CHP 9 LAPLACE TRANSFORMS

Definition:
$$\boxed{F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt} = \boxed{\mathcal{L}\{f(t)\}}$$

- existence. $f(\cdot)$ is piecewise continuous, $|f(t)| < Ke^{\alpha t}$ for some $t \geq 0$. If integral converges for some s, we say that $F(s)$ exists.
- Lower limit 0^-, to capture ~~discontinuity~~ impulse at $t=0$ at $t=0$, in case $f(\cdot)$ is not continuous at $t=0$. If $f(\cdot)$ is continuous, it is not important.

unit step.

$\rightarrow$ $u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

$F(s) = \frac{1}{s}$. $\qquad F(s) = \int_{0^-}^{\infty} e^{-st} dt = -\frac{e^{-st}}{s}\Big|_{0^-}^{\infty} = \frac{1}{s}$ for $Re\{s\} > 0$.

$\rightarrow$ $f(t) = e^{-\alpha t} \Rightarrow F(s) = \int_{0^-}^{\infty} e^{-st} \cdot e^{-\alpha t} dt = -\frac{e^{-(s+\alpha)t}}{s+\alpha}\Big|_{0^-}^{\infty} = \frac{1}{s+\alpha}$

ALL TIME FUNCTIONS ARE DEFINED FOR $t \geq 0$!

BASIC PROPERTIES:

1) Uniqueness:

$x(t) \Leftrightarrow X(s) \qquad y(t) \Leftrightarrow Y(s) \qquad X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$

$x(t) = y(t)$ iff $X(s) = Y(s)$.

2) Linearity:

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \int_{0^-}^{\infty} [\alpha x(t) + \beta y(t)]\, e^{-st} dt = \alpha \int_{0^-}^{\infty} x(t) e^{-st} dt + \beta \int_{0^-}^{\infty} y(t) e^{-st} dt$$
$$= \alpha X(s) + \beta Y(s)$$

(α may be real or complex)

$$e^{j\omega t} \longleftrightarrow \frac{1}{s-j\omega} \quad , \quad e^{-j\omega t} \longleftrightarrow \frac{1}{s+j\omega}$$

$$\cos\omega t = \frac{e^{j\omega t}+e^{-j\omega t}}{2} \longleftrightarrow \frac{1}{2}\cdot\frac{1}{s-j\omega}+\frac{1}{2}\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2}$$

$$\sin\omega t = \frac{e^{j\omega t}-e^{-j\omega t}}{2j} \longleftrightarrow \frac{1}{2j}\cdot\frac{1}{s-j\omega}-\frac{1}{2j}\cdot\frac{1}{s+j\omega} = \frac{\omega}{s^2+\omega^2}.$$

i) <u>Differentiation Rule.</u>

$f(t) \longleftrightarrow F(s)$, what is $\mathcal{L}\left\{\frac{df}{dt}\right\} = ?$

$$\boxed{\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)}$$

$\Downarrow$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s\,\mathcal{L}\left\{\frac{df}{dt}\right\} - f'(0^-) = s\left(sF(s)-f(0^-)\right)-f'(0)$$

$$= s^2F(s) - sf(0^-) - f'(0)$$

$\Downarrow$

$$\boxed{\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)}$$

USEFUL IN SOLVING DIFFERENTIAL EQUATIONS

j) <u>Integration Rule.</u>

$f(t) \longleftrightarrow F(s)$ $\qquad x(t) = \int_0^t f(\tau)d\tau \qquad X(s) = ?$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s).$$

$$r(t) = t \quad t \geq 0 \quad = \int_0^t 1\,d\tau = \int_0^t u(\tau)\,d\tau$$

$$* \quad \mathcal{L}\{t\,u(t)\} = \frac{1}{s}\mathcal{L}\{u(t)\} = \frac{1}{s^2}.$$

$$* \quad t^2 = 2\int_0^t \tau\,d\tau \Rightarrow \mathcal{L}\{t^2u(t)\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\Rightarrow \quad \mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

s-domain translation:

$$\mathcal{L}\{f(t)\} = F(s) \iff \mathcal{L}\{e^{-\alpha t} f(t)\} = \underline{\underline{F(s+\alpha)}}$$

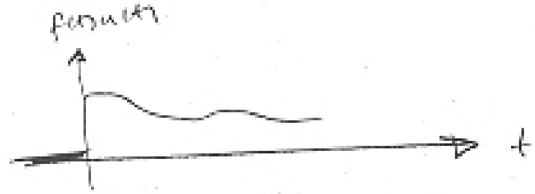
$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \iff \mathcal{L}\{t e^{-\alpha t}\} = \frac{1}{(s+\alpha)^2}.$$

