Circuit Theory

Chapter 3 : Circuit Analysis Techniques

• **Node Analysis:** $\Longrightarrow$. Variables are node voltages.

• Choose a DATUM NODE (**Ground**) $\rightarrow$ Arbitrary node. Let us assume that this node is node G.

• Let A be any other node. Then the node voltage of A, denoted as v_A is the potential difference between nodes A and G, + side being at node A, $-$ side being at G.

• **Example : Fig. 3.2**. Let A, B, G be arbitrary nodes, G being ground. Let v_1 be the voltage between nodes A and B, + side being at node A, $-$ side being at node B.

• **KVL** : $v_1 = v_A - v_B$.

• **Proof :** Choose the loop between the nodes (sequence) A-B-G-A. Loop equation $\rightarrow v_1 + v_B - v_A = 0 \rightarrow v_1 = v_A - v_B$.

• **Remark 1 :** $v_G = v_G - v_G = 0$ according to this definition.

• **Remark 2 :** This form of KVL is equivalent to KVL for loops.

• **Basic Idea :** We have Combined Constraint Equations : KCL + KVL + Element Relations.

• Use **Node Voltages** as basic variables (there are $n-1$ of them). By using KVL, express each element voltage in terms of node voltage. By using Elemnet Relations, express element currents in terms of node voltages. Use them in KCL equations (there are $n-1$ of them).

• **Result :** If everything goes right, we will have $n-1$ equations in terms of $n-1$ node voltages. If something goes wrong, we will modify this technique $\rightarrow$ Modified Node Analysis.

• **Simplest Case :** Circuits with Independent Current Sources and Resistors only.

• **Basic Steps :**

• **Step 1 :** Choose a Reference Node (Datum = Ground). Specify reference directions for element currents and hence voltages. Now we can write each element voltage in terms of node voltages.

• **Step 2 :** Write KCL equations at all nodes except for the reference node.

• **Step 3 :** Use element relations (i.e. $i = Gv, \quad i - I$) to express currents in terms of node voltages.

- **Step 4 :** Substitute these in KCL equations.

- **Result :** We will have $n-1$ equations in terms of $n-1$ node voltages.

- **Example :** See Fig. 3-4.

- **Step 1 :** Choose G as the ground node. Choose the reference directions.

Then we have by KVL : $v_1 = v_A,\ v_2 = v_A - v_B,\ v_3 = v_B$.

- **Step 2 :** KCL at A : $-i_1 - i_2 - i_0 = 0$, at B : $i_2 - i_3 = 0$.

- **Step 3 :** Element Relations : $i_0 = -i_s,\ i_1 = G_1 v_A,\ i_2 = G_2(v_A - v_B)$,
- $i_3 = G_3 v_B$

- **Step 4 :** KCL at A : $i_s - (G_1 + G_2)v_A + G_2 v_B = 0$

- KCL at B : $G_2 v_A - (G_1 + G_2)v_B = 0$

- These equations can equivalently be written as :

- **Step 4 :** KCL at A : $(G_1 + G_2)v_A - G_2 v_B = i_s$

- KCL at B : $-G_2 v_A + (G_2 + G_3)v_B = 0$

- $$\begin{pmatrix} (G_1 + G_2) & -G_2 \\ -G_2 & (G_2 + G_3) \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{pmatrix} i_s \\ 0 \end{pmatrix}$$

• **Example :** See Fig. 3-5.

• **Step 1 :** Choose G as the ground node. Choose the reference directions. Then we have by KVL : $v_1 = v_A - v_B,\ v_2 = v_A - v_C,\ v_3 = v_B,\ v_4 = v_C.$

• **Step 2 :** KCL at A : $-i_1 - i_2 + i_0 = 0$, at B : $i_1 + i_5 - i_3 = 0$,

• at C : $i_2 - i_4 - i_5 = 0$.

• **Step 3 :** Element Relations : $i_0 = i_{s1},\ i_1 = G_1(v_A - v_B),,$

• $i_2 = G_2(v_A - v_C),\ i_3 = G_3 v_B,\ i_4 = G_4 v_C,\ i_5 = i_{s2}$

• **Step 4 :** KCL at A : $i_{s1} - (G_1 + G_2)v_A + G_1 v_B + G_2 v_C = 0$

• KCL at B : $G_1 v_A - (G_1 + G_3)v_B + i_{s2} = 0$

• KCL at C : $G_2 v_A - (G_2 + G_4)v_C - i_{s2} = 0$

• These equations can equivalently be written as :

• **Step 4 :** KCL at A : $(G_1 + G_2)v_A - G_1 v_B - G_2 v_C = i_{s1}$

• KCL at B : $-G_1 v_A + (G_1 + G_3)v_B = i_{s2}$

• KCL at C : $-G_2 v_A + (G_2 + G_4)v_C = -i_{s2}$

• $$\begin{pmatrix} (G_1 + G_2) & -G_1 & -G_2 \\ -G_1 & (G_1 + G_3) & 0 \\ -G_2 & 0 & (G_2 + G_4) \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{s1} \\ i_{s2} \\ -i_{s2} \end{pmatrix}$$

- We can write these equations (in the simple case) in matrix form as :

- $Gv = u$

- G : Conductance (admittance matrix) It is $n - 1 \times n - 1$. v is the vector of node voltages. u is the current source vector.

- **Basic Properties : Writing by inspection**

- G_{ii} : + (sum of conductances connected to node **i**)

- $G_{ij},\ i \neq j$: - (sum of conductances connected **directly** between nodes **i** and **j**)

- **Result :** Since $G_{ij} = G_{ji},\ i \neq j$, G is a **symmetric** matrix.

- $u(i)$: + (sum of independent current sources **entering** to node **i**)

- **Example :** See Fig. 3-6.

- $$\begin{pmatrix} (G_1 + G_2) & -G_2 \\ -G_2 & (G_2 + G_3 + G_4) \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{pmatrix} i_{s1} - i_{s2} \\ i_{s2} \end{pmatrix}$$

- **Example :** See Fig. 3-7.

- $$\begin{pmatrix} 2.5G & -0.5G & -2G \\ -0.5G & 3G & -0.5G \\ -2G & -0.5G & 3.5G) \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_s \\ 0 \\ 0 \end{pmatrix}$$

• What happens when we have **voltage sources as well ?** $\rightarrow$ modify the same idea.

