Recitation 10: The Z-Transform

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Z-Transform: Definition

Analysis (Forward Z-Transform):

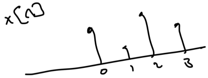
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ ✓

Synthesis (Inverse Z-Transform):

$$x[n] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{n-1} dz$$ ✓

Interpretation: Z-Transform generalizes the DTFT by allowing complex values of $z = re^{j\omega}$, enabling analysis of stability and convergence.

What Does the Z-Transform Represent?



- A discrete signal $x[n]$ is a list of values over time.
- The Z-transform gives a new view of the signal by analyzing it in the **complex-frequency domain**.
- Write $z = re^{j\omega}$:
  - Angle $\omega \rightarrow$ frequency content (like DTFT)
  - Radius $r \rightarrow$ growth or decay of the signal
- So the Z-transform shows:
  - What frequencies are present ✓
  - Whether components grow, decay, or stay steady ✓
  - Whether a system using this signal will be stable ✓

ROC, Poles, and Zeros (Simple Explanation)

- The Z-transform is an infinite sum:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

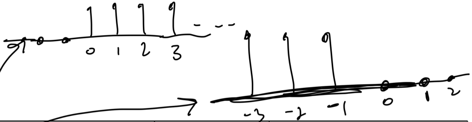
- The **Region of Convergence (ROC)** tells us for which values of z this sum converges.
- ROC determines:
  - Whether $X(z)$ exists ✓
  - Whether the **Fourier transform exists** (the ROC must include the unit circle) ✓
  - Stability (poles must be inside the unit circle) ✓
- **Poles** $\rightarrow$ where $X(z)$ blows up (strong resonances) $\rightarrow$ 1, 2
- **Zeros** $\rightarrow$ where $X(z)$ becomes zero (supressed frequencies) $\rightarrow$ 3, 4

$\uparrow \infty$

$H(z) = \frac{(z-3)(z-4)}{(z-1)(z-2)}$

$z=1$ $z=2$

Z-Transform: Common Transform Pairs



$x[n]$	$X(z)$	**ROC**
$\delta[n]$	1	entire z-plane
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 = X(z) \quad n=0$$

$$\sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \quad \left|\frac{1}{z}\right|<1$$

$$|z| > 1$$

$$\sum_{n=-\infty}^{\infty} -u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{0} z^{-n} \overset{n=-m}{=} -\sum_{m=0}^{\infty} z^{m} \quad |z|<1$$

$$= -\frac{1}{1-z}$$

They have different Z-transforms.

Z-Transform: Key Properties

- **Linearity:** $ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$
- **Time Shift:** $x[n - n_0] \leftrightarrow z^{-n_0}X(z)$ $x[n + n_0] \leftrightarrow z^{n_0}X(z)$
- **Time Reversal:** $x[-n] \leftrightarrow X(z^{-1})$
- **Exponential Scaling:** $a^n x[n] \leftrightarrow X(a^{-1}z)$
- **Convolution:** $x[n] * h[n] \leftrightarrow X(z)\,H(z)$
- **Multiplication in Time = Convolution in Z:**
  $x[n]\,h[n] \leftrightarrow \frac{1}{2\pi j}\oint X(\zeta)\,H(z/\zeta)\frac{d\zeta}{\zeta}$
- **Differencing:** $x[n] - x[n-1] \leftrightarrow (1 - z^{-1})X(z)$
- **Z-Domain Differentiation:**

t $\longrightarrow$ $\frac{\partial}{\partial w}$ $\longrightarrow$ FT

$$-nx[n] \leftrightarrow z\frac{dX(z)}{dz}$$

Example: Z-Transform of $a^n u[n]$

$$\sum_{n=0}^{\infty} 3^n \rightarrow \text{diverges}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \rightarrow \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$a^n \rightarrow \frac{1}{1-a}$$

$$\boxed{x[n] = a^n u[n]}$$

$$\boxed{X(z) = \sum_{n=0}^{\infty} a^n z^{-n}} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$\left|\frac{a}{z}\right| < 1$$

