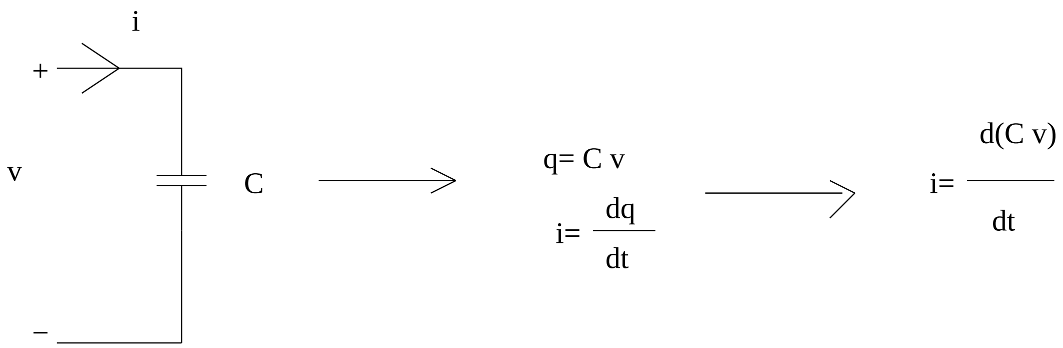
Circuit Theory

Chapter 6 : Capacitors and Inductors

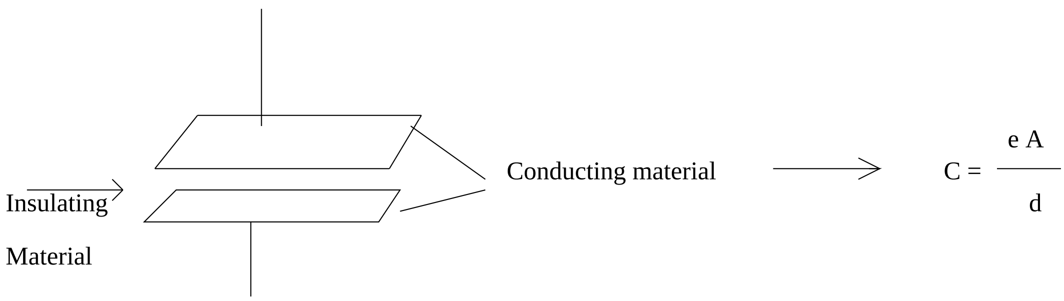
• Linear capacitor



• If C is time-invariant (i.e. a **constant**) $\Rightarrow$ $i = C\frac{dv}{dt}$

The unit of C : Capacitance $\Rightarrow$ [C]= $\frac{As}{V} = \frac{s}{\Omega}$= F (Farad).

• Typical example : parallel conducting plates separated by insulating material



• Here A is the area, d is the distance between plates, and ϵ is the dielectric constant of the insulating material.

- $i = C\frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{dt} = \frac{i}{C} \quad \Rightarrow \quad \int_{t_0}^{t_1} \dot{v}(\tau)d\tau = \frac{1}{C} \int_{t_0}^{t_1} i(\tau)d\tau \quad \Rightarrow \quad \dot{()} = \frac{d}{dt}$

- $v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^{t_1} i(\tau)d\tau$

- **Result 1 :** If $i(\cdot)$ is bounded on an interval, then $v(\cdot)$ is continuous...

- $v(t+\Delta t) = v(t) + \frac{1}{C} \int_{t}^{t+\Delta t} i(\tau)d\tau \quad \Rightarrow \quad v(t+\Delta t) - v(t) = \frac{1}{C} \int_{t}^{t+\Delta t} i(\tau)d\tau$

- $\mid i(\tau) \mid < M \quad \Rightarrow \quad \mid v(t+\Delta t) - v(t) \mid \leq \frac{1}{C} \int_{t}^{t+\Delta t} M d\tau = \frac{M}{C} \Delta t$

- $\lim_{\Delta t \to 0} v(t+\Delta t) = v(t) \quad \Rightarrow \quad v(\cdot)$ is **continuous**.