• **Example :** See Fig. 3-11.

• **Case 1 :** If a voltage source is in **series** with a resistor $\rightarrow$ use equivalence (see previous lecture notes)

• **Example :** See Fig. 3-12.

• After the equivalence transformation $\rightarrow (G_1+G_2+G_3)v_A = G_1v_{s1}+G_2v_{s2}$

• $\rightarrow v_A = v_0 = \frac{G_1v_{s1}+G_2v_{s2}}{G_1+G_2+G_3}$

• $\rightarrow$ Can find any remaining voltage/current by KCL/KVL. e.g. $v_1 = v_s - v_0$, $i_1 = G_1v_1$, etc...

• **Case 2 :** If a voltage source is not in **series** with a resistor $\rightarrow$ We may choose one end of the voltage source as Reference Node...

• Do not write Node equation at node A. There remains $n-2$ nodes, write Node Equations for these as usual. $\rightarrow$ We get $n-2$ equations + $v_A = v_s$ $\rightarrow$ We can find all the node voltages....

• **Example :** See Fig. 3-13.

• Do not write KCL at A. We have $v_A = v_s$

• KCL at B : $-0.5Gv_A + 3Gv_B - 0.5Gv_C = 0$

• KCL at C : $-2Gv_A - 0.5Gv_B + 3.5Gv_C = 0$

- $\begin{pmatrix} 3G & -0.5G \\ -0.5G & 3.5G \end{pmatrix} \begin{pmatrix} v_B \\ v_C \end{pmatrix} = \begin{pmatrix} 0.5Gv_s \\ 2Gv_s \end{pmatrix} \rightarrow \begin{pmatrix} v_B \\ v_C \end{pmatrix} = \begin{pmatrix} 0.2683v_s \\ 0.6098v_s \end{pmatrix}$

- $i_{in} = 0.5G(v_A - v_B) + 2G(v_A - v_C)$

  $= G(0.5 - 0.2683 + 2 - 2 \times 0.6098)v_s = 1.1463Gv_s$

- $R_{in} = \frac{v_s}{i_{in}} = 0.8724R$

- **Case 3 :** If a voltage source is not in **series** with a resistor and we cannot (or don't want) choose one end of the voltage source as Reference Node...

- Sum the KCL equations for both nodes of voltage source $\rightarrow$ Current of the voltage source cancels $\rightarrow$ Supernode .....

- **Example :** See Fig. 3-14.

- Call the current of the voltage source v_{s1} as i_x.

- KCL at A : $-i_1 - i_2 - i_x = 0$, KCL at C : $i_x - i_3 - i_4 = 0$

- -A -B : $i_1 + i_2 + i_3 + i_4 = 0$ (Could also be obtained by using the Gaussian Volume ..)

- $G_1 v_A + G_2(v_A - v_B) + G_3(v_C - v_B) + G_4 v_C = 0$

- $v_B = v_{s2}, \rightarrow (G_1 + G_2)v_A + (G_3 + G_4)v_C = (G_2 + G_3)v_{s2}$

- KVL at A and C $\rightarrow v_A - v_C = v_{s1}$

- Two equations and two unknowns.....

• **Mesh Analysis:** $\Longrightarrow$. Variables are **mesh** currents.

• **Planar Circuit :** A circuit which can be drawn on a plane such that two elements only cut themselves at nodes.

• **Mesh :** Given a planar circuit, a mesh is a loop which does not contain any element inside. See Fig. 2.16.

• **Fact :** Given a planar circuit which has b two terminal elements, n nodes, there are exactly $b - n + 1$ meshes.

• **Mesh Current :** In each mesh, define a circulation direction, necessary to write KVL equation. To this direction, assign a circulating current, which is called **mesh current**.

• Note that mesh current is conceptually dual of node voltages. But they are not as natural, because node voltages can be measured easily, whereas mesh current is basically a variable which simplifies analysis. In most of the cases, they cannot be measured directly, but computed by using some element currents.

• Mesh currents are either element currents, or can be written in terms of element currents. Conversely, each element current can be written in terms of mesh currents.

• As a convention, we will choose **clockwise** circulation direction for defining mesh currents. It will simplify the analysis.

• If a branch current i_x is common between mesh A and mesh B, and its reference direction is in the same direction for mesh current A : i_A and hence necessarily in the opposite direction for the mesh current B : i_B (due to clockwise direction convention) $\longrightarrow i_x = i_A - i_B$

• Hence any element current, therefore KCL equations can be written in terms of mesh currents.

• **Basic Idea :** We have Combined Constraint Equations : KCL + KVL + Element Relations.

• Use **Mesh currents** as basic variables (there are $b - n + 1$ of them). By using KCL, express each element current in terms of mesh currents. By using Element Relations, express element voltages in terms of mesh currents (e.g. $v = Ri$). Use them in KVL equations (there are $b - n + 1$ of them).

• **Result :** If everything goes right, we will have $b - n - +1$ equations in terms of $b - n + 1$ mesh currents. If something goes wrong, we will modify this technique → Modified Mesh Analysis.

• **Simplest Case :** Circuits with Independent Voltage Sources and Resistors only.

- **Basic Steps :**

- **Step 1 :** Choose meshes. Specify reference directions for element currents and hence voltages. Now we can write each element current in terms of mesh currents.

- **Step 2 :** Write KVL equations at all meshes.

- **Step 3 :** Use element relations (i.e. $v = Ri, v = E$ to express voltages in terms of mesh currents.

- **Step 4 :** Substitute these in KVL equations.

- **Result :** We will have $b - n + 1$ equations in terms of $b - n + 1$ mesh currents.

- **Example** Fig. 3-18

- **Step 1 :** Choose meshes A and B as given in the figure and assign the mesh currents i_A and i_B. Now we have $i_1 = i_A,\ i_2 = i_B,\ i_3 = i_A - i_B$.

- **Step 2 :** KVL at mesh A : $v_1 + v_3 - v_0 = 0$, at mesh B : $v_2 + v_4 - v_3 = 0$.