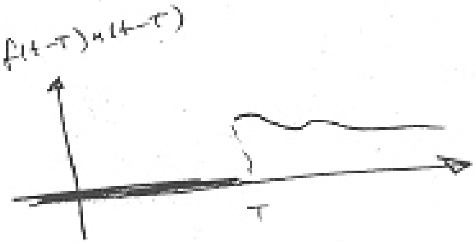
$$\mathcal{L}\{\cos\beta t\, u(t)\} = \frac{s}{s^2+\beta^2} \iff \mathcal{L}\{e^{-\alpha t}\cos\beta t\, u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

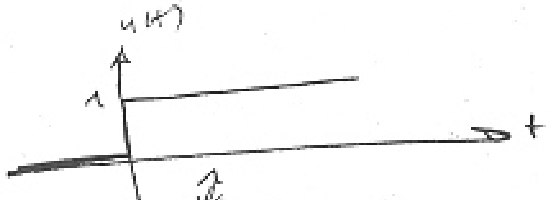
$$\mathcal{L}\{\sin\beta t\, u(t)\} = \frac{\beta}{s^2+\beta^2} \iff \mathcal{L}\{e^{-\alpha t}\sin\beta t\, u(t)\} = \frac{\beta}{(s+\alpha)^2+\beta^2}.$$

time-domain translation:

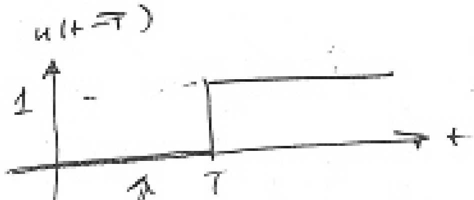
$$\mathcal{L}\{f(t)\underset{u(t)}{}\} = F(s) \iff \mathcal{L}\{f(t-T)\,u(t-T)\} = e^{-sT}F(s) \;!$$





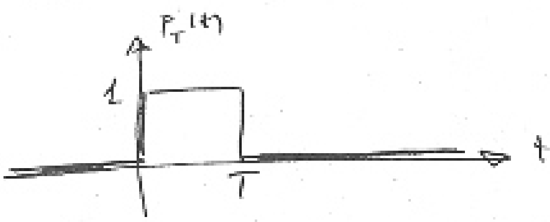


$F(s) = \frac{1}{s}$



$e^{-sT} \cdot \frac{1}{s}$

PULSE:



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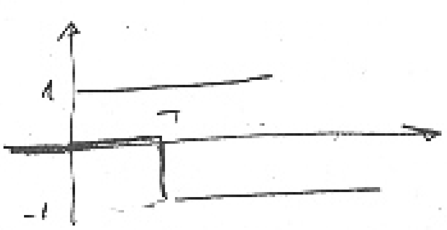


$P_T(t) = u(t) - u(t-T)$

$$\ldots e^{-sT} = \frac{1-e^{-sT}}{\ldots} \;!$$

Exercise p. 443

$f(t) = \left(e^{-2t} + 4t - 1\right) u(t)$.

$$F(s) = \frac{1}{s+2} + \frac{4}{s^2} - \frac{1}{s} = \frac{s^2+4s+8-\cancel{s^2}-2s}{s^2(s+2)} = \frac{2(s+4)}{s^2(s+2)}$$

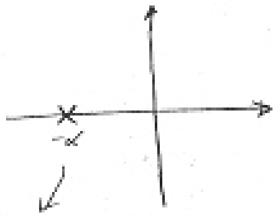
$s^2 \quad (s+2) \quad s(s+2)$

– POLES – ZEROES.

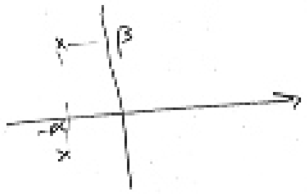
$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = K \cdot \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

zeroes

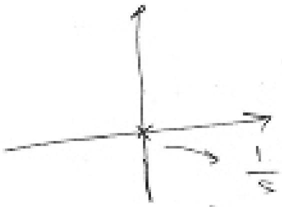
→ poles



$e^{-\alpha t} \longleftrightarrow \frac{1}{s+\alpha}$



$\frac{1}{(s+\alpha)^2+\beta^2} \leftarrow e^{-\alpha t}\cos\beta t$, $e^{-\alpha t}\sin\beta t$



Au(t) ?