- **Result 2 :** If $i(t) = 0$ on $[t_0, t_1] \quad \Rightarrow \quad v(t) = v(t_0) =$ **constant** on $[t_0, t_1]$.

- **Result 3 :** If $v(t) = v(t_0) =$ **constant** on $[t_0, t_1] \quad \Rightarrow \quad i(t) = 0$ on $[t_0, t_1]$ (**open circuit**)

- **Power and Energy**

- Here we assume that C is constant.

- $p(t) = v(t)i(t) = v(t)C\frac{dv}{dt} \quad \Rightarrow \quad p(t) = \frac{d}{dt} \left(\frac{1}{2} Cv^2(t) \right)$

- Since $p(t) = \frac{d}{dt}$ (Energy) $\quad \Rightarrow \quad E(t) = \frac{1}{2} Cv^2(t)$

- Analogous with kinetic energy of a particle mass m with velocity v.

- $W(t_0, t_1) =$ Total power received by the element in the interval $[t_0, t_1]$

- $W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt$

- For a resistor : $W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt = R \int_{t_0}^{t_1} i^2(t)dt > 0$

- For a capacitor : $W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt = \frac{1}{2}\ Cv^2(t_1) - \frac{1}{2}\ Cv^2(t_0) = E(t_1) - E(t_0)$

- Hence, if $v(t_1) = v(t_0) \quad \Rightarrow \quad W(t_0, t_1) = 0$

- Assume that the capacitor operates under the periodic regime

- $\Rightarrow \quad i(t+T) = i(t)\ \forall t \quad \Rightarrow \quad v(t+T) = v(t)\ \forall t$

- $\Rightarrow \quad W(t_0, t_0 + T) = 0$!

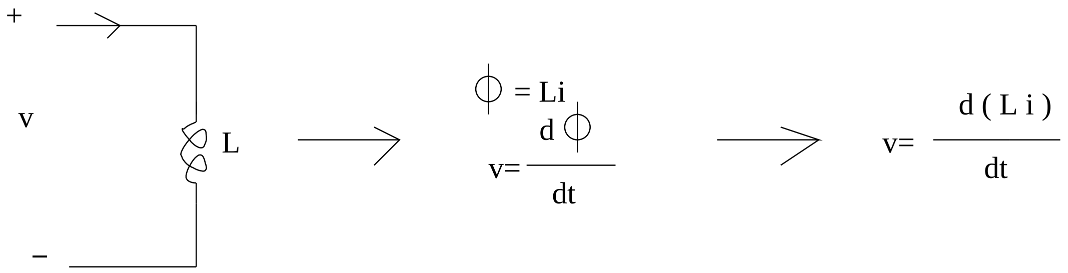
- Under the periodic regime, within periods, capacitor does not receive total power

- $\Rightarrow$ **Lossless device**

- Half of the period receives energy, the next half it gives it back.

- As a result, there is always a **phase difference** between v and i.

- Linear Inductor



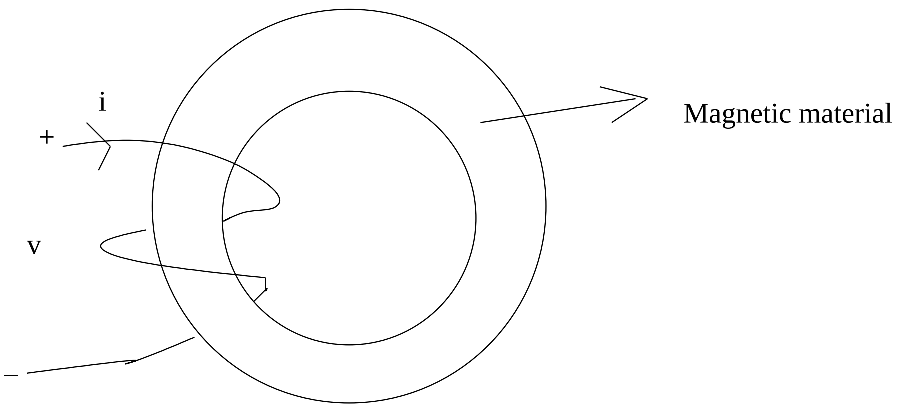
- If L is time-invariant (i.e. a **constant**) $\Rightarrow$ $v = L\frac{di}{dt}$

