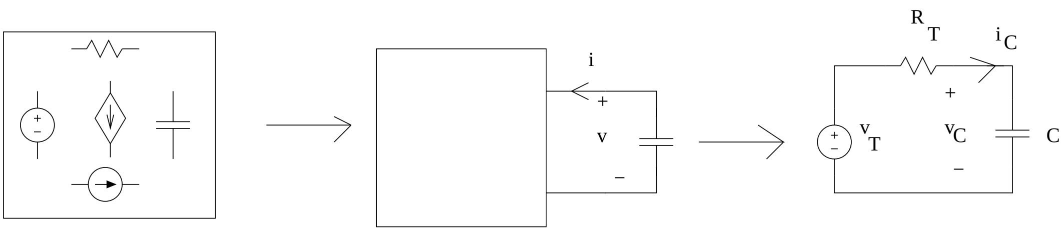
Circuit Theory

Chapter 7 : First and Second Order Circuits

• Here, order refers to the number of capacitors and inductors

• **First Order Circuits :** The circuits which contain only one capacitor or only one inductor.

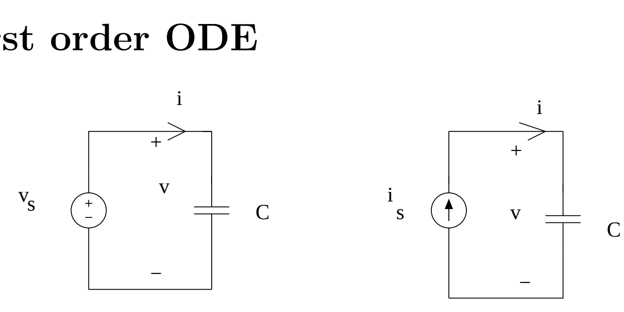
• **First Order RC Circuits :** Such circuits contain independent and dependent sources, resistors + one capacitor.



• KVL : $v_T = R_T i_C + v_C \qquad i_C = C\frac{dv_C}{dt} \Rightarrow v_T = R_T C\frac{dv_C}{dt} + v_C$

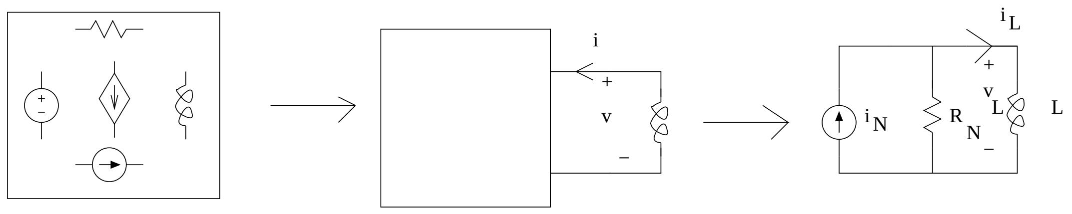
• $$\frac{dv_C}{dt} + \frac{1}{R_T C} v_C = \frac{1}{R_T C} V_T$$

• Result is a **first order ODE**



• $v_C = v_s,\ i_C = C\frac{dv_s}{dt} \qquad i_C = i_s,\ v_C = v_C(t_0) + \frac{1}{C}\int_{t_0}^{t} i_s(\tau)d\tau$

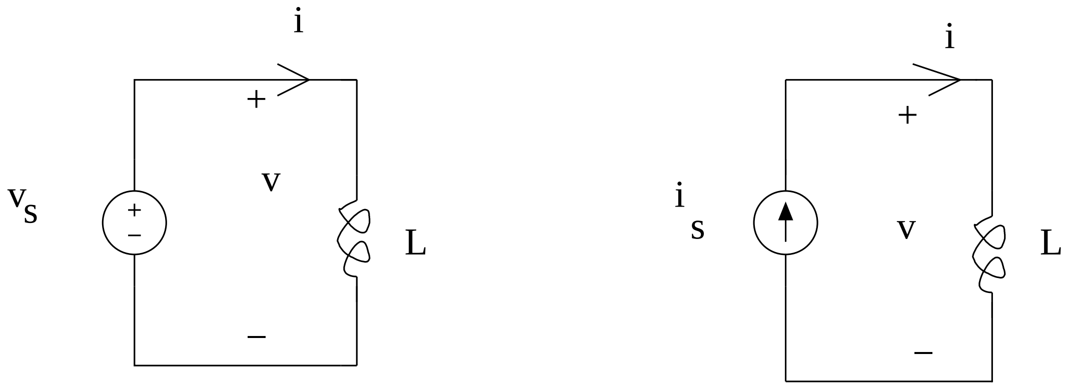
- **First Order RL Circuits :** Such circuits contain independent and dependent sources, resistors + one inductor.



- KCL : $i_N = G_N v_L + i_L \qquad v_L = L\frac{di_L}{dt} \Rightarrow i_N = G_N L\frac{di_L}{dt} + i_L$

- $$\frac{di_L}{dt} + \frac{1}{G_N L} i_L = \frac{1}{G_N L} i_N \qquad G_N = \frac{1}{R_N}$$

- Result is a **first order ODE**



- $v_L = v_s,\ i_L = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} v_s(\tau)d\tau \qquad i_L = i_s,\ v_L = L\frac{di_s}{dt}$

- **Step Response of First Order RC / RL Circuits :**

- Here we have DC sources, i.e. V_T and i_N are **constants**

- $\frac{dv_C}{dt}+\frac{1}{R_TC}v_C = \frac{1}{R_TC}V_T$ $\qquad$ $\frac{di_L}{dt}+\frac{1}{G_NL}i_L = \frac{1}{G_NL}i_N$

- These two equations can be combined into a single equation :

- $$\frac{dx}{dt}+\frac{1}{\tau}x = \frac{1}{\tau}x_\infty$$

- For RC circuits, $x = v_C$, $\tau = R_TC$, $x_\infty = V_T$

- For RL circuits, $x = i_L$, $\tau = G_NL = \frac{L}{R_N}$, $x_\infty = i_N$

- τ : Time Constant.
- Unit : $[\tau] = \Omega\frac{sec}{\Omega}$ (for RC case) $=\frac{\Omega sec}{\Omega}$ (for RL case) $= sec$

- Solution of the ODE : Since x_∞ is constant, $\frac{dx_\infty}{dt} = 0$

- $$\frac{d(x-x_\infty)}{dt}+\frac{1}{\tau}(x-x_\infty) = 0$$

- Define a new variable $y(t) = x(t) - x_\infty$ $\quad\Rightarrow\quad$ $\frac{dy}{dt}+\frac{1}{\tau}y = 0$

