Properties of Systems / Convolution

Memoryless: $y[n]$, $y(t)$ $\longleftarrow$ $x[n]$, $y(t)$

Causality: $y[n] \longleftarrow x[n], x[n-1], \ldots$

Stability (BIBO) $|x[n]| < B$, $|y[n]| < C$

$$x \longrightarrow \boxed{T} \longrightarrow y$$

$$x_1 \rightarrow y_1 \quad , \quad x_2 \rightarrow y_2$$

Linearity

$$\alpha_1 x_1 + \alpha_2 x_2 \longrightarrow \alpha_1 y_1 + \alpha_2 y_2$$

$$\forall\, \alpha_1, \alpha_2, x_1, x_2$$

Time-Invariant

$$x_1[n] \rightarrow y_1[n]$$

$$x_1(t) \rightarrow y_1(t)$$

$$x_1[n-n_0] \longrightarrow y_1[n-n_0] \qquad \forall\, n_0, x_1, t_0$$

$$x_1(t-t_0) \longrightarrow y_1(t-t_0]$$

LTI

Ex

$y[n] = x[2n]$

Linear: $\alpha x[2n] \rightarrow$

$\alpha y[n] = \alpha x[2n]$ ✓

Time Invariant

$x[2(n-n_0)] = x[2n-2n_0]$ $\neq$ Time variant

$y[n-n_0] = x[2n-n_0]$

Memoryless

The output depends on the past values. Not memoryless.

Ex

$$y(t) = \underline{x(t)} - \underline{x(1-t)}$$

Not memoryless (depends on past values)

$t \to t - t_0$

$x(t) \to x(t-t_0) \Rightarrow x(t-t_0) - x(1-t-t_0)$

$\neq$

$$y(t-t_0) = x(t-t_0) - x(1-(t-t_0))$$

Time-variant

Linear

Ex

$$y[n] = \sum_{k=-2}^{3} x[n+k]$$

Not memoryless
Not causal
Linear
Time invariant

Ex

$y[n] = 3\underline{x[n]} - 2$

Linearity

$x[n] \rightarrow 3x[n] - 2$

$\alpha x[n] \rightarrow 3\alpha x[n] - 2$

$\alpha y[n] \rightarrow 3\alpha x[n] - 2\alpha$

$\neq$

Not linear

Time-invariant ✓

$x[n] \rightarrow x[n-n_0] \Rightarrow 3x[n-n_0] - 2$

$y[n] \rightarrow y[n-n_0] = 3x[n-n_0] - 2$

$=$

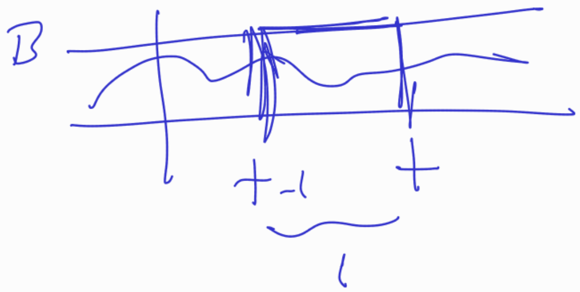
$|x[n]| < B$

$y[n] = 3\underline{x[n]} - 2 < 3B - 2 = \underline{C}$

Biba Stable

Ex

$$y(t) = \int_{t-1}^{t} x(\tau)\, d\tau$$



$|x(\tau)| < B,$

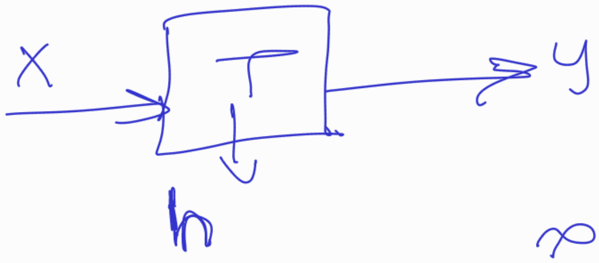
$|y(t)| < B$

BIBO stable

Integral is linear $\longrightarrow$ System is linear

$$\left.\begin{aligned} &x(t) \longrightarrow y(t) \\ &x(t-t_0) \longrightarrow \int_{t-t_0-1}^{t-t_0} x(\tau)\, d\tau \\ &y(t-t_0) = \int_{t-t_0-1}^{t-t_0} x(\tau)\, d\tau \end{aligned}\right\} \text{Time invariant}$$

Convolution



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\Updownarrow$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