– INVERSE TRANSFORMATION (PARTIAL FRACTION EXPANSION)

$$F(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

Assume proper rational function : $\deg(q) > \deg(p)$ $[n > m]$

• SIMPLE POLES : (no repeated poles)

$$F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \dots + \frac{k_n}{s-p_n}$$

$$f(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \dots + k_n e^{p_n t}$$

$k_1 = (s-p_1) F(s) \Big|_{s=p_1}$ -- $k_i = (s-p_i) F(s) \Big|_{s=p_i}$

$i = 1, 2, \dots, n$.

Example.

$$F(s) = 2\,\frac{s+3}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$k_1 = sF(s)\Big|_{s=0} = \cancel{s}\cdot 2\,\frac{s+3}{\cancel{s}(s+1)(s+2)}\Big|_{s=0} = \cancel{2}\cdot\frac{3}{1\cdot\cancel{2}} = 3$$

$$k_3 = (s+2)F(s)\Big|_{s=-2} = \cancel{(s+2)}\,2\cdot\frac{s+3}{s(s+1)\cancel{(s+2)}}\Big|_{s=-2} = 2\cdot\frac{1}{-2\cdot(-1)} = 1$$

$$k_2 = (s+1)\,F(s)\Big|_{s=-1} = \cancel{(s+1)}\,2\,\frac{(s+3)}{s\cancel{(s+1)}(s+2)}\Big|_{s=-1} = 2\cdot\frac{2}{(-1)\cdot(1)} = -4$$

$$\boxed{f(t) = \left(3 - 4e^{-t} + e^{-2t}\right)u(t)}$$

COMPLEX ROOTS.

$$= \cdots + \frac{k_1}{s-(-\alpha+j\beta)} + \frac{k_2}{s-(-\alpha+j\beta)} + \cdots \qquad k_2 = k_1^*$$

$$k_1 = |k|\,e^{j\theta} \qquad k_2 = |k|\,e^{-j\theta}$$

$$f(t) = \cdots\ |k|\,e^{j\theta}\,e^{(-\alpha+j\beta)t} + |k|\,e^{-j\theta}\,e^{(-\alpha-j\beta)t}$$

$$= \cdots + |k|e^{-\alpha t}\,e^{j(\beta t+\theta)} + |k|\,e^{-\alpha t}\,e^{-j(\beta t+\theta)} + \cdots$$

$$= \cdots + 2\,|k|e^{-\alpha t}\cos(\beta t+\theta) + \cdots$$

AMPLE: (p. 451)

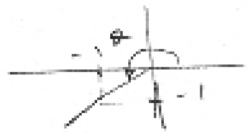
$$F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)} = \frac{k_1}{s+1} + \frac{k}{s-(-1+j2)} + \frac{k^*}{s-(-1-j2)}$$

$$s^2+2s+5=0 \Rightarrow \begin{matrix} s_1 = -1+j2 \\ s_2 = -1-j2 \end{matrix}$$

$$k_1 = (s+1)\,F(s)\Big|_{s=-1} = 20\,\frac{s+3}{s^2+2s+5}\Big|_{s=-1} = 20\cdot\frac{2}{1-2+5} = 10$$

$$k = (s-(-1+j2))F(s)\Big|_{s=-1+j2} = 20\,\frac{s+3}{(s+1)(s-(-1-j2))}\Big|_{s=-1+j2} = 20\cdot\frac{-1+j2+3}{j2\cdot j4}$$

$$= 20\cdot\frac{2(1+j)}{-8} = -5(1+j) = 5\sqrt{2}\,e^{j\frac{5\pi}{4}}$$



$$f(t) = \left[10e^{-t} + 10\sqrt{2}\cos\left(2t+\frac{5\pi}{4}\right)\right]u(t).$$

MULTIPLE POLES:

$$F(s) = \frac{n(s)}{(s-p_1)^m(\cdots)} = \frac{k_{11}}{(s-p_1)^m} + \frac{k_{12}}{(s-p_1)^{m-1}} + \cdots + \frac{k_{1m}}{s-p_1}$$

$$(s-p_1)^m F(s) = k_{11} + k_{12}(s-p_1) + k_{13}(s-p_1)^2 + \cdots + k_{1m}(s-p_1)^{m-1} + \cdots$$

$$k_{11} = (s-p_1)^m F(s)\Big|_{s=p_1}$$

$$k_{12} = \left\{\frac{d}{ds}(s-p_1)^m F(s)\right\}\Big|_{s=p_1} \quad \cdots \quad k_{1m} = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{ds^{m-1}}(s-p_1)^m F(s)\right\}\Big|_{s=p_1}.$$

$$k_{13} = \frac{1}{2}\left\{\frac{d^2}{ds^2}(s-p_1)^m F(s)\right\}\Big|_{s=p_1}$$

How to find the inverse transform?