- Solution is : $\quad y(t) = y(t_0)e^{-\frac{t-t_0}{\tau}}$

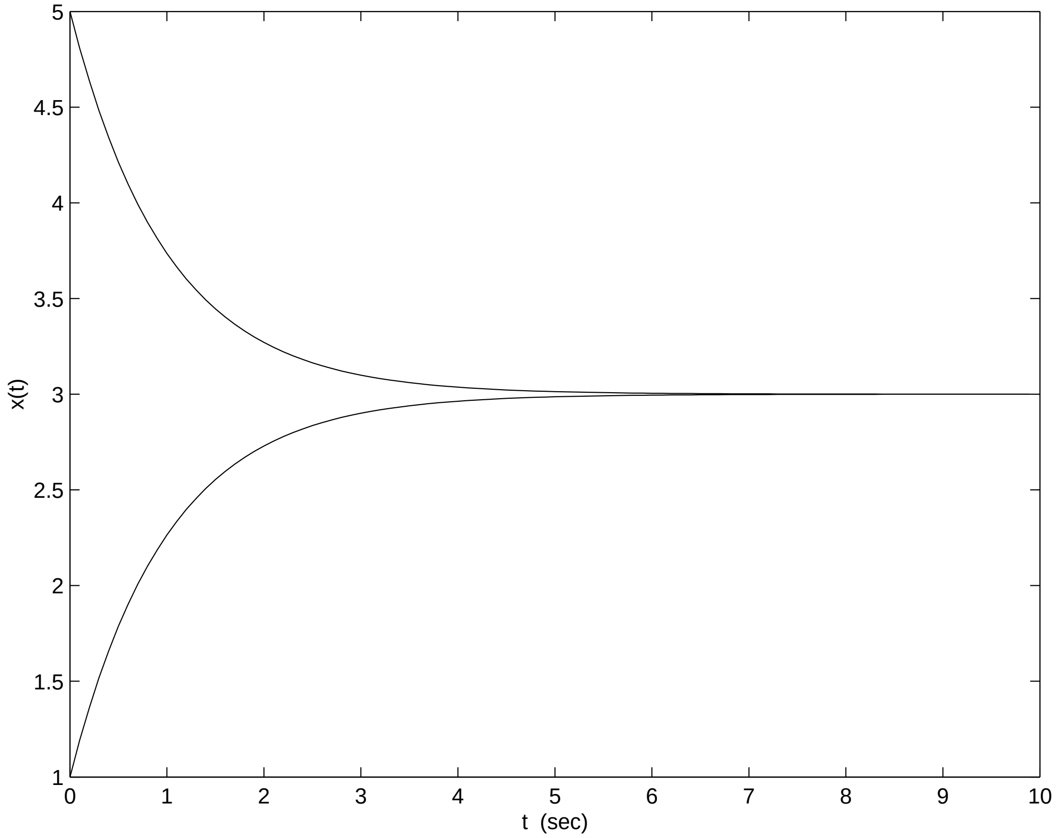
- $$x(t) - x_\infty = (x(t_0) - x_\infty)e^{-\frac{t-t_0}{\tau}}$$

- For RC Case : $\quad v_C(t) - V_T = (v_C(t_0) - V_T)e^{-\frac{t-t_0}{R_TC}}$

- For RL Case : $\quad i_L(t) - i_N = (i_L(t_0) - i_N)e^{-\frac{t-t_0}{G_NL}}$

- Why x_∞ ? assume that $\tau > 0$ and take limit as $t \to \infty$

- $\lim_{t\to\infty} x(t) = x_\infty + \lim_{t\to\infty}(x(t_0) - x_\infty)e^{-\frac{t-t_0}{\tau}} = x_\infty$



- Question : Practically, how long should we wait till we safely assume that $x(t) \approx x_\infty$. This is called **the steady state**

- Practically, we have $x(t) \approx x_\infty$, for $t - t_0 > 5\tau$

- $$x(t) - x_\infty \approx (x(t_0) - x_\infty)e^{-5} \approx 0.006(x(t_0) - x_\infty)$$

- In the above figure, we have $\tau = 1$ *sec*.

• For RC Case : $\lim_{t\to\infty} v_C(t) = V_T \quad \Rightarrow \quad \lim_{t\to\infty} i_C(t) = 0$

• In the steady state, capacitor behaves like an **open circuit**.

• For RL Case : $\lim_{t\to\infty} i_L(t) = i_N \quad \Rightarrow \quad \lim_{t\to\infty} v_L(t) = 0$

• In the steady state, inductor behaves like a **short circuit**.

• **Example 1 :** Example 7.8. p. 300.

• Since the switch is open for a long time, we may assume that inductor reaches its steady state → **short circuit**!

• By shorting the inductor → $i_L(0) = \frac{V_A}{R_1+R_2}$

• This shows a way to set up the initial condition for inductors.

• For $t > 0$, the switch is closed. If we wait long enough, we may assume that inductor reaches its steady state → **short circuit**!

• $i_L(\infty) = x_\infty = i_N = \frac{V_A}{R_1}$

• For $t > 0$, the switch is closed → $\tau = G_1 L = \frac{L}{R_1}$

• $i_L(t) = \frac{V_A}{R_1} + (\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1})e^{-\frac{t-t_0}{G_1 L}} = \frac{V_A}{R_1} + (\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1})e^{-\frac{R_1(t-t_0)}{L}}$

• $v_L(t) = L\ \frac{di_L(t)}{dt} = -\ \frac{1}{G_1}(\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1})e^{-\frac{t-t_0}{G_1 L}} \to 0$

- **Example 2 :** Example 7.9. p. 301.

- Since the switch is closed for a long time, we may assume that capacitor reaches its steady state $\rightarrow$ **open circuit**!

- By opening the capacitor $\rightarrow v_C(0) = \frac{V_A R_1}{R_1 + R_2}$

- For $t > 0$, the switch is opened. If we wait long enough, we may assume that capacitor reaches its steady state $\rightarrow$ **open circuit**!

- $v_C(\infty) = x_\infty = V_T = V_A$

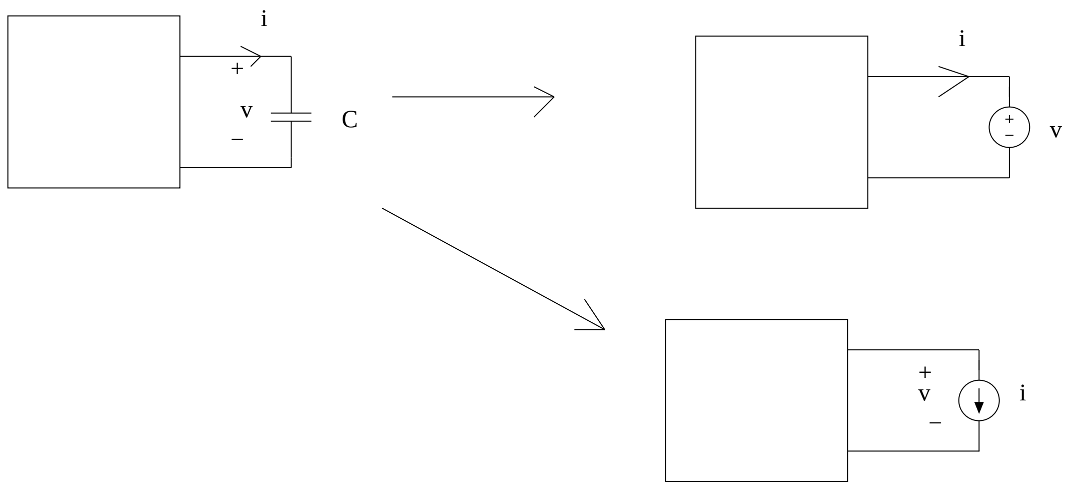
- For $t > 0$, the switch is opened $\rightarrow \tau = R_2 C$

- $v_C(t) = V_A + (\frac{V_A R_1}{R_1 + R_2} - V_A) e^{-\frac{t - t_0}{R_2 C}} \rightarrow V_A$