The unit of L : Inductance $\Rightarrow$ [L]= $\frac{Vs}{A} = \Omega s =$ H (Henry).

- Typical example : Toroidal Inductance

- Here, $L = \mu \, \frac{N^2 A}{l}$

- μ : Magnetic permeability, A : Cross-sectional area
- N : Number of turns, l : mid circumference



- $v = L\frac{di}{dt} \quad \Rightarrow \quad \frac{di}{dt} = \frac{v}{L} \quad \Rightarrow \quad \int_{t_0}^{t_1} \dot{i}(\tau)d\tau = \frac{1}{L}\int_{t_0}^{t_1} v(\tau)d\tau \quad \Rightarrow \quad \dot{()} = \frac{d}{dt}$

- $i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t_1} v(\tau)d\tau$

- **Result 1 :** If $v(\cdot)$ is bounded on an interval, then $i(\cdot)$ is continuous...

- $i(t+\Delta t) = i(t) + \frac{1}{L}\int_{t}^{t+\Delta t} v(\tau)d\tau \quad \Rightarrow \quad i(t+\Delta t) - i(t) = \frac{1}{L}\int_{t}^{t+\Delta t} v(\tau)d\tau$

- $\mid v(\tau) \mid < M \quad \Rightarrow \quad \mid i(t+\Delta t) - i(t) \mid \leq \frac{1}{L}\int_{t}^{t+\Delta t} M d\tau = \frac{M}{L}\Delta t$

- $\lim_{\Delta t \to 0} i(t+\Delta t) = i(t) \quad \Rightarrow \quad i(\cdot)$ is **continuous**.

- **Result 2 :** If $v(t) = 0$ on $[t_0, t_1] \quad \Rightarrow \quad i(t) = i(t_0) =$ **constant** on $[t_0, t_1]$.

- **Result 3 :** If $i(t) = i(t_0) =$ **constant** on $[t_0, t_1] \quad \Rightarrow \quad v(t) = 0$ on $[t_0, t_1]$ (**short circuit**)

- **Power and Energy**

- Here we assume that L is constant.

- $p(t) = v(t)i(t) = i(t)L\frac{di}{dt} \quad \Rightarrow \quad p(t) = \frac{d}{dt}\left(\frac{1}{2}Li^2(t)\right)$

- Since $p(t) = \frac{d}{dt}$ (Energy) $\quad \Rightarrow \quad E(t) = \frac{1}{2}Li^2(t)$

- Analogous with kinetic energy of a particle mass m with velocity v.

- $W(t_0, t_1) =$ Total power received by the element in the interval $[t_0, t_1]$

- $W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt$

- For an inductor : $W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt = \frac{1}{2}\ Li^2(t_1) - \frac{1}{2}\ Li^2(t_0) = E(t_1) - E(t_0)$

- Hence, if $i(t_1) = i(t_0) \quad \Rightarrow \quad W(t_0, t_1) = 0$

- Assume that the inductor operates under the periodic regime

- $\Rightarrow \quad v(t+T) = v(t)\ \forall t \quad \Rightarrow \quad i(t+T) = i(t)\ \forall t$

- $\Rightarrow \quad W(t_0, t_0 + T) = 0$!