- **Step 3 :** Element Relations : $v_0 = v_{s1},\ v_1 = R_1 i_A,\ v_2 = R_2 i_B,$
- $v_3 = R_3(i_A - i_B) \quad , \quad v_4 = v_{s2}$

- **Step 4 :** KVL at mesh A : $-v_{s1} + (R_1 + R_3)i_A - R_3 i_B = 0$

- KVL at mesh B : $-R_3 i_A + (R_1 + R_3)i_B + v_{s2} = 0$

- These equations can equivalently be written as :

- **Step 4 :** KVL at mesh A : $(R_1 + R_3)i_A - R_3 i_B = v_{s1}$

- KVL at B : $-R_3 i_A + (R_1 + R_3)i_B = -v_{s2}$

- $$\begin{pmatrix} (R_1 + R_3) & -R_3 \\ -R_3 & (R_3 + R_2) \end{pmatrix} \begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} v_{s1} \\ -v_{s2} \end{pmatrix}$$

- **Example** Fig. 3-24

- **Step 1 :** Choose meshes A and B as given in the figure and assign the mesh currents i_A and i_B. Now we have $i_3 = i_A,\ i_4 = i_5 = i_B,\ i_6 = i_A - i_B$.

- **Step 2 :** KVL at mesh A : $v_3 + v_6 + v_2 - v_1 = 0$, at mesh B : $v_4 + v_5 - v_2 - v_6 = 0$.

- **Step 3 :** Element Relations : $v_3 = 2Ri_A,\ v_4 = Ri_B,\ v_5 = 2Ri_B,$
- $v_6 = 2R(i_A - i_B)$

- **Step 4 :** KVL at mesh A : $4Ri_A - 2Ri_B = v_1 - v_2$

- KVL at mesh B : $-2Ri_A + 5Ri_B = v_2$

- $$\begin{pmatrix} 4R & -2R \\ -2R & 5R \end{pmatrix} \begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_2 \end{pmatrix}$$

- $8Ri_B = v_1 + v_2 \rightarrow v_0 = 2Ri_B = \frac{v_1+v_2}{4}$

- We can write these equations (in the simple case) in matrix form as :

- $Ri = u$

- R : Resistance (impedance matrix) It is $b - n + 1 \times b - n + 1$. i is the vector of mesh current vector. u is the voltage source vector.

- **Basic Properties : Writing by inspection**

- R_{ii} : + (sum of resistances in the mesh **i**)

- $R_{ij},\ i \neq j$: - (sum of resistances connected **directly** between meshes **i** and **j**)

- **Result :** Since $R_{ij} = R_{ji},\ i \neq j$, R is a **symmetric** matrix.

- $u(i)$: Algebraic sum of independent voltage sources in the mesh **i**; if first $-$ terminal is encountered in clockwise direction, that source enters the sum with + sign; otherwise with $-$ sign.

- **Example :** See Fig. 3-19.

- $$\begin{pmatrix} R_1 + R_2 & 0 & -R_2 \\ 0 & R_3 + R_4 & -R_3 \\ -R_2 & -R_3 & R_2 + R_3 \end{pmatrix} \begin{pmatrix} i_A \\ i_B \\ i_C \end{pmatrix} = \begin{pmatrix} -v_{s2} \\ v_{s2} \\ -v_{s1} \end{pmatrix}$$

- **Mesh Equations with Current Sources :**

- We have 3 different cases :

- **Case 1 :** If a current source is in series with a resistor $\rightarrow$ simplify....

- **Case 2 :** If a current source is in only one mesh $\rightarrow$ that mesh current is already known. Hence write the remaining mesh equations.

- **Case 3 :** If a current source is between two meshes $\rightarrow$ We can combine these two mesh equations into a single mesh equation $\rightarrow$ supermesh

- See Fig. 3.20. Let v_x be the voltage of the branch of the current source

- Mesh A : .... $+ v_x +$ .... $= 0$, Mesh B : .... $- v_x +$ .... $= 0$

- Mesh A + mesh B : v_x is eliminated ...Supermesh equation...

- We need one more equation $\rightarrow i_A - i_B = i_s$...

- We again get $b - n + 1$ equations and that may unknowns...

- **Example 3.8:** Fig. 3.21

- **Regular Way:** Note that $i_C = -2\ mA$

- Use resistors as $K\Omega$, voltages as Volts $\rightarrow$ Currents as mA ...

- Mesh A : $6i_A - 2i_B = 5$, Mesh B : $-2i_A + 11i_B - 4i_C = 0$
- $\rightarrow -2i_A + 11i_B = 4i_C = -8$

- Two equations, two unknowns....

- **By circuit reduction:** Replace Current source and resistor parallel combination with the voltage source series with the resistor one as shown in Fig. 3.21b

- Mesh A : $6i_A - 2i_B = 5$, Mesh B : $-2i_A + 11i_B = -8$

- Same equations ...Solve them ...$i_A = 0.6290\ mA,\ i_B = -0.6129\ mA$
- $\rightarrow i_0 = i_A - i_B = 1.2419\ mA$.

- **Example 3.9:** Fig. 3.22

- Note that $i_A = i_{s1}$, i_{s2} forms a supermesh between meshes B and C.

- Mesh B + Mesh C : Supermesh : $R_1(i_B - i_A) + R_2 i_B + R_4 i_C + R_3(i_C - i_A) = 0$

- Mesh B + Mesh C : $-(R_1 + R_3)i_A + (R_1 + R_2)i_B + (R_3 + R_4)i_C = 0$
- $\rightarrow (R_1 + R_2)i_B + (R_3 + R_4)i_C = (R_1 + R_3)i_{s1}$

- $i_B - i_C = i_{s2} \longrightarrow i_B = i_C + i_{s2}$

- $(R_1 + R_2 + R_3 + R_4)i_C = (R_1 + R_3)i_{s1} - (R_1 + R_2)i_{s2}$

- $v_0 = R_4 i_C = R_4 \frac{(R_1+R_3)i_{s1} - (R_1+R_2)i_{s2}}{(R_1+R_2+R_3+R_4)}$

• **Linear Circuits:** $\Longrightarrow$ Superposition, Thevenin, Norton equivalent Circuits.

• A circuit is called **linear** if it contains linear elements + independent sources...

• Circuit equations : Combined Constraints

$$\begin{array}{rc} \text{KCL :} & Ai = 0 \\ \text{KVL :} & Bv = 0 \\ \text{Element :} & Mv + Ni = u \end{array} \quad \longrightarrow \quad \begin{pmatrix} 0 & A \\ B & 0 \\ M & N \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

• Resulting equations are **linear :** $\longrightarrow$ $Tw = u_s$

• T: An $2b \times 2b$ matrix which depends on circuit elements. but **not** on independent sources, w: a $2b$ vector of unknowns, u_s : a $2b$ vector which depends **only** on the independent sources.