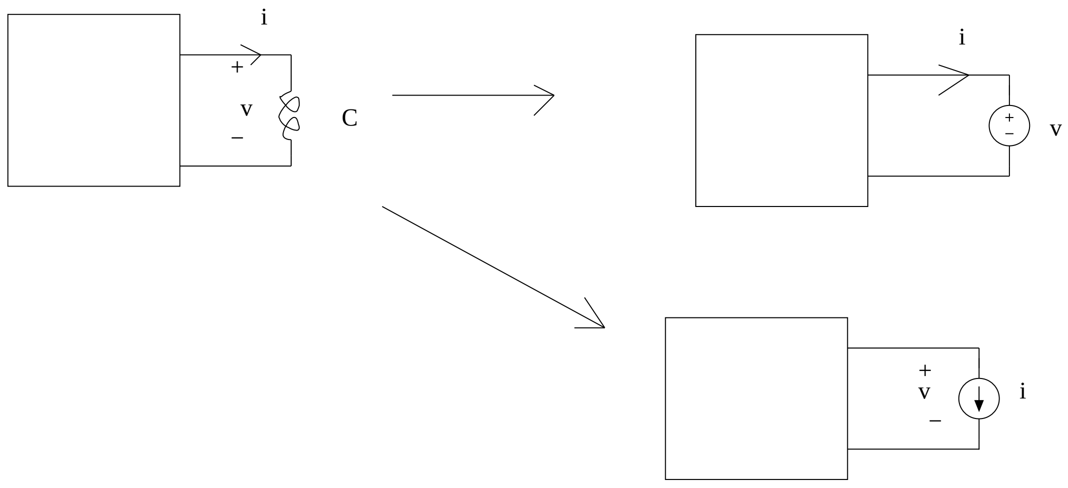
- $i_C(t) = C \frac{dv_C(t)}{dt} = -\frac{1}{R_2} (\frac{V_A R_1}{R_1 + R_2} - V_A) e^{-\frac{t - t_0}{R_2 C}} \rightarrow 0$

- After finding $v_C(t)$, $i_C(t)$, $v_L(t)$, $i_L(t)$, how can we find the remaining voltages or currents ?

- By **substitution**. i.e. Replace the capacitor/inductor by a voltage and/or current source with the found solution.

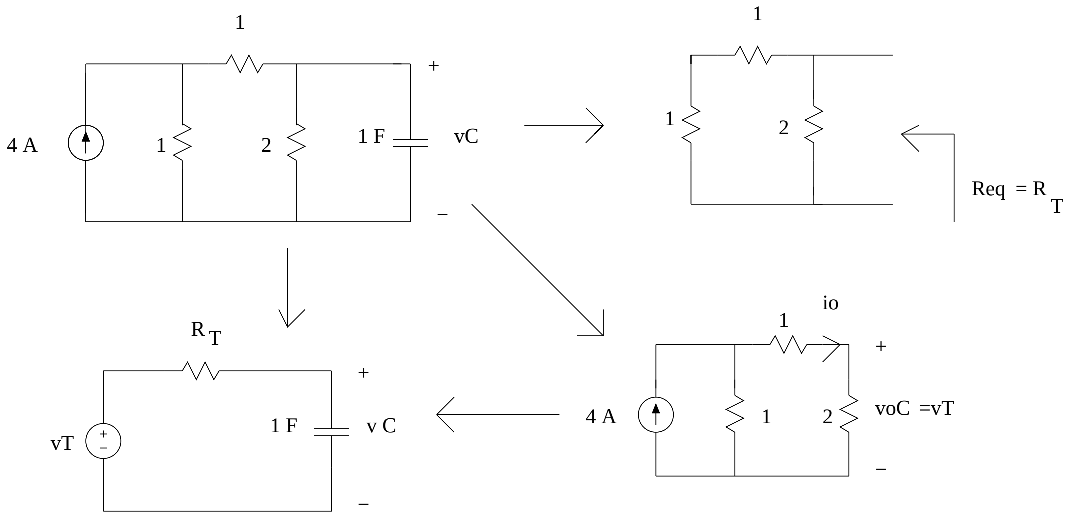


- Here $v(t) = v_C(t)$, $i(t) = i_C(t)$, which are already found.

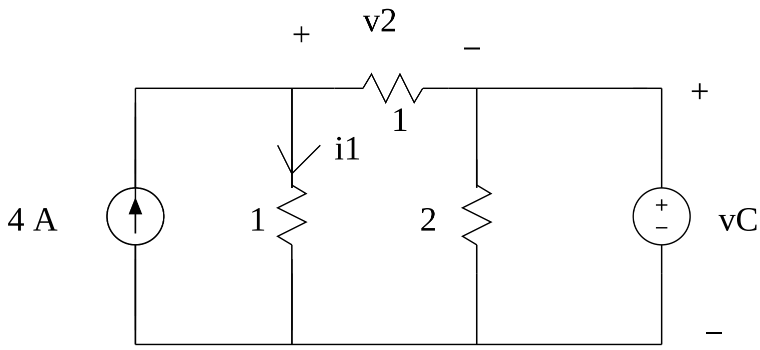


- Here $v(t) = v_L(t)$, $i(t) = i_L(t)$, which are already found.

- **Example :**

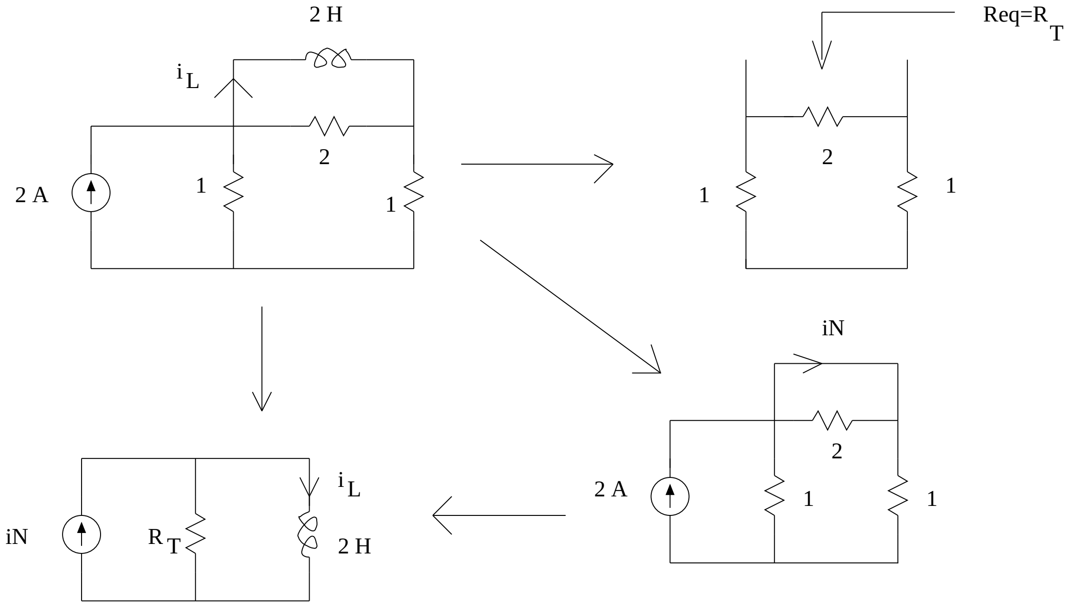


- $R_T = 1\ \Omega$, By current division $\Rightarrow i_0 = \frac{1}{4}4 = 1\ A \Rightarrow v_{oc} = v_T = 2\ V$
- $\tau = R_T C = 1\ sec$, $v_C(\infty) = v_T = 2\ V$. Assume that $v_C(0) = 1\ V$.
- $v_C(t) - v_C(\infty) = (v_C(0) - v_C(\infty))e^{\frac{t-t_0}{\tau}}$
- $\Rightarrow v_C(t) = 2 - e^{-t} \Rightarrow i_C(t) = C\frac{dv_C(t)}{dt} = e^{-t}$
- Suppose that we want to find out, say, i_1 and v_2. $\Rightarrow$ Use substitution.



- Here, $v_C = 2 - e^{-t}$. Simple node analysis yields :
- $i_1(t) = 3 - 0.5e^{-t}$, $v_2(t) = 1 + 0.5e^{-t}$.