This is a geometric series:

$$X(z) = \frac{1}{1 - az^{-1}}$$

ROC: $|z| > |a|$

Z-Transform: ROC, Poles/Zeros, and Existence of DTFT

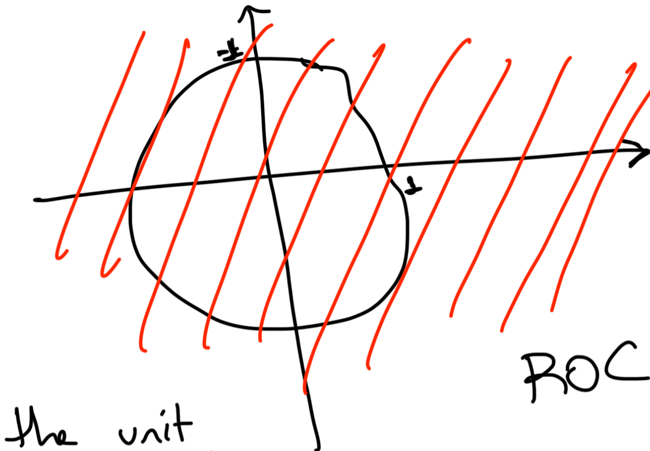
Problem:
Determine the **Z-transform** for each of the following sequences. Sketch the **pole-zero plot** and indicate the **region of convergence (ROC)**. Indicate whether or not the **Fourier transform** of the sequence exists.

(a) $\delta[n+3]$

(b) $3^n u[-n] \; + \; \left(\frac{1}{4}\right)^n u[n+1]$

a) $\sum_{n=-\infty}^{\infty} \underbrace{\delta[n+3]}_{n=-3} z^{-n} = z^3 = X(z)$

for all z values, its valid



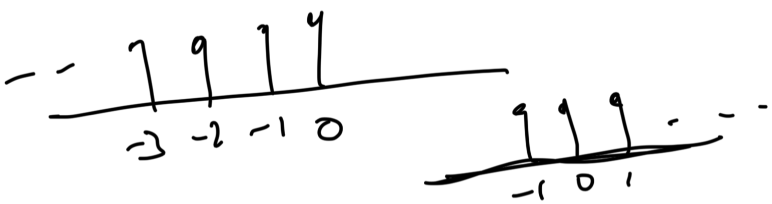
ROC: entire z-plane

Fourier transform exists, ROC covers the unit circle.

b) $\sum_{n=-\infty}^{\infty} 3^n u[-n] z^{-n} = \sum_{n=-\infty}^{0} 3^n z^{-n} \overset{n=-m}{=} \sum_{m=0}^{\infty} 3^{-m} z^m = \sum_{m=0}^{\infty} \left(\frac{z}{3}\right)^m \longrightarrow \left|\frac{z}{3}\right| < 1$

$ROC_1: |z| < 3$

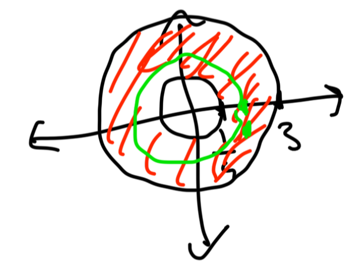
$X_1(z) = \frac{1}{1 - z/3} = \frac{-3z^{-1}}{1 - 3z^{-1}}$



$\sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n+1] z^{-n} = \sum_{n=-1}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n + \overbrace{\left(\frac{1}{4}\right)^{-1} z^{1}}^{4z}$

$\left|\frac{1}{4z}\right| < 1$

$ROC_2: |z| > \frac{1}{4}$



$\frac{1}{1 - \frac{1}{4z}} = \frac{4}{4 - z^{-1}} = X_2(z)$

ROC: $3 > |z| > \frac{1}{4}$

FT exists as ROC covers the unit circle

$$X(z) = X_1(z) + X_2(z) = -\frac{3z^{-1}}{1-3z^{-1}} + 4z + \frac{4}{4-z^{-1}} \quad , \quad \text{ROC}: 3 > |z| > \frac{1}{4}$$

Problem

$x[n] - x[n-1] \rightarrow X(z) - z^{-1}X(z)$

Consider a discrete time LTI system whose algebraic form of its system function is given by:

$$H(z) = \frac{z^2 - 2az + 1}{z^2 + \frac{1}{6}z - \frac{1}{6}}, \quad = \frac{Y(z)}{X(z)}$$

where a is a real positive number with $0 < a < 1$.

a) Write the difference equation for this system.