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{t^n e^{pt}\} = \frac{n!}{(s-p)^{n+1}} \Rightarrow \mathcal{L}\{t^{n-1}e^{pt}\} = \frac{(n-1)!}{(s-p)^n}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{k}{(s-p)^n}\right\} = \frac{k}{(n-1)!}e^{pt}\cdot t^{n-1}}$$

$$f(t) = k_{1m}e^{p_1 t} + \cdots + \frac{k_{12}}{(m-2)!}e^{p_1 t}t^{m-2} + \frac{k_{11}}{(m-1)!}e^{p_1 t}t^{m-1} + \cdots$$

$$F(s) = \frac{4(s+3)}{s(s+2)^2} = \frac{k_2}{s} + \frac{k_{11}}{(s+2)^2} + \frac{k_{12}}{(s+2)}$$

$$k_2 = sF(s)\Big|_{s=0} = \frac{4(s+3)}{(s+2)^2}\Big|_{s=0} = \frac{4\cdot 3}{4} = 3$$

$$k_{11} = (s+2)^2 F(s)\Big|_{s=-2} = \frac{4(s+3)}{s}\Big|_{s=-2} = \frac{4\cdot 1}{-2} = -2$$

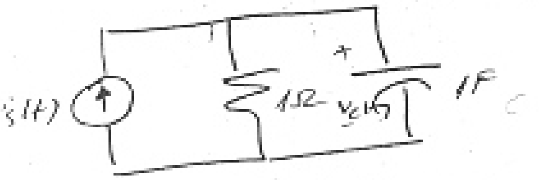
$$k_{12} = \frac{d}{ds}\left\{\frac{4(s+3)}{s}\right\}\Big|_{s=-2} = \frac{4s-(4s+12)}{s^2}\Big|_{s=-2} = \frac{-12}{s^2}\Big|_{s=-2} = -3$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = \frac{1}{1!} e^{-2t}\, t$$

$$F(s) = \frac{3}{s} - \frac{2}{(s+2)^2} - \frac{3}{s+2}$$

$$\boxed{f(t) = \left(3 - 3e^{-2t} - 2te^{-2t}\right) u(t)}$$

EXAMPLE:



$v_c(0^-) = 1V.$ $\quad i_s(t) = \cos t$

$$\boxed{i_s = \frac{v_c}{1} + C\,\dot{v}_c}$$

$$I_s = V_c(s) + sV_c(s) - v_c(0^-)$$

$$\frac{s}{s^2+1} = (s+1)V_c(s) - 1$$

$$\Rightarrow \quad V_c(s) = \left[\frac{s}{s^2+1} + 1\right]\frac{1}{s+1} = \frac{s^2+s+1}{(s+1)(s^2+1)} = \frac{k_1}{s+1} + \frac{k}{s-j} + \frac{k^*}{s+j}$$

$$k_1 = (s+1)V_c(s)\Big|_{s=-1} = \frac{s^2+s+1}{s^2+1}\Big|_{s=-1} = \frac{1}{2}$$

$$k = (s-j)V_c(s)\Big|_{s=j} = \frac{s^2+s+1}{(s+1)(s+j)}\Big|_{s=j} = \frac{-1+j+1}{(1+j)\,2j} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}\,e^{j45°}} = \frac{1}{2\sqrt{2}}\, e^{-j45°}$$

$$\boxed{v_c(t) = \frac{1}{2} e^{-t} + \frac{1}{\sqrt{2}} \cos(t - 45°)}$$

If we used phasors $\Rightarrow$ $\quad I_s = V_c + j\cdot 1\cdot 1\cdot V_c = (1+j)V_c = 1 \qquad V_c = \frac{1}{\sqrt{2}\, e^{j45°}}$

$$v_c(t) = Re\left\{\frac{1}{\sqrt{2}} e^{-j45°} e^{jt}\right\} = \frac{1}{\sqrt{2}} \cos(t-45°)$$