- **Example :**



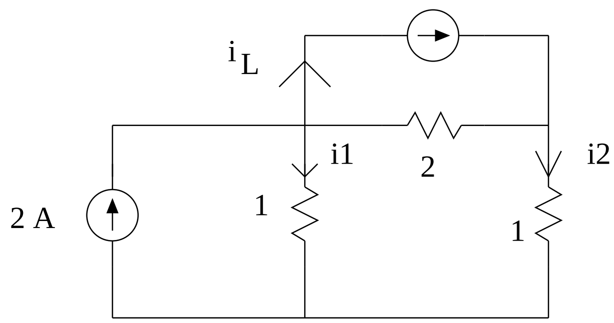
- $R_T = 1\ \Omega$, By KCL $\Rightarrow 2 = v_1 + i_N,\ i_N = v_1 \Rightarrow i_N = 1\ A.$

- $\tau = G_T L = 2\ sec$, $i_L(\infty) = i_N = 1\ A.$ Assume that $i_L(0) = 3\ V$.

- $i_L(t) - i_L(\infty) = (i_L(0) - i_L(\infty))e^{\frac{t-t_0}{\tau}}$

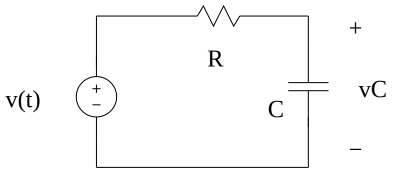
- $\Rightarrow i_L(t) = 1 + e^{-\frac{t}{2}} \Rightarrow v_L(t) = L\frac{di_L(t)}{dt} = -2e^{-\frac{t}{2}}$

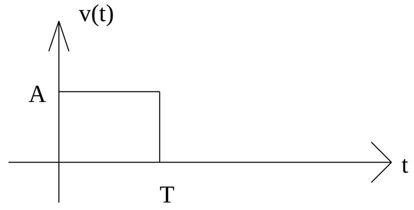
- Suppose that we want to find out, say, i_1 and i_2. $\Rightarrow$ Use substitution.



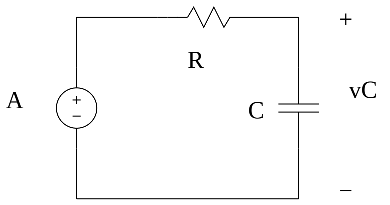
- Here, $i_L = 1 + e^{-\frac{t}{2}}$. Simple node analysis yields :
- $i_1(t) = 1 - e^{-\frac{t}{2}}$, $i_2(t) = 1 + e^{-\frac{t}{2}}$.

• **Response to a Pulse :** This will depend on the relation between τ and T :

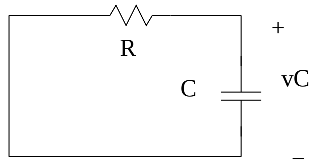




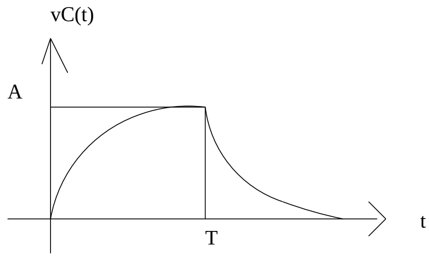
$0 < t < T$



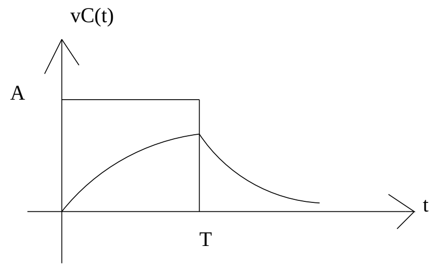
$t > T$



• If $T \gg \tau$, $v_C(T) \simeq v_C(\infty) = A$. Otherwise, $v_C(T) \neq v_C(\infty) = A$



Time Constant < < T



Time Constant comparable with T

• **Example :** $R = 1\ k\Omega$, $C = 1\ \mu F$, $A = 10\ V$, $T = 10\ ms$, $v_C(0) = 0\ V$.

• $\Rightarrow \tau = RC = 1\ ms << T$

• $0 < t < 10 \Rightarrow v_C(t) = 10 - 10e^{-t} \Rightarrow v_C(10) \simeq 10\ V$

• $t > 10 \Rightarrow v_C(t) = 10e^{-(t-10)}$

• **Example :** $R = 10\ k\Omega$, $C = 1\ \mu F$, $A = 10\ V$, $T = 10\ ms$, $v_C(0) = 0\ V$.

• $\Rightarrow \tau = RC = 10\ ms = T$

• $0 < t < 10 \Rightarrow v_C(t) = 10 - 10e^{-0.1t} \Rightarrow v_C(10) = 10 - 10e^{-1} = 6.32$

• $t > 10 \Rightarrow v_C(t) = 6.32e^{-0.1(t-10)}$

• **Sinusoidal Response of First Order RC / RL Circuits :**

• Here, the source term in the Thévenin/Norton equivalent circuit is sinusoidal.

• $$\frac{dx}{dt}+\frac{1}{\tau}x = \frac{1}{\tau}x_\infty \qquad x_\infty = V_A \cos\omega t$$

• $x(t) = x_N(t) + x_F(t) =$ Natural Response + Forced Response

• This is the same as SUPERPOSITION. Natural response is due to initial condition $x(0)$ and is called ZERO INPUT RESPONSE. This is the solution of ODE when the source term is set to ZERO.

• Forced response is due to source term and is called ZERO STATE RESPONSE. This is the solution of ODE when $x(0) = 0$.

• $$\frac{dx_N}{dt}+\frac{1}{\tau}x_N = 0 \qquad x_N(t) = Ke^{-\frac{t}{\tau}}$$

• $$\frac{dx_F}{dt}+\frac{1}{\tau}x_F = \frac{1}{\tau}V_A \cos\omega t$$

• $x_F(t) = V_F \cos(\omega t + \phi) = a \cos\omega t + b \sin\omega t$

• $a = V_F \cos\phi \quad , \quad b = -V_F \sin\phi$

• $\dot{x}_F(t) = -a\omega \sin\omega t + b\omega \cos\omega t$

• $(b\omega+\frac{1}{\tau}\ a) \cos\omega t + (-a\omega+\frac{1}{\tau}\ b) \sin\omega t = \frac{1}{\tau}V_A \cos\omega t$

• $(b\omega+\frac{1}{\tau}\ a) = \frac{1}{\tau}V_A \qquad (-a\omega+\frac{1}{\tau}\ b) = 0$

• Given $\tau,\ V_A$ and ω Find a and $b \Rightarrow V_F$ and ϕ

- **Example 7-12**, p. 307.

- Note that since the switch is open for a long time $\rightarrow v(0) = 0$.

- For Thévenin Equivalent circuit : $R_T = 4\ k\Omega \parallel 4\ k\Omega = 2\ k\Omega$

- $v_T = 0.5 v_s(t) = 10 \sin 1000t \rightarrow \tau = 2\ .10^3 \times 10^{-6} = 2\ .10^{-3}$ sec.

- $x_N(t) = Ke^{-\frac{t}{\tau}} = Ke^{-500t}$

- $x_F(t) = V_F \cos(\omega t + \phi) = a \cos \omega t + b \sin \omega t \qquad \omega = 1000$ Hz.

