $\rightarrow$ **Circuit Reduction**

• Sometimes, by replacing certain parts of a given circuits by their equivalent circuits, by using series/parallel combinations, we may simplify a given circuit. This is called circuit reduction technique. Works in some simple circuits, and some parts of complicated circuits, but may not be applicable to some complex circuits. Typical application $\rightarrow$ ladder circuits, see Fig. 2.48.

- **Example 2.22, p. 49-50**

- Note that not all resistors are linear. A typical example is a diode, which is a nonlinear resistor. ($i = I_s(e^{v/v_T} - 1)$). This model can be simplified (piecewise linear-switch)

- A resistor is called **passive** if we have $p = vi \geq 0$ for all possible cases. Otherwise, it is called **active**.

- A resistor is called **time varying**, if its $i - v$ behaviour changes with time.

- A resistor is called **bilateral**, if nothing changes when we replace the terminal connections $\rightarrow$ its $i - v$ relation is symmetric with respect to the origin.