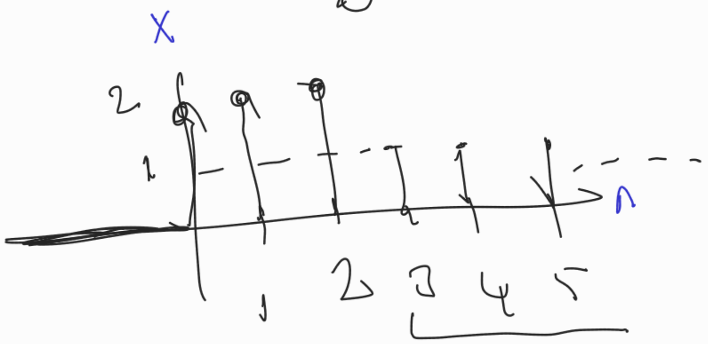
Ex

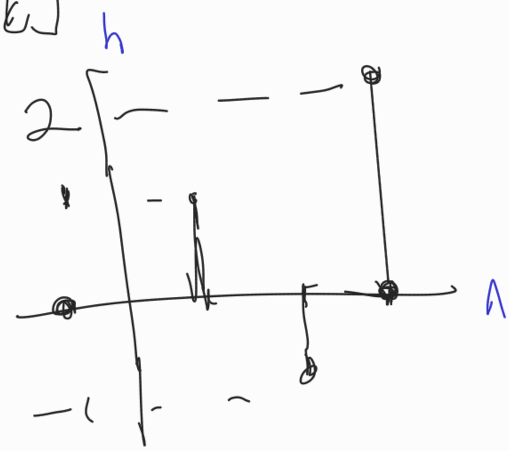
$$x[n] = 2u[n] - u[n-3]$$

$$h[n] = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 2, & n=2 \\ 0, & \text{else} \end{cases}$$

$$y[n] = x[n] \circledast h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$





$y[n] = 0$ if $n < 0$

for $y[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k]$

$$= h[0]\,x[0] + h[1]\,\cancel{x[-1]} + h[2]\,\cancel{x[-2]}$$

$$= 2$$

for $y[1] = \sum_{k} h[k]\, x[1-k]$

$$= h[1]\,x[0] + h[0]\,x[1]$$

$$= -2 + 2 = 0$$

$$y[2] = \sum_{k=-\infty}^{\infty} h[k]\,x[2-k]$$

$$= h[2]x[0] + h[1]x[1] + h[0]x[2]$$

$$= 4 + (-2) + 2 = 4$$

$\vdots$

$$y[7] = \sum h[k]\,x[7-k]$$

$$= \sum h[k]$$

$$= 2$$

$$y[8],\ y[9] \ldots = 2$$

$$y[n] = \sum_{k=-\infty}^{\infty} \underline{h[k]}\,x[n-k]$$

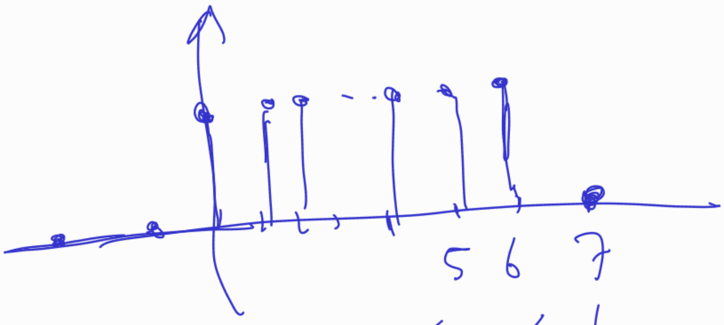
$$= h[0]\,\underline{x[n]} + h[1]\,\underline{x[n-1]} + h[2]\,\underline{x[n-2]}$$

if $n \geq 5$, the result will be 2.

$$y[n] = \begin{cases} 0, & n < 0 \\ 2, & n = 0 \\ 0, & n = 1 \\ 4 & n = 2 \\ \vdots & \\ 2 & n \geq 5 \end{cases}$$

Ex

$$x[n] = u[n] - u[n-7]$$



$$x[n] = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{else} \end{cases}$$

$$h[n] = (0.4)^n u[n]$$

$$y[n] = \sum_{k=0}^{6} (0.4)^{n-k} u[n-k]$$

for $n > 6$

$$y[n] = \sum_{k=0}^{6} (0.4)^{n-k} \quad *$$

if $0 \le n < 6$

$$y[n] = \sum_{k=0}^{n} (0.4)^{n-k} \quad **$$

Be careful with the limits of the summation. $u[n-k]$ has to be non zero. $(n > k)$

$$y[n] = \begin{cases} 0, & n < 0 \\ **, & 0 \le n < 6 \\ *, & 6 < n \end{cases}$$

~~ex~~

$$x(t) = \begin{cases} t, & 0 \leq t \leq 2 \\ 0, & \text{else} \end{cases}$$

$$h(t) = e^{-t} u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{2} t\, e^{-(t-\tau)}\, u(t-\tau)\, \underline{d\tau}$$

if $t < 2$

$$\underline{y(t) = \int_{0}^{t} t\, e^{-(t-\tau)}\, d\tau} \quad *$$

if $t > 2$

$$\underline{y(t) = \int_{0}^{2} t\, e^{-(t-\tau)}\, d\tau} \quad **$$

Again, Careful with the limits of the integral

if $t < 0$

$$y(t) = 0$$

$$y(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ *, & \text{if } 0 \leq t < 2 \\ **, & \text{if } 2 \leq t \end{cases}$$