- $(b\omega + \frac{1}{\tau}\ a) = 0 \qquad (-a\omega + \frac{1}{\tau}\ b) = \frac{1}{\tau} V_A$

- $1000b + 500a = 0 \qquad -1000a + 500b = 5000 \longrightarrow a = -4\ ,\ b = 2$

- $v(t) = Ke^{-500t} - 4 \cos 1000t + 2 \sin 1000t$

- $v(0) = 0 = K - 4 \longrightarrow K = 4$

- $a = V_F \cos \phi = -4 \ \ , \ \ b = -V_F \sin \phi = 2$

- $V_F = \sqrt{a^2 + b^2} = 4.47, \qquad \tan \phi = -2/-4 \longrightarrow \phi = -153$ deg.

- $v(t) = 4e^{-500t} + 4.47 \cos(1000t - 153)$

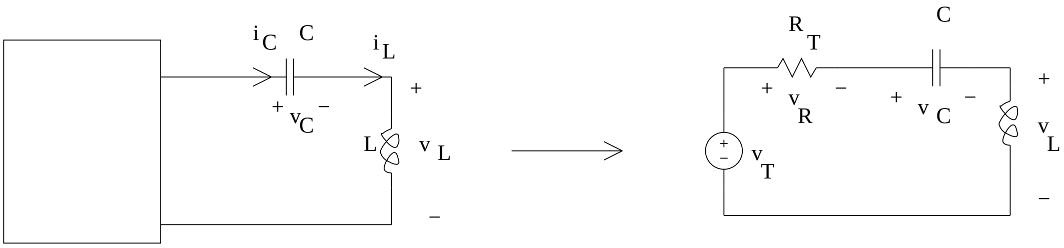
- Note that as $t \rightarrow \infty, \longrightarrow v(t) \rightarrow 4.47 \cos(1000t - 153)$

- This is called **Sinusoidal Steady State**. We will find it directly in Chp. 8 in a simple way → Phasor Analysis.

• **Second Order Circuits :** The circuits which contain total of two capacitor- inductor combination. → 2 capacitors, 2 inductors, or 1 capacitor+ 1 inductor.

• Before considering the most general case, we will first consider two important cases : series and parallel RLC circuits.

• **Series RLC circuits**



• KVL : $v_T = v_R + v_C + v_L$ KCL : $i_R = i_C = i_L$

• From the first order case, we know that v_C and i_L are **state variables** (i.e. variables of ODE)

• $i_C = C\frac{dv_C}{dt} = i_L \longrightarrow \quad \frac{dv_C}{dt} = i_L/C$ (1)

• $v_T = R_T i_R + v_C + v_L = Ri_L + v_C + L\frac{di_L}{dt}$

• $\frac{di_L}{dt} = -\frac{1}{L}v_C - \frac{R_T}{L}i_L + \frac{1}{L}v_T$ (2)

• (1) and (2) are called **state equations**. They are first order coupled ODE's. They can be written in matrix form as :

- $\frac{d}{dt}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 & 1/C \\ -1/L & -R_T/L \end{pmatrix}\begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} V_T$ (*)

- General form : $\longrightarrow$ $\dot{x} = Ax + bu$

- x :(vector) state variable, u : input, A : matrix, b : vector.

- Alternative form : Scalar second order equation. Differentiate (1), use (1) and (2) to eliminate i_L :

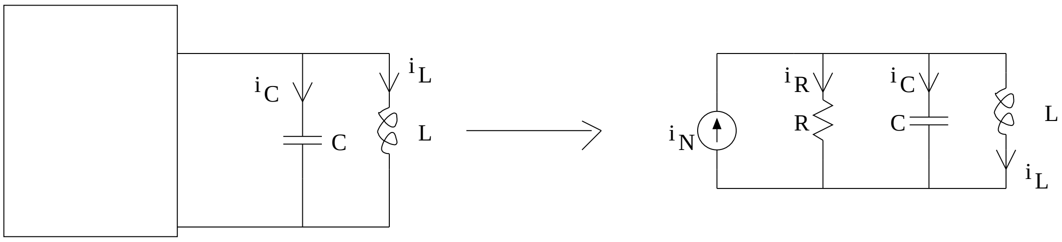
- $\ddot{v}_C + \frac{R_T}{L}\dot{v}_C + \frac{1}{LC}v_C = \frac{1}{LC}v_T$ (**)

- So we have to either solve (*) or (**). Since they are equivalent, they give the same result. Usually we are given $v_C(0)$ and $i_L(0)$.

- To solve (**), we need $\dot{v}_C(0)$. This comes from (1) : $\dot{v}_C(0) = i_L(0)/C$

- Before finding the solution of (**), let us consider the other case :

- **Parallel RLC circuits**



- KCL : $i_N = i_R + i_C + i_L,$ KVL : $v_R = v_C = v_L,$

- From the first order case, we know that v_C and i_L are **state variables** (i.e. variables of ODE)

- $v_L = L\frac{di_L}{dt} = v_C \longrightarrow \quad \dot{i}_L = v_C/L \qquad (1)$

- $i_N = Gv_R + C\dot{v}_C + i_L = Gv_C + C\dot{v}_C + i_L$

- $\dot{v}_C = -G/Cv_C - 1/Ci_L + 1/Ci_N \qquad (2)$

- (1) and (2) are called **state equations** and can be written also as :

- $\frac{d}{dt}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} -G/C & -1/C \\ 1/L & 0 \end{pmatrix}\begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} 1/C \\ 0 \end{pmatrix} i_N \qquad (*)$

- Alternative form : Scalar second order equation. Differentiate (1), use (1) and (2) to eliminate v_C :

• $\ddot{i}_L + \frac{G}{C}\,\dot{i}_L + \frac{1}{LC} i_L = \frac{1}{LC} i_N \qquad (**)$

• So we have to either solve (*) or (**). Since they are equivalent, they give the same result. Usually we are given $v_C(0)$ and $i_L(0)$.

• To solve (**), we need $\dot{i}_L(0)$. This comes from (1) : $\quad \dot{i}_L(0) = v_C(0)/L$

• Solution = zero input response + zero state response.

• **Zero Input Response** ($v_T = 0$, $i_N = 0$)

•
$$\ddot{x} + a\dot{x} + bx = 0$$

• Series RLC : $x = v_C$, $a = \frac{R_T}{L}$, $b = \frac{1}{LC}$

• Parallel RLC : $x = i_L$, $a = \frac{G}{C}$, $b = \frac{1}{LC}$

• $x(0)$ and $\dot{x}(0)$ are given.

• Assume solution of the form $x(t) = e^{st} \Rightarrow \dot{x}(t) = se^{st} \Rightarrow \ddot{x}(t) = s^2 e^{st}$

• $\Rightarrow (s^2 + as + b)e^{st} = 0 \longrightarrow (s^2 + as + b) = 0$

• This is called the **characteristic polynomial** of the circuit, here s could be real of complex.

- $(s^2 + as + b) = 0 \longrightarrow$ roots are s_1 and s_2

- Roots can be found as : $s_{1,2} = \pm\ \frac{-a \pm \sqrt{a^2-4b}}{2}$

- **Case 1 :** Distinct Roots, $s_1 \neq s_2$

- $x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

- Here K_1 and K_2 are constants (real or complex) to be found from the initial conditions.

- $x(0) = K_1 + K_2, \qquad \dot{x}(0) = s_1 K_1 + s_2 K_2$

- $K_1 = \frac{s_2 x(0) - \dot{x}(0)}{s_2 - s_1} \qquad K_2 = \frac{s_1 x(0) - \dot{x}(0)}{s_1 - s_2}$

- These formulas are valid even if s_1 and s_2 are complex.

- What if the roots are complex ?