• **Consequences of linearity :**

• **Homogeneity :** $f(\alpha x) = \alpha f(x) \quad , \quad \alpha \in \mathbf{R}$

• **Additivity :** $f(x_1 + x_2) = f(x_1) + f(x_2)$

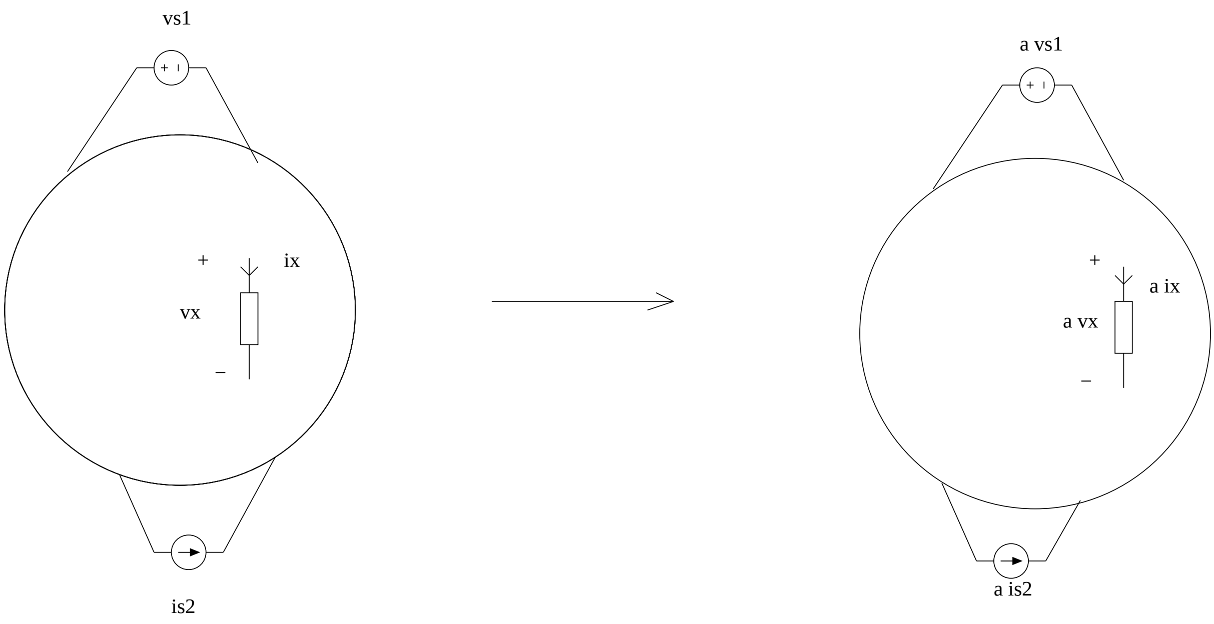
• **Typical Example :** Linear Functions : $\rightarrow f(x) = Ax$ in our case $f(w) = Tw$

• **Homogeneity :** $u_s \rightarrow w \qquad \Rightarrow \alpha u_s \rightarrow \alpha w$

• **Proof :** $u_s \rightarrow w \qquad \Rightarrow \qquad Tw = u_s$

• $\alpha Tw = \alpha u_s = T\alpha w \qquad \Rightarrow \qquad \alpha u_s \rightarrow \alpha w$

• **Circuit interpretation**



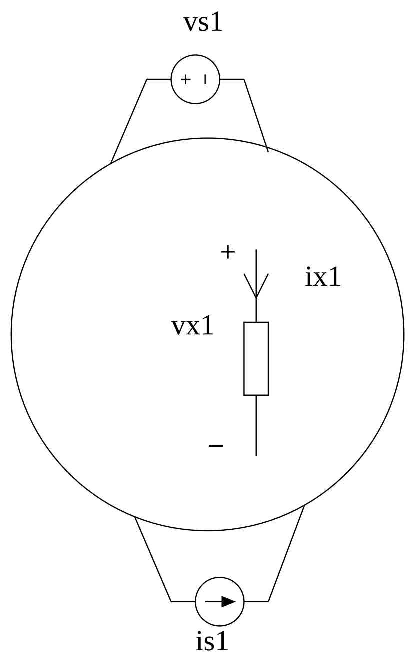
• Note that all independent sources must be multiplied by the same constant.

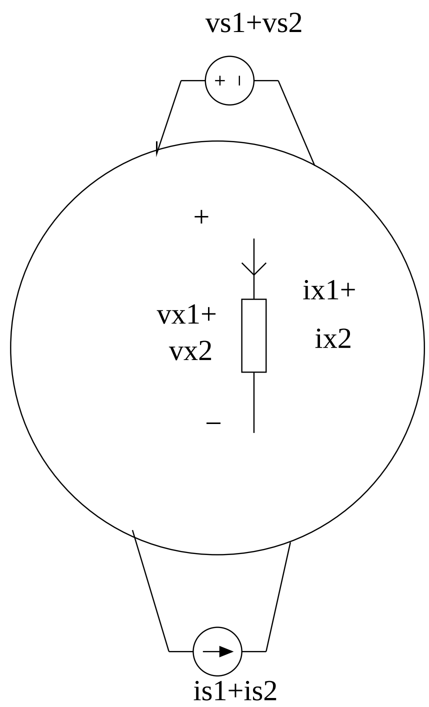
• **Additivity :** $u_{s1} \rightarrow w_1, \quad u_{s2} \rightarrow w_2 \qquad \Rightarrow \qquad u_{s1} + u_{s2} \rightarrow w_1 + w_2,$

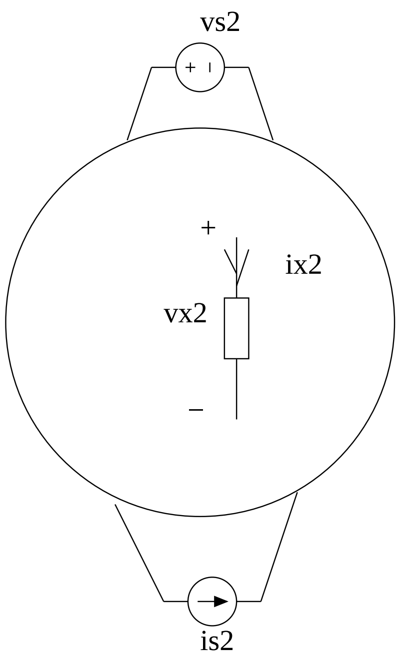
• **Proof :** $u_{s1} \rightarrow w_1, \quad u_{s2} \rightarrow w_2 \qquad \Rightarrow \qquad Tw_1 = u_{s1}, \quad Tw_2 = u_{s2}$