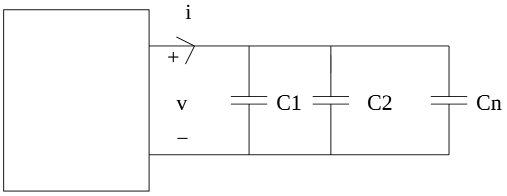
- Under the periodic regime, within periods, inductor does not receive total power

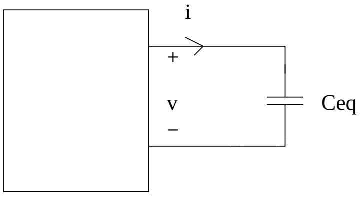
- $\Rightarrow$ **Lossless device**

- Half of the period receives energy, the next half it gives it back.

- As a result, there is always a **phase difference** between v and i.

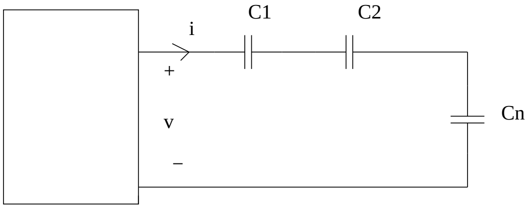
- **Parallel Connection of Capacitors**

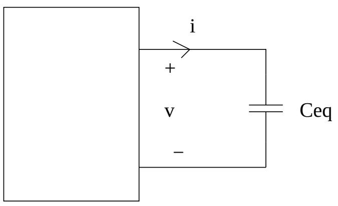




- KVL : $v_1 = v_2 = \ldots = v_n = v$,   KCL : $i = i_1 + i_2 + \ldots + i_n$
- $i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \ldots + C_n \frac{dv}{dt} = (C_1 + C_2 + \ldots + C_n) \frac{dv}{dt}$
- $\Rightarrow C_{eq} = C_1 + C_2 + \ldots + C_n$

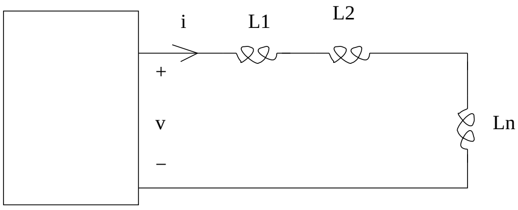
- **Series Connection of Capacitors**

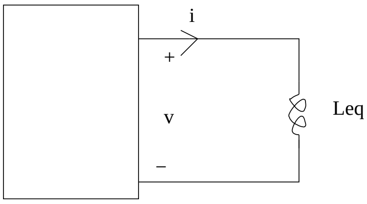




- KCL : $i_1 = i_2 = \ldots = i_n = i$,   KVL : $v = v_1 + v_2 + \ldots + v_n$
- $v(t) = v_1(t_0) + \frac{1}{C_1} \int_{t_0}^{t_1} i(\tau)d\tau + v_2(t_0) + \frac{1}{C_2} \int_{t_0}^{t_1} i(\tau)d\tau + \ldots + v_n(t_0) + \frac{1}{C_n} \int_{t_0}^{t_1} i(\tau)d\tau = (v_1(t_0) + v_2(t_0) + \ldots + v_n(t_0)) + (\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}) \int_{t_0}^{t_1} i(\tau)d\tau$
- $v(t) = v(t_0) + (\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}) \int_{t_0}^{t_1} i(\tau)d\tau$
- $\frac{1}{C_{eq}} = (\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n})$

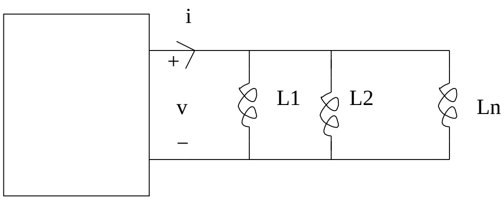
- **Series Connection of Inductors**

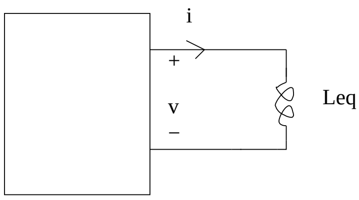




- KCL : $i_1 = i_2 = \ldots = i_n = i,$ KVL : $v = v_1 + v_2 + \ldots + v_n$
- $v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_n \frac{di}{dt} = (L_1 + L_2 + \ldots + L_n) \frac{di}{dt}$
- $\Rightarrow L_{eq} = L_1 + L_2 + \ldots + L_n$