- $s = s_1 = \bar{s}_2 = -\alpha + \jmath\beta \Rightarrow K_1 = \bar{K}_2 = r/2 e^{-\jmath\theta}$

- $x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} = K_1 e^{st} + \bar{K}_1 e^{\bar{s}t} = 2\Re(\ K_1 e^{st}\ )$

- $x(t) = 2\Re((\ r/2) e^{-\alpha t + \jmath(\beta t - \theta)}\ ) = r e^{-\alpha t} \cos(\beta t - \theta)$

- Need to find r and θ by using initial conditions.

- $x(t) = re^{-\alpha t}\cos(\beta t - \theta)$

- $x(0) = r\cos\theta \longrightarrow \dot{x}(0) = -\alpha r\cos\theta + \beta r\sin\theta$

- $r\cos\theta = x(0) \longrightarrow r\sin\theta = \frac{\dot{x}(0)+\alpha x(0)}{\beta}$

- Alternatively we could find the solution as :

- $x(t) = C_1 e^{-\alpha t}\cos\beta t + C_2 e^{-\alpha t}\sin\beta t$

- $x(0) = C_1 \quad , \quad \dot{x}(0) = -\alpha C_1 + \beta C_2 \longrightarrow C_2 = \frac{\dot{x}(0)+\alpha x(0)}{\beta}$

- **Case 2 :** Repeated Roots, $s_1 = s_2 = s$

- $x(t) = K_1 e^{st} + K_2 t e^{st}$

- $x(0) = K_1 \quad , \quad \dot{x}(0) = sK_1 + K_2 \longrightarrow K_2 = \dot{x}(0) - sx(0)$

• **Stability of solutions**

• $s_1 \neq s_2 \longrightarrow x_N(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

• $s_1 = s_2 = s \longrightarrow x_N(t) = K_1 e^{st} + K_2 t e^{st}$

• If $\Re(s_i) < 0 \longrightarrow x_N(t) \to 0$ as $t \to \infty \longrightarrow$ **Stable Case**.

• In this case, if the independent sources are bounded $\longrightarrow$ solutions are bounded.

• If $\Re(s_i) > 0$, or $s_1 = s_2 \geq 0 \longrightarrow x_N(t) \to \infty$ as $t \to \infty \longrightarrow$ **Unstable Case**.

• In this case, solutions are unbounded.

• For practical reasons, we want **stable systems**.

• From geometrical point of view, the roots of characteristic polynomial are in the open left half of the complex plane in the stable case.

• We will do further classification for the stable case, as explained in the next example.

• **Example 7-15**, p. 318.

• Consider a series RLC circuit with no independent sources. (Hence we find zero input response, i.e. only the natural response). Let $C = 0.25\ \mu F$, $L = 1H$, $v_C(0) = 15\ V$, $i_L(0) = 0\ A$. Find the solution for : (a) $R = 8.5\ k\Omega$, (b) $R = 4\ k\Omega$, (c) : $R = 1\ k\Omega$.

- $\ddot{v}_C + \frac{R_T}{L}\ \dot{v}_C + \frac{1}{LC} v_C = \frac{1}{LC} v_T = 0$ (**)

- Characteristic polynomial :
- $\Rightarrow (s^2 + as + b) = 0 \longrightarrow a = \frac{R_T}{L},\ b = \frac{1}{LC}$

- Roots can be found as : $s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

- **Case 1 :** $a = \frac{R_T}{L} = 8500,\ b = \frac{1}{LC} = 4.10^6 \Rightarrow s_1 = -500,\ s_2 = -8000.$

- Roots are negative $\Rightarrow$ Stable case.

- $v_C(t) = K_1 e^{-500t} + K_2 e^{-8000t}$

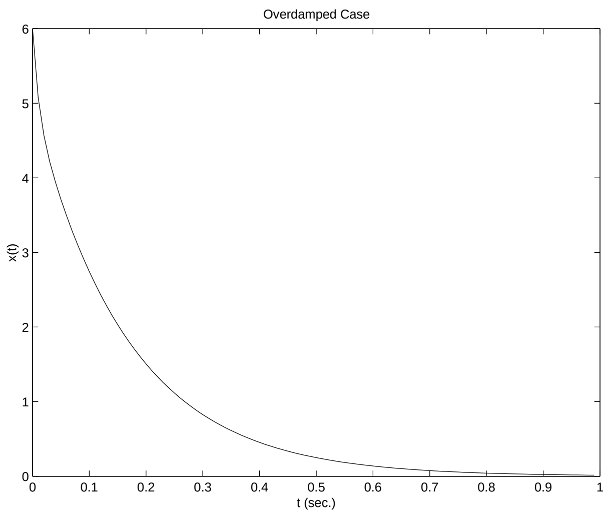
- $v_C(0) = K_1 + K_2 = 15.$

- $\dot{v}_C(0) = i_L(0)/C = 0 \Rightarrow -500K_1 - 8000K_2 = 0$

- $\Rightarrow K_1 = 16 \ ,\quad K_2 = -1 \qquad \Rightarrow \qquad v_C(t) = 16e^{-500t} - e^{-8000t} \quad V$

- C$\frac{dv_C}{dt} = i_L = -2.10^{-3} e^{-500t} + 2.10^{-3} e^{-8000t} \quad A$

- This is called **Overdamped Case** (i.e. two distinct real-negative roots)



- **Case 2 :** $a = \frac{R_T}{L} = 4000,\ b = \frac{1}{LC} = 4.10^6 \Rightarrow s_1 = s_2 = -2000.$

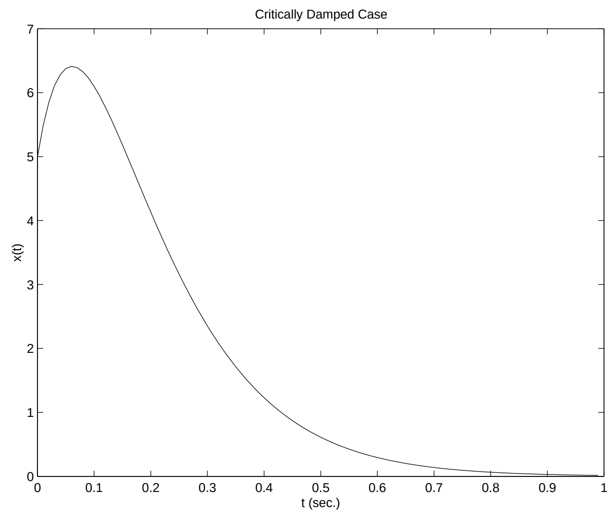
- $v_C(t) = K_1 e^{-2000t} + K_2 t e^{-2000t}$

- $v_C(0) = K_1 = 15,$
- $\dot{v}_C(0) = i_L(0)/C = 0 \Rightarrow -2000K_1 + K_2 = 0 \Rightarrow K_2 = 30000$

- $v_C(t) = (15 + 30000t)e^{-2000t} \quad V$

- $C\frac{dv_C}{dt} = i_L = 15te^{-2000t} \quad A$

• This is called **Critically Damped Case** (i.e. two repeated real-negative roots)



- **Case 3 :** $a = \frac{R_T}{L} = 1000,\ b = \frac{1}{LC} = 4.10^6 \Rightarrow s_{1,2} = -500 \pm \jmath 500\sqrt{15}.$

- $\alpha = 500 \quad , \quad \beta = 500\sqrt{15}$

- $v_C(t) = Ke^{-500t}\cos(500\sqrt{15}t + \phi)$
- $\quad\quad = K_1 e^{-500t}\cos(500\sqrt{15}t) + K_2 e^{-500t}\sin(500\sqrt{15}t)$

- $v_C(0) = 15 = K_1$
- $\dot{v}_C(0) = i_L(0)/C = 0 \Rightarrow -500K_1 + 500\sqrt{15}K_2 = 0 \Rightarrow K_2 = \sqrt{15}$