$Tw_1 + Tw_2 = T(w_1 + w_2) = u_{s1} + u_{s2} \qquad \Rightarrow \qquad u_{s1} + u_{s2} \rightarrow w_1 + w_2,$

• **Circuit interpretation**







• **Superposition :**

• is a consequence of linearity and hence the ideas given above. Assume that the source vector u_s is given as below (for simplicity, we assume that we have only one voltage source v_{s1} and one current source i_{s1}) :

$$\bullet\ u_s = \begin{pmatrix} 0 \\ \vdots \\ v_{s1} \\ i_{s1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ v_{s1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i_{s1} \end{pmatrix} = u_{s1} + u_{s2} \longrightarrow Tw = u_s$$

• Note that for u_{s1}, only v_{s1} is operational $\rightarrow i_{s1} = 0 \rightarrow i_{s1}$ is **OPEN CIRCUIT**.

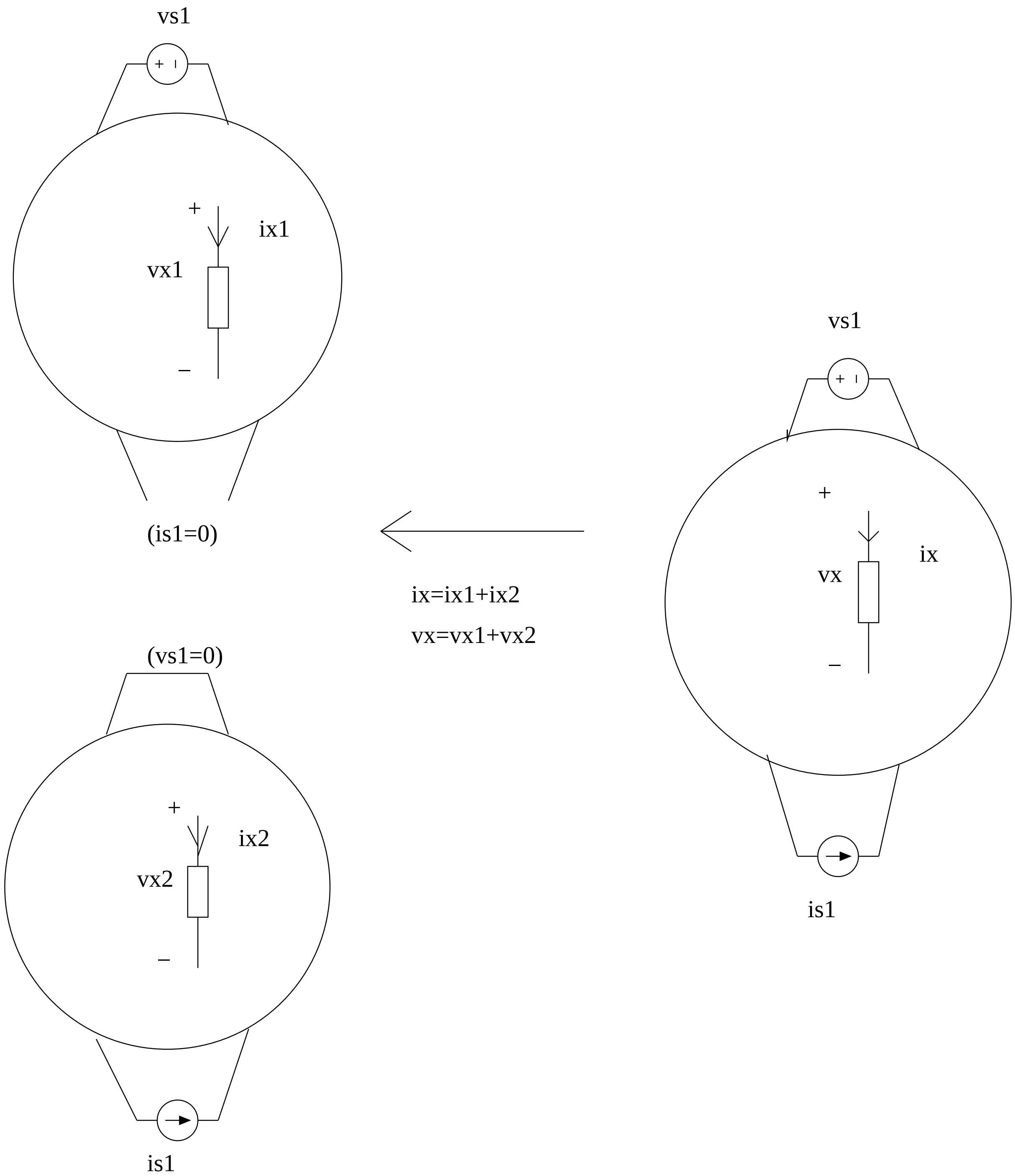
• Similarly, for u_{s2}, only i_{s1} is operational $\rightarrow v_{s1} = 0 \rightarrow v_{s1}$ is **SHORT CIRCUIT**.

$$\bullet\ u_{s1} \rightarrow w_1, \quad u_{s2} \rightarrow w_2 \qquad \Rightarrow \qquad Tw_1 = u_{s1}, \quad Tw_2 = u_{s2}$$

$$Tw_1 + Tw_2 = T(w_1 + w_2) = u_{s1} + u_{s2} = u_s$$

$$\Rightarrow \qquad u_s = u_{s1} + u_{s2} \rightarrow w = w_1 + w_2,$$

- **Circuit interpretation**



- **Example** Fig. 3.31, p. 98..