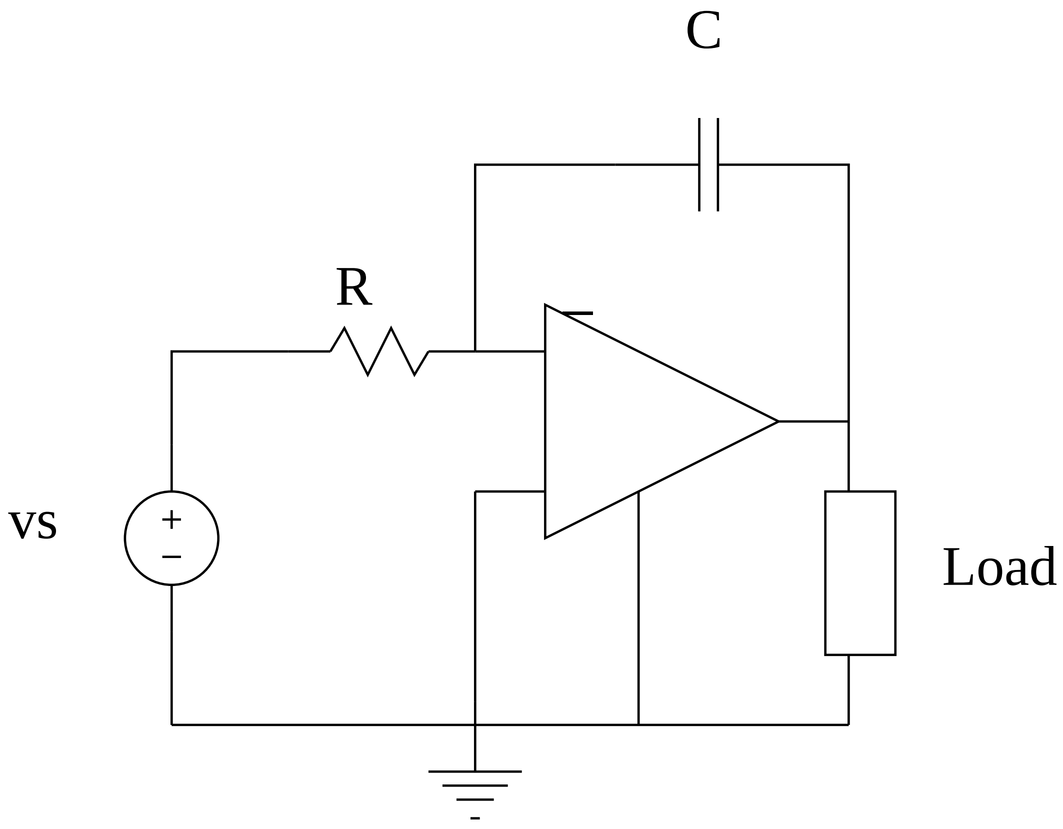
- **Parallel Connection of Inductors**





- KVL : $v_1 = v_2 = \ldots = v_n = v,$ KCL : $i = i_1 + i_2 + \ldots + i_n$
- $i(t) = i_1(t_0) + \frac{1}{L_1} \int_{t_0}^{t_1} v(\tau)d\tau + i_2(t_0) + \frac{1}{L_2} \int_{t_0}^{t_1} v(\tau)d\tau + \ldots + i_n(t_0) + \frac{1}{L_n} \int_{t_0}^{t_1} v(\tau)d\tau = (i_1(t_0) + i_2(t_0) + \ldots + i_n(t_0)) + (\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n}) \int_{t_0}^{t_1} v(\tau)d\tau$
- $i(t) = i(t_0) + (\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n}) \int_{t_0}^{t_1} v(\tau)d\tau$
- $\frac{1}{L_{eq}} = (\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n})$