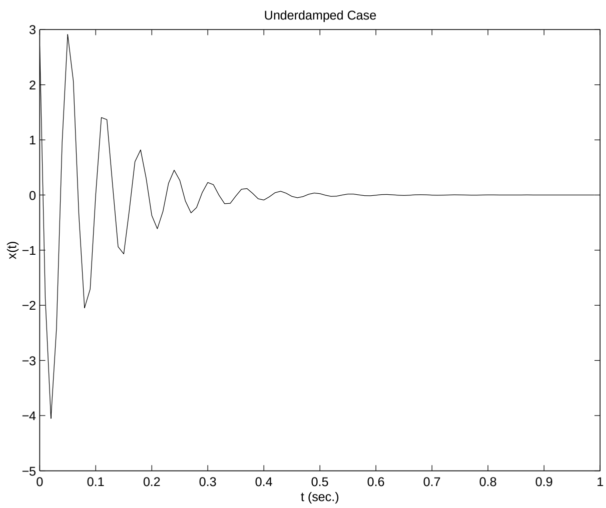
- $v_C(t) = 15e^{-500t}\cos(500\sqrt{15}t) + \sqrt{15}e^{-500t}\sin(500\sqrt{15}t)$

- $K\cos\phi = K_1 = 15 \quad , \quad K\sin\phi = -K_2 = -\sqrt{15}$

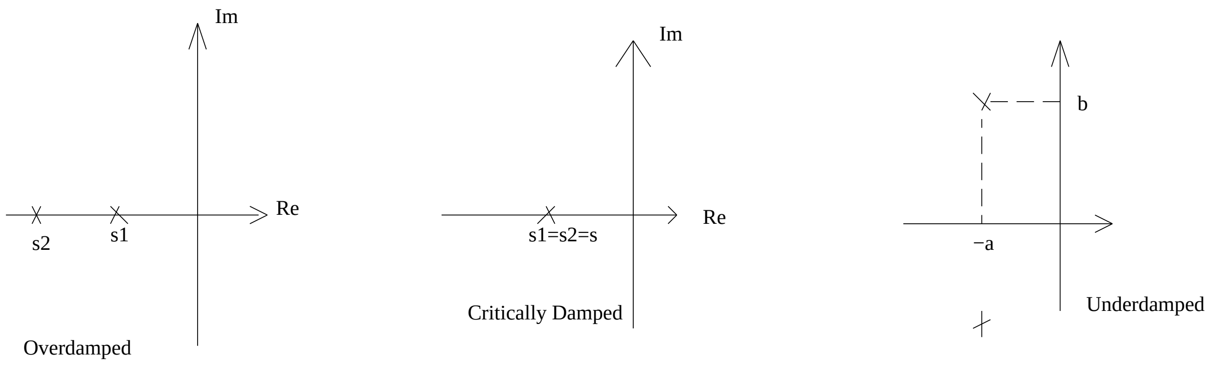
- $K = \sqrt{15^2 + 15} = 15.5 \quad , \quad \tan\phi = -\sqrt{15} \Rightarrow \phi = -75.52$ deg.$= -1.32$ rad.

- $v_C(t) = 15.5e^{-500t}\cos(500\sqrt{15}t - 1.32)$ (True notation)
- $\quad\quad = 15.5e^{-500t}\cos(500\sqrt{15}t - 75.52)$ (False but acceptable)

- This is called **Underdamped Case** (i.e. two complex conjugate roots with negative real part)



- Here $s = -\alpha + \jmath\beta$; α determines the decay, and β determines the frequency.



- **Step Response**

- In this case, the standard ODE becomes :

- $\ddot{x}(t) + a\dot{x}(t) + bx(t) = A$

- $x(t) = x_N(t) + x_F$

- $x_N(t)$ is the natural response and can be found as before.

- $x_F = B \longrightarrow \ddot{x}_F = \dot{x}_F = 0 \Rightarrow x_F = A/b$

- **Example 7-19, p. 328**. Series RLC circuit with $V_T = 10\ V$, $C = 0.5\ \mu F$, $L = 2\ H$, $R = 1\ k\Omega$, $v_C(0) = 0$, $i_L(0) = 0$.

- $\ddot{v}_C + \frac{R_T}{L}\ \dot{v}_C + \frac{1}{LC} v_C = \frac{1}{LC} v_T \qquad (**)$

- $\ddot{v}_C(t) + 500\dot{v}_C(t) + 10^6 v_C(t) = 10^7$

- For x_N, the characteristic polynomial is : $s^2 + 500s + 10^6 = 0$

- Roots : $s_{1,2} = -250 \pm \jmath 968 \longrightarrow$ Underdamped case.

- $v_{CN}(t) = K_1 e^{-250t} \cos 968t + K_2 e^{-250t} \sin 968t = K e^{-250t} \cos(968t + \phi)$

- $V_{CF} = 10^7/10^6 = 10$

- $v_C(t) = 10 + K_1 e^{-250t} \cos 968t + K_2 e^{-250t} \sin 968t$

- $v_C(0) = 10 + K_1 = 0 \longrightarrow K_1 = -10$
- $\dot{v}_C(0) = i_L(0)/C = 0 \Rightarrow -250K_1 + 968K_2 = 0 \longrightarrow K_2 = -2.58$

- $v_C(t) = 10 - 10e^{-250t} \cos 968t - 2.58e^{-250t} \sin 968t$

- $K \cos \phi = K_1 = -10 \quad , \quad K \sin \phi = -K_2 = 2.58$

- $K = \sqrt{100 + 2.58^2} = 10.12$, $\phi = 104.46$ deg.

- The characteristic polynomial can also be written as :

- $s^2 + 2\xi\omega_0 s + \omega_0^2 = 0$
- ξ : damping ratio, ω_0 : undamped natural frequency.

- $s_{1,2} = \omega_0(-\xi \pm \sqrt{\xi^2 - 1})$

- $\xi > 1 \longrightarrow$ Overdamped Case.
- $\xi = 1 \longrightarrow$ Critically damped Case.
- $0 < \xi < 1 \longrightarrow$ Underdamped case.
- $\xi = 0 \longrightarrow$ Lossless Case. In this case, the natural response is a **pure sinusoid**.

- **General Second Order Circuits**

- Note that we have v_C and i_L as variables, and by element relations we have $i_C = C\dot{v}_C$ and $v_L = L\dot{i}_L$.

- Hence the general strategy is as follows :

- **Step 1 :** Use any method we have seen (node, mesh, combined constraints etc.) to obtain i_C and v_L in terms of v_C, i_L and independent sources only. (This is a process of writing equations, and eliminating undesired variables $\Rightarrow$ Linear Algebra)

- At the end of step 1 , we obtain the following equations :

- $i_C = c_{11}v_C + c_{12}i_L + d_1u_1$
- $v_L = c_{21}v_C + c_{22}i_L + d_2u_2$

- Here, coefficients c_{ij}, d_i depend on circuit parameters, u_i depend on independent sources.