- **Direct Way :** Use any method (node, mesh analysis)

- Node Eqn. at A : $i_s = G_2 v_A + G_1(v_A - v_s) \rightarrow (G_1 + G_2)v_A = i_s + G_1 v_s$

- $v_A = v_0 = \frac{1}{G_1+G_2} i_s + \frac{G_1}{G_1+G_2} v_s = \frac{R_1 R_2}{R_1+R_2} i_s + \frac{R_2}{R_1+R_2} v_s$

- **Superposition :**

- **Step 1 :** Set $i_s = 0 \rightarrow$ OPEN CIRCUIT...

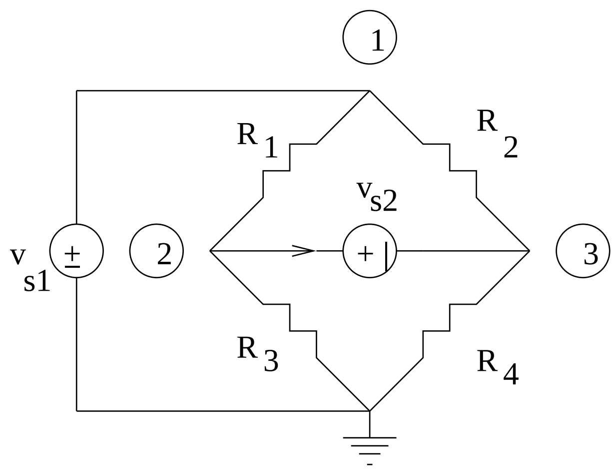
- Voltage Divider $\rightarrow v_{01} = \frac{R_2}{R_1+R_2} v_s$

- **Step 2 :** Set $v_s = 0 \rightarrow$ SHORT CIRCUIT...

- Current Divider $\rightarrow i_{02} = \frac{R_1}{R_1+R_2} i_s \rightarrow v_{02} = R_2 i_{02} = \frac{R_1 R_2}{R_1+R_2} i_s$

- **Step 3 :** $v_0 = v_{01} + v_{02} = \frac{R_1 R_2}{R_1+R_2} i_s + \frac{R_2}{R_1+R_2} v_s$

- **Example** Find v_2, node voltage 2, for the following circuit..



• **Superposition :**

• **Step 1 :** Set $v_{s2} = 0 \rightarrow$ SHORT CIRCUIT...

• **Result :** $R_1 \parallel R_2$, $R_3 \parallel R_4$

• Voltage Divider $\rightarrow v_{01} = \frac{R_3 \| R_4}{(R_1 \| R_2) + (R_3 \| R_4)} v_{s1} = \frac{R_3 R_4 (R_1 + R_2)}{R_1 R_2 (R_3 + R_4) + R_3 R_4 (R_1 + R_2)} v_{s1}$

• **Step 2 :** Set $v_{s1} = 0 \rightarrow$ SHORT CIRCUIT...

• **Result :** $R_1 \parallel R_3$, $R_2 \parallel R_4$

• Voltage Divider $\rightarrow v_{02} = \frac{R_1 \| R_3}{(R_1 \| R_3) + (R_2 \| R_4)} v_{s2} = \frac{R_1 R_3 (R_2 + R_4)}{R_1 R_3 (R_2 + R_4) + R_2 R_4 (R_1 + R_3)} v_{s2}$

• **Step 3 :** $v_0 = v_{01} + v_{02} = K_1 v_{s1} + K_2 v_{s2}$

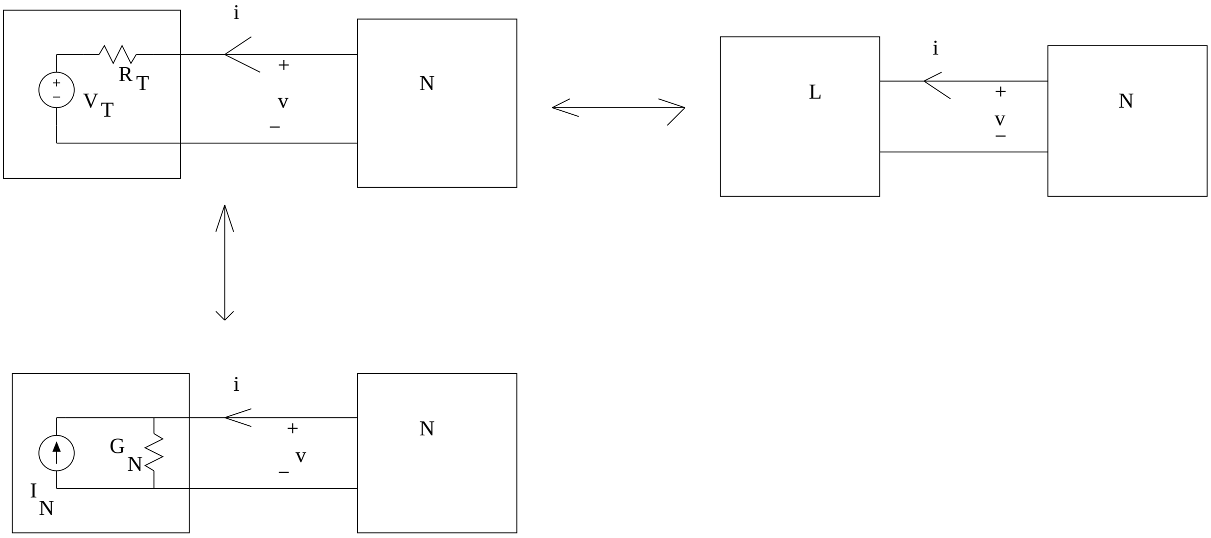
• In general, if we have a linear circuit with n independent voltage sources $v_{s1}, \ldots, v_{sn}$ and m independent current sources $i_{s1}, \ldots, i_{sm}$, if y denotes an arbitrary branch voltage or current...

• $y = K_1 v_{s1} + \ldots + K_n v_{sn} + H_1 i_{s1} + \ldots + H_m i_{sm}$

• Here K_i and H_j are constants which depends only on the circuit elements but not on the independent sources themselves...

- **Thévenin and Norton Equivalent Circuits :**

- Here N is **any** circuit (which may be nonlinear, L is any **linear** circuit. The total circuit is assumed to have **unique solution**.