- **Op-amp Integrator**



- $v_- = v_+ = 0 \rightarrow v_s = Ri_R$, $v_C = -v_0$

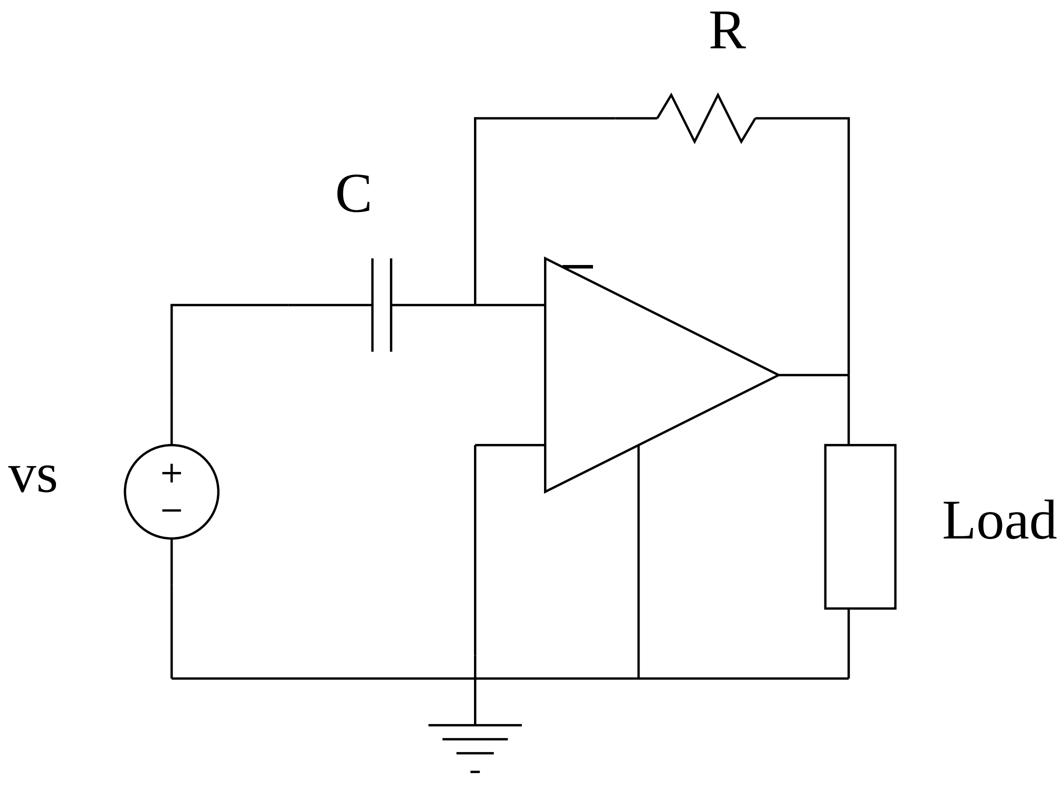
- $i_- = 0 \rightarrow i_R = i_C \rightarrow \frac{v_s}{R} = -C\ \frac{dv_0}{dt}$

- $\frac{dv_0}{dt} = -\frac{v_s}{RC}$

- $v_0(t) = v_0(t_0) - \frac{1}{RC} \int_{t_0}^{t} v_s(\tau)d\tau$

- Output is the integral of the input.

- **Op-amp Differentiator**



- $v_{-} = v_{+} = 0 \longrightarrow i_C = C\frac{dv_s}{dt}$, $v_R = -v_0$

- $i_C = i_R = -\ \frac{v_0}{R}$

- $v_0 = -RC\ \frac{dv_s}{dt}$

- Output is the differential of the input.