- **Step 2 :** Use $i_C = C\dot{v}_C$ and $v_L = L\dot{i}_L$:

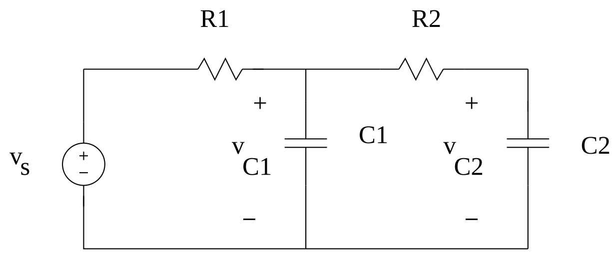
- $C\dot{v}_C = c_{11}v_C + c_{12}i_L + d_1u_1$
- $L\dot{i}_L = c_{21}v_C + c_{22}i_L + d_2u_2$

- These equations can be written either in component form :

- $\dot{v}_C = a_{11}v_C + a_{12}i_L + b_1u_1 \quad (*)$
- $\dot{i}_L = a_{21}v_C + a_{22}i_L + b_2u_2 \quad (**)$

- Or in matrix form :

- $\frac{d}{dt}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} b_1 u_1 \\ b_2 u_2 \end{pmatrix}$ (*)
- General form : $\longrightarrow$ $\dot{x} = Ax + bu$
- x :(vector) state variable, u : input, A : matrix, b : vector or matrix.
- Alternative form : Scalar second order equation. Differentiate $(*)$ or $(**)$, use these equations to eliminate i_L or v_C :
- **Case 1 :** $a_{12} \neq 0$
- $\ddot{v}_C = a_{11}\dot{v}_C + a_{12}\dot{i}_L + b_1\dot{u}_1 = a_{11}\dot{v}_C + a_{12}\ (a_{21}v_C + a_{22}i_L + b_2u_2) + b_1\dot{u}_1$
- $= a_{11}\dot{v}_C + a_{12}a_{21}v_C + a_{22}a_{12}i_L + a_{12}b_2u_2 + b_1\dot{u}_1$
- $= a_{11}\dot{v}_C + a_{12}a_{21}v_C + a_{22}\ (\dot{v}_C - a_{11}v_C - b_1u_1) + a_{12}b_2u_2 + b_1\dot{u}_1$
- $= (a_{11} + a_{22})\dot{v}_C + (a_{12}a_{21} - a_{11}a_{22})v_C - a_{22}b_1u_1 + a_{12}b_2u_2 + b_1\dot{u}_1$
- $\ddot{v}_C - (a_{11} + a_{22})\dot{v}_C + (a_{11}a_{22} - a_{12}a_{21})v_C = -a_{22}b_1u_1 + a_{12}b_2u_2 + b_1\dot{u}_1$
- **Case 2 :** $a_{21} \neq 0$. Change v_C with i_L and indexes $1 \longleftrightarrow 2$, we obtain :
- $\ddot{i}_L - (a_{22} + a_{11})\dot{i}_L + (a_{22}a_{11} - a_{21}a_{12})i_L = -a_{11}b_2u_2 + a_{21}b_1u_1 + b_2\dot{u}_2$
- Let us see the relation with A :
- $(a_{11} + a_{22}) = \text{Trace}(A) = T$
- $(a_{11}a_{22} - a_{12}a_{21}) = \det A = D$
- $$\ddot{x} - T\dot{x} + Dx = u_s$$
- v_C Case : $u_s = -a_{22}b_1u_1 + a_{12}b_2u_2 + b_1\dot{u}_1$
- i_L Case : $u_s = -a_{11}b_2u_2 + a_{21}b_1u_1 + b_2\dot{u}_2$



• KCL at C_1 : $i_{C1} + G_1(v_{C1} - v_s) + G_2(v_{C1} - v_{C2}) = 0$

• $\Rightarrow C_1\dot{v}_{C1} = -(G_1 + G_2)v_{C1} + G_2v_{C2} + G_1v_s$

• KCL at C_2 : $i_{C2} + G_2(v_{C2} - v_{C1}) = 0$

• $\Rightarrow C_2\dot{v}_{C2} = G_2v_{C1} - G_2v_{C2}$

• $\Rightarrow a_{11} = -(G_1 + G_2)/C_1,\ a_{12} = G_2/C_1,\ a_{21} = G_2/C_2,\ a_{22} = -G_2/C_2,$ $b_1 = G_1/C_1,\ u_1 = v_s,\ b_2 = 0,\ u_2 = 0$

• $(a_{11} + a_{22}) = \text{Trace}(A) = T = -((G_1 + G_2)/C_1 + G_2/C_2)$

• $(a_{11}a_{22} - a_{12}a_{21}) = \det A = D = G_1G_2/C_1C_2$

• By using Case 2 : $u_s = -a_{11}b_2u_2 + a_{21}b_1u_1 + b_2\dot{u}_2 = (G_1G_2/C_1C_2)v_s$

• $\ddot{v}_{C2} + ((G_1 + G_2)/C_1 + G_2/C_2)\dot{v}_{C2} + (G_1G_2/C_1C_2)v_{C2} = (G_1G_2/C_1C_2)v_s$

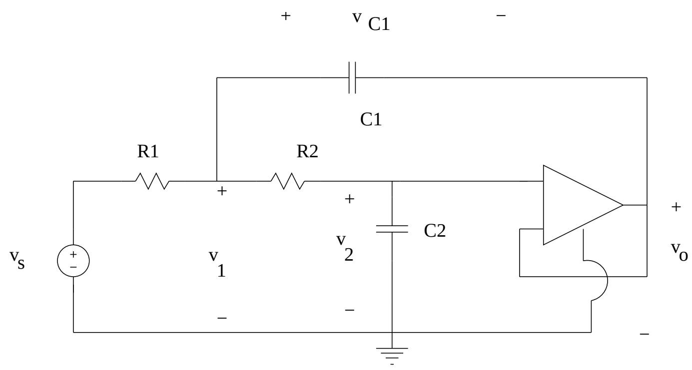
• Given initial conditions $v_{C1}(0)$ and $v_{C2}(0)$, we can calculate $\dot{v}_{C2}(0)$ from the second state equation given above :

• $\Rightarrow C_2\dot{v}_{C2}(0) = G_2v_{C1}(0) - G_2v_{C2}(0)$

• Given the parameters G_i, R_i and the source term v_s, we can solve this ODE to find v_{C2}.

• Then we can find $\dot{v}_{C2}$ by differentiation. By using the second equation, we can find v_{C1} ...etc...

- **Example on p. 334**, Fig. 7-14



- KCL for Node 1 : $i_{C1} + G_1(v_1 - v_s) + G_2(v_1 - v_2) = 0$
- KCL for Node 2 : $i_{C2} + G_2(v_2 - v_1) = 0$
- Op-amp eqn : $v_2 = v_+ = v_- = v_o \longrightarrow v_{C1} = v_1 - v_o = v_1 - v_2$, $v_{C2} = v_2$

- $C_1 \dot{v_{C1}} = C_1 \dot{v}_1 - C_1 \dot{v}_2 = -(G_1 + G_2)v_1 + G_2 v_2 + G_2 v_s$
- $C_2 \dot{v_{C2}} = G_2 v_1 - G_2 v_2$

- $\dot{v}_2 = (G_2/C_2)v_1 - (G_2/C_2)v_2$
- $C_1 \dot{v}_1 = C_1(G_2/C_2)v_1 - C_1(G_2/C_2)v_2 - (G_1 + G_2)v_1 + G_2 v_2 + G_2 v_s$
- $\dot{v}_1 = (G_2/C_2 - ((G_1 + G_2)C_1)\ )v_1 + (\ (G_2/C_1) - (G_2/C_2))v_2 + (G_2/C_1)v_s$

- applying Case 2, we obtain :

- $\ddot{v}_2 + ((G_1 + G_2)/C_1)\dot{v}_2 + (G_1 G_2/C_1 C_2)v_2 = (G_1 G_2/C_1 C_2)v_s$

• As before, given the parameters, and the initial conditions, we can solve this ODE. Then, we can find v_1. Then we can find all of the remaining voltages and currents ...etc...