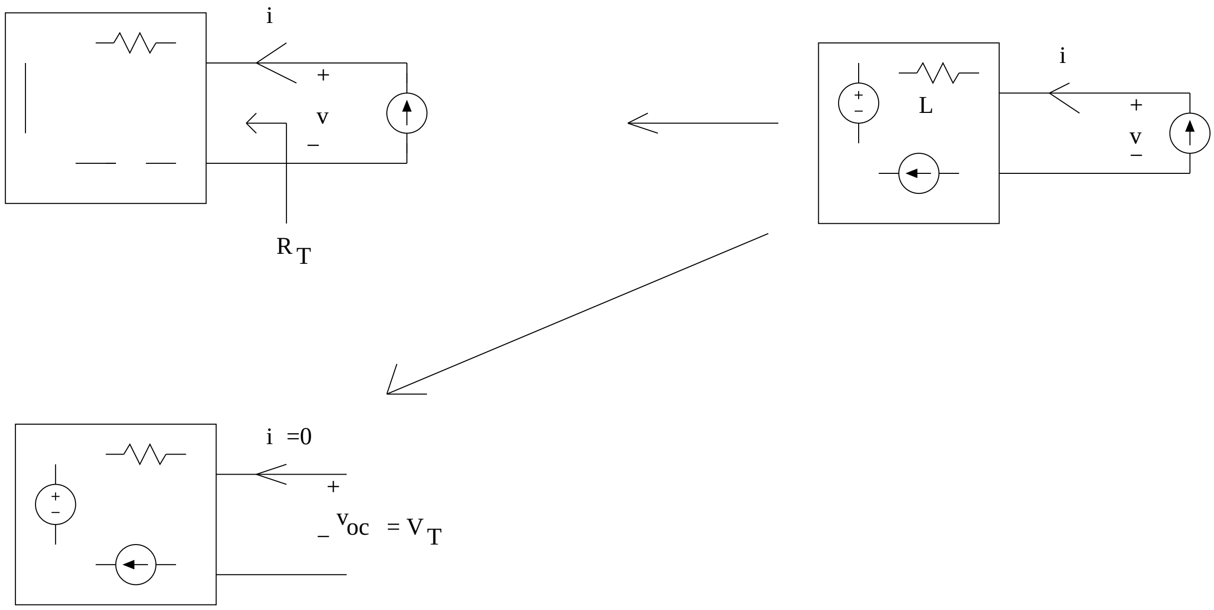


- v_T : Thévenin equivalent voltage source, R_T : Thévenin equivalent resistance, i_N : Norton equivalent current source, G_N : Norton equivalent conductance.

- $v = R_T i + V_T$, $i = G_N v - i_N \rightarrow i = v/R_T - V_T/R_T$

- Hence Thévenin and Norton equivalent circuits are equivalent to each other if $\quad R_N = R_T, i_N = V_T/R_T$

- **Proof of Thévenin equivalent circuit :**



- Because of unique solvability, we can replace N by a current source (this is called substitution). This circuit is uniquely solvable. Hence apply superposition :

- $v = K_1 i +$(contribution of sources inside L)

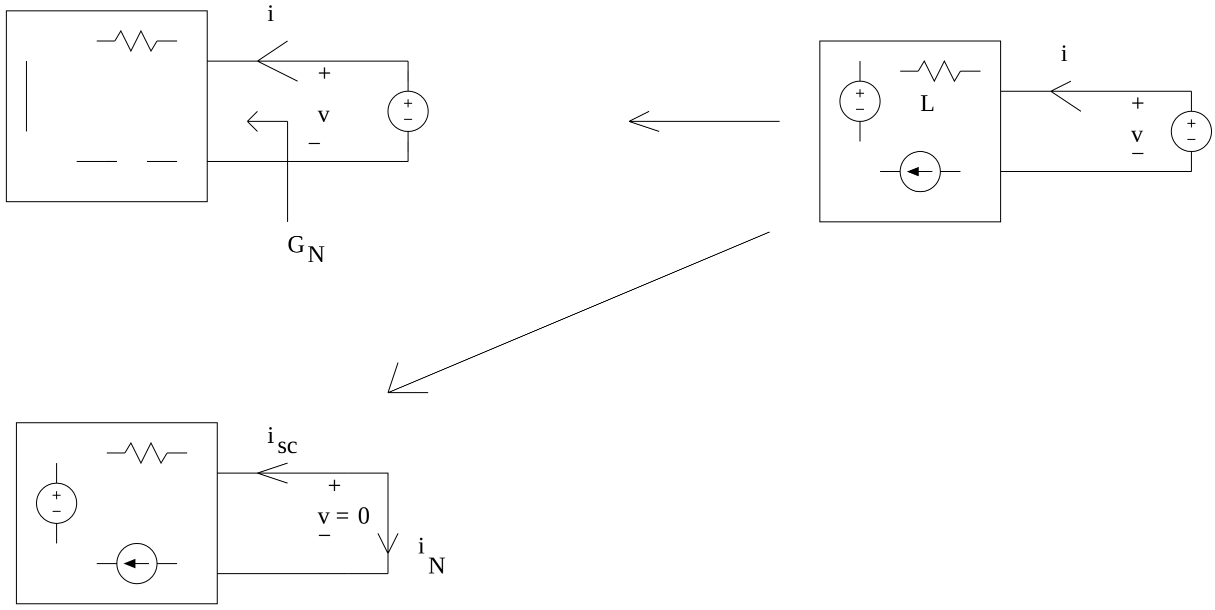
- $K_1 = R_T$, $V_T =$ contribution of sources inside L.

- How to find R_T ? Set $V_T = 0 \rightarrow$ all sources inside L to zero (voltage sources OPEN, current sources SHORT), then $v = K_1 i \rightarrow K_1 = R_T = v/i$

- How to find V_T ? Set $i = 0 \rightarrow$ OPEN CIRCUIT L. Find v_{oc} We have $V_T = v_{oc}$.

- Hence Thévenin equivalent circuit is a direct consequence of superposition and unique solvability....

• **Proof of Norton equivalent circuit :**



• Because of unique solvability, we can replace N by a voltage source (this is called substitution). This circuit is uniquely solvable. Hence apply superposition :

• $i = H_1 v +$(contribution of sources inside L)

• $H_1 = G_N$, $i_N = -$ contribution of sources inside L.

• How to find G_N ? Set $i_N = 0 \rightarrow$ all sources inside L to zero (voltage sources OPEN, current sources SHORT), then $i = H_1 v \rightarrow H_1 = G_N = i/v$

• How to find i_N ? Set $v = 0 \rightarrow$ SHORT CIRCUIT L. Find i_{sc} We have $i_N = -i_{sc}$.