b) It is known that the output of this system to the input

$$x[n] = \cos\left(\frac{\pi n}{5} + \frac{\pi}{3}\right)$$

is $y[n] = 0$. Is this system causal? Is it stable? Justify your answers.

c) What is the value of a?

a) $H(z) = \dfrac{\cancel{z^2}\left(1 - 2az^{-1} + z^{-2}\right)}{\cancel{z^2}\left(1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right)} \Rightarrow \dfrac{Y(z)}{X(z)}$

$$X(z) - 2a\underline{z^{-1}X(z)} + z^{-2}X(z) = Y(z) + \frac{1}{6}Y(z)z^{-1} - \frac{1}{6}z^{-2}Y(z)$$

$\hookrightarrow z^{-1}$

$$x[n] - 2a\,x[n-1] + x[n-2] = y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2]$$

b) $x[n] = \cos\left(\frac{\pi n}{5} + \frac{\pi}{3}\right) = \frac{1}{2}\left(e^{j\frac{\pi}{3}}e^{j\frac{n\pi}{5}} + e^{-j\frac{\pi}{3}}e^{-j\frac{\pi n}{5}}\right)$

$$y[n] = 0 = \frac{e^{j\frac{\pi}{3}}}{2}\underline{H\left(e^{j\frac{\pi}{5}}\right)}e^{j\frac{\pi n}{5}} + \frac{e^{-j\frac{\pi}{3}}}{2}\underline{H\left(e^{-j\frac{\pi}{5}}\right)}e^{-j\frac{\pi n}{5}})$$

$$\underline{H\left(e^{j\frac{\pi}{5}}\right) = H\left(e^{-j\frac{\pi}{5}}\right) = 0}$$

$$H(z) = \frac{z^2 - 2az + 1}{z^2 + \frac{1}{6}z - \frac{1}{6}} = \frac{z^2 - 2az + 1}{\underline{\left(z+\frac{1}{2}\right)}\left(z - \frac{1}{3}\right)}$$

poles: $z = -\frac{1}{2}$
$z = \frac{1}{3}$

potential ROCs: $|z| < \frac{1}{3}$
$\frac{1}{3} < |z| < \frac{1}{2}$

System is causal,

System is stable.

$|z| > \frac{1}{2}$

c) $x[n] = \cos\left(\frac{\pi}{5}n + \frac{\pi}{3}\right) \to y[n] = 0$

$$H\left(e^{j\frac{\pi}{5}}\right) = H\left(e^{-j\frac{\pi}{5}}\right) = 0$$

zeros: $z = e^{j\frac{\pi}{5}}$ $\quad z = e^{-j\pi/5}$

$$H(z) = \frac{\left(z - e^{j\frac{\pi}{5}}\right)\left(z - e^{-j\frac{\pi}{5}}\right)}{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{z^2 - 2az + 1}{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

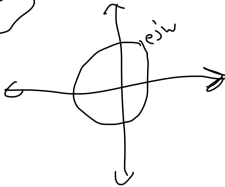
$$z^2 \underbrace{- e^{-j\frac{\pi}{5}} z - z e^{j\frac{\pi}{5}}}_{2a\cancel{z} = \left(e^{j\frac{\pi}{5}} + e^{-j\frac{\pi}{5}}\right)\cancel{z}} + 1$$

$$a = \cos\left(\frac{\pi}{5}\right)$$

Relationship Between Z-Transform and DTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$z = e^{j\omega}$



Let $z = e^{j\omega}$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Thus, DTFT is the Z-transform evaluated on the unit circle.

Key Insight: Pole-Zero Interpretation

- Poles close to the unit circle $\rightarrow$ long-lasting oscillations.
- Poles inside the unit circle $\rightarrow$ stable system.
- Zeros determine frequency notches.
- Frequency response is $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$.

Z-Transform Problem

Given:

$$X(z) = \frac{z^5}{z^5 - \frac{1}{2}}, \qquad \text{with ROC: } \left(\frac{1}{2}\right)^{1/5} < |z|.$$

Tasks:

- Determine the sequence $x[n]$ corresponding to $X(z)$.