• Hence Norton equivalent circuit is a direct consequence of superposition and unique solvability....

- **Example :** Fig 3.39, p.104

- **Direct Way :**

- Node eqn a X : $(v_X - 15)/5 + (v_X - v)/15 + (v_X)/10 = 0$
- $\Rightarrow (11v_X)/30 = 3 + v/15 \Rightarrow v_X = (2v + 90)/11$
- $\Rightarrow v = 15i + v_X \Rightarrow v = 18.3i + 10$
- $\Rightarrow R_T = 18.3\ \Omega,\ V_T = 10\ V,\ i_N = V_T/R_T = 545\ mA$

- **Indirect Way :**

- **To find R_T :** Set 15 V source to ZERO $\Rightarrow$ SHORT. Then the equivalent resistance is : $R_T = (5\ \Omega \parallel 10\ \Omega) + 15 = 18.3\ \Omega$

- **To find V_T :** OPEN CIRCUIT N
- $\Rightarrow$ Voltage Divider $\Rightarrow V_T = \frac{10}{15}15 = 10\ V$

- **To find i_N :** SHORT CIRCUIT N
- $\Rightarrow R_{EQ} = 15\ \Omega \parallel 10\ \Omega = 6\ \Omega \Rightarrow i_X = 15/(5+6) = 1.36\ A$
- $\Rightarrow$ Current Divider $i_X = \frac{10}{10+15} i_X = 545\ mA$

- **Example :** Fig 3.40, p.106

- **Example :** Fig 3.42, p.107
- **Direct Way :**

- Note that $i_C = -2\ A$. Also note the direction of our i which is the **opposite** of what is used in the book. Hence we have $i_B = -i$

- Mesh Eqn. at A : $240i_A - 180i_B - 60i_C = 40$,
- Mesh Eqn. at B :$-180i_A + 195i_B - 15i_C + v = 0$

- $240i_A + 180i = -80,\ -180i_A - 195i + v = -30$

- Eliminate $i_A \Rightarrow -240i + 4v = -360 \Rightarrow v = 60i - 90$

- $R_T = 60\ \Omega,\ V_T = -90\ V,\ i_N = V_T/R_T = -1.5\ A$

- **Indirect Way :**

- **To find** R_T **:** Set sources to ZERO. Then the equivalent resistance is : $R_T = (60\ \Omega \parallel 180\ \Omega) + 15 = 60\ \Omega$

- **To find** V_T **:** OPEN CIRCUIT N

- Note that $i_C = -2\ A$, $i_B = 0$. Mesh A : $240i_A - 60i_C = 40 \Rightarrow 240i_A = -80 \Rightarrow i_A = -80/240\ A \Rightarrow V_T = v_{oc} = 15i_C + 180i_A = -90\ V$

- **To find** i_N **:** SHORT CIRCUIT N

- Note that $i_C = -2\ A$, $i_B = i_N$

- Mesh Eqn. at A : $240i_A - 180i_B - 60i_C = 40$,
- Mesh Eqn. at B :$-180i_A + 195i_B - 15i_C = 0$

- $240i_A - 180i_N = -80, \; -180i_A + 195i_N = -30$

- Eliminate $i_A \Rightarrow 240i_N = -360 \Rightarrow i_N = -1.5 \; A$

- **Typical Application :** when N is a nonlinear device (e.g. a diode). See Figure 3.48, p. 111

- L : $v_T = R_T i + v$, NLE : $v = f(i)$

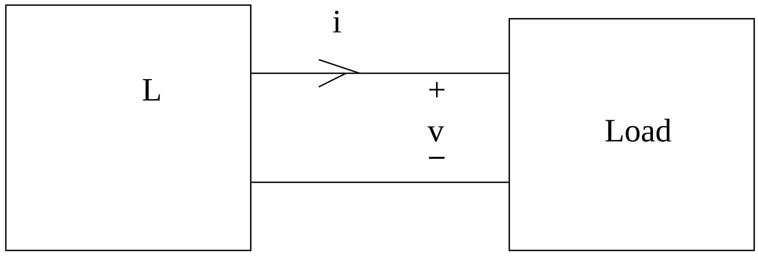
- Analytical solution : $v_T = R_T i + f(i) \Rightarrow F(i) = 0$

- Nonlinear Equation $\Rightarrow$ Finding solution is usually difficult.

- Graphical solution : Draw the equation for L and NLE on the same $i - v$ plane $\Rightarrow$ Intersection point will be the solution for v and i.

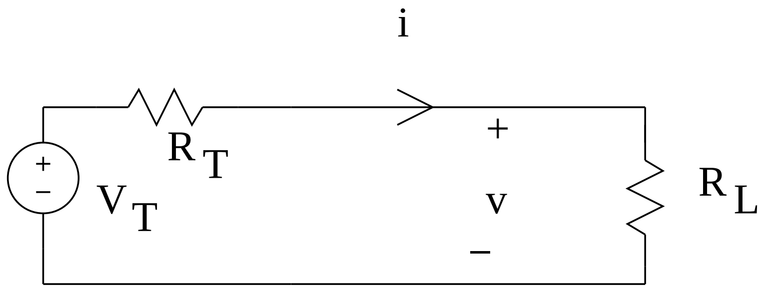
- In this case, the line representing L is called the **load line** for NLE.

- **Maximum Power Transfer :**



- Here, L is a linear circuit, Load represents a linear resistor. Aim is **given L**, find a load such that L transfers maximum power to Load.

• By taking the Thévenin equivalent circuit of L, and representing the Load as a resistor R_L we obtain :



• Question : Given V_T and R_T, find R_L so that $p = vi$ is **maximum**.

• $v = \frac{R_L}{R_T+R_L} V_T,\qquad i = \frac{1}{R_T+R_L} V_T$

• $\Rightarrow p = vi = \frac{R_L}{(R_T+R_L)^2} V_T^2$

• For maximum $\Rightarrow \frac{dp}{dR_L} = 0$ must hold

• $\frac{dp}{dR_L} = \frac{R_T - R_L}{(R_T+R_L)^3} V_T^2 = 0 \qquad \Rightarrow \qquad R_T = R_L$

• This is called **impedance matching**.

• $p_{max} = \frac{R_T}{4R_T^2} V_T^2 = \frac{V_T^2}{4R_T}$