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \longrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}, \quad \text{ROC: } \frac{1}{2} < |z|$$

$$X(z) = X_1(z^5) \quad \text{with ROC: } \left(\frac{1}{2}\right)^{1/5} < |z|$$

$$x[n] = \begin{cases} x_1[k], & n = 5k \\ 0, & n \neq 5k \end{cases} \qquad k \in \mathbb{Z}$$

$$\Rightarrow x[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} \delta[n-5k]$$

Inverse z-Transform: Partial Fraction Method

Goal: Given $X(z)$, find the time-domain sequence $x[n]$.

Key Idea: Rewrite $X(z)$ as a sum of simpler rational functions whose inverse z-transform is known:

$$X(z) = \sum_k \frac{A_k}{1 - p_k z^{-1}} + \sum_m B_m z^{-m}.$$

Steps:

1. **Factor the denominator** of $X(z)$ to find poles.
2. **Express** $X(z)$ **as partial fractions:**

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots$$

3. **Invert each term individually** using known pairs:

$$\mathcal{Z}^{-1}\left\{\frac{1}{1 - pz^{-1}}\right\} = p^n u[n]$$

4. **Apply ROC** to determine whether sequences are causal, anti-causal, or two-sided.

Result:

$$x[n] = \sum_k A_k p_k^n u[n] + \text{(any shifted delta terms)}.$$

Inverse z-Transform Question

Consider a left-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

z^{-1}

- (a) Write $X(z)$ as a ratio of polynomials in z instead of z^{-1}.
- (b) Using a partial-fraction expression, express $X(z)$ as a sum of terms, where each term represents a pole from your answer in part (a).
- (c) Determine $x[n]$.

a) $X(z) = \dfrac{1}{z^{-1}\left(z-\frac{1}{2}\right) z^{-1}\left(z-1\right)} = \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-1\right)} = \dfrac{z^2}{z^2-\frac{3}{2}z+\frac{1}{2}}$

b)

$$\begin{array}{r|l} z^2 & z^2-\frac{3}{2}z+1/2 \\ \hline -\;\; z^2-\frac{3}{2}z+\frac{1}{2} & 1 \\ \hline \frac{3}{2}z-\frac{1}{2} & \end{array}$$

$\Rightarrow \dfrac{z^2}{z^2-\frac{3}{2}z+\frac{1}{2}} = 1 + \dfrac{3/2\, z - 1/2}{z^2 - 3/2\, z + 1/2}$

$$\frac{\frac{3}{2}z-\frac{1}{2}}{(z-1/2)(z-1)} = \frac{A}{z-1/2}_{(z-1)} + \frac{B}{z-1}_{(z-1/2)} = \frac{Az-\cancel{A}+Bz-B/2}{(z-1/2)(z-1)}$$

$$Az - \frac{A}{2} + Bz - B = \frac{3}{2}z - \frac{1}{2} \Rightarrow$$

$A + B = 3/2$

$-A - \frac{B}{2} = -1/2$

$-2A - B = -1$

$A + B = 3/2$

$-A = +1/2$

$A = -1/2$

$B = 2$

$$X(z) = \underbrace{1}_{x_1(z)} + \underbrace{\frac{2}{z-1}}_{X_2(z)} - \underbrace{\frac{1/2}{z-1/2}}_{X_3(z)}$$

c) $x[n] = ?$

$X_1(z) = 1 \Rightarrow x_1[n] = \delta[n]$

$$X_2(z) = \frac{2}{z-1} = \frac{2z^{-1}}{1-z^{-1}} \Rightarrow z^{-1}\left\{\frac{2z^{-1}}{1-z^{-1}}\right\} = 2(-1)^{n-1}u[-n]$$

$$z^{-1}\left\{\frac{1}{1-z^{-1}}\right\} = (-1)^n u[-n-1]$$

$n-1$

$$X_3(z) = -\frac{1/2}{z - 1/2} = \frac{-1/2\, z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$\mathcal{Z}^{-1}\left\{\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right\} = -\left(\frac{1}{2}\right)^{n-1} u[-(n-1)-1] = -\left(\frac{1}{2}\right)^{n-1} u[-n]$$

$$x[n] = \delta[n] + 2(-1)^{n-1} u[-n] + \left(\frac{1}{2}\right)^n u[-n]$$

$$= \overbrace{\delta[n]}^{x_1} + \left[\overbrace{\left(\frac{1}{2}\right)^n}^{x_3} + \overbrace{2(-1)^{n-1}}^{x_2}\right] u[-n]$$