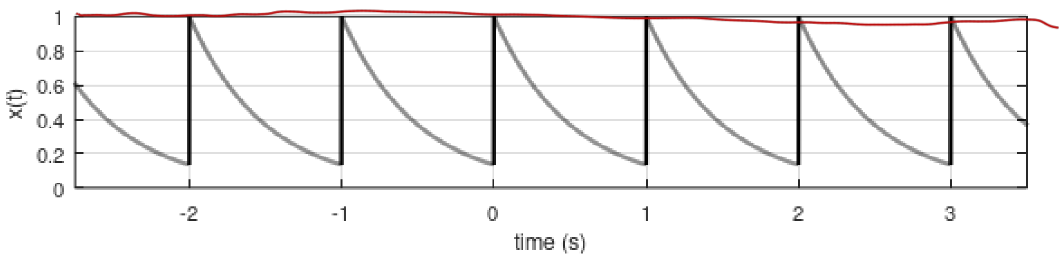
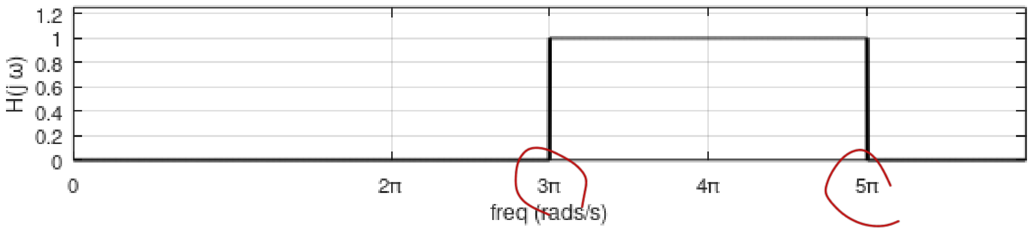
Consider a periodic CT signal defined as $x(t) = e^{\alpha t}u(t),\ 0 \le t < 1$ with a period of T=1 seconds, where $\alpha < 0$.



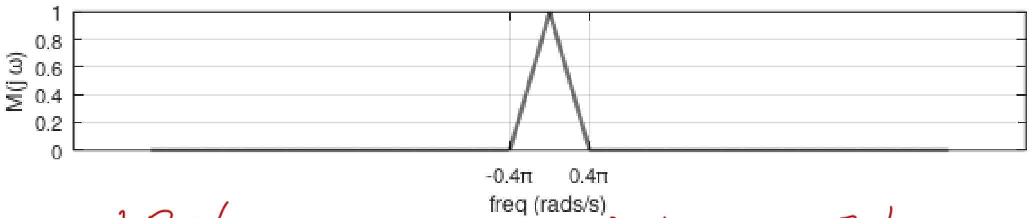
a) Determine the Fourier transform of $x(t)$. Roughly sketch $|X(j\omega)|$.

b) Check what becomes of $x(t)$ as α goes to 0. Is your result in part (a) valid in this particular case?

c) Consider an LTI system that possesses a **real** impulse response. Its frequency response $H(j\omega)$ is depicted below **for positive frequencies**. Let the output be $y(t)$ when $x(t)$ is the input to this system. Find and roughly sketch $|Y(j\omega)|$ by clearly indicating the frequency values (without explicit numerical evaluation of the amplitude).



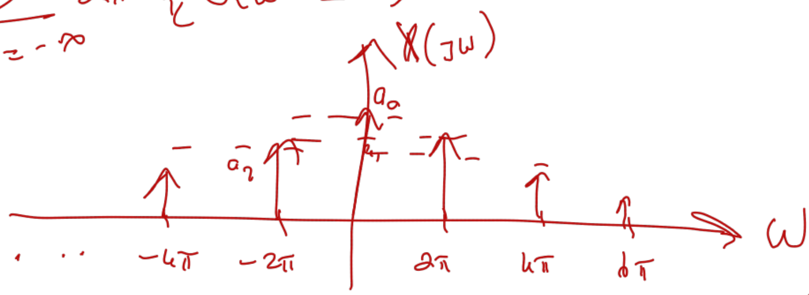
d) A signal $m(t)$ with the spectrum below is multiplied by $x(t)$ and passed through the LTI system in part (c). Let the output be $z(t)$ when $m(t)x(t)$ is the input to this system. Find and roughly sketch $|Z(j\omega)|$ by clearly indicating the frequency values (without explicit numerical evaluation of the amplitude).



(a)

$$a_k = \frac{1}{1}\int_0^1 e^{\alpha t} e^{-jk2\pi t}\,dt = \frac{e^{t(\alpha - j2\pi k)}}{\alpha - j2\pi k}\Bigg|_0^1 = \frac{e^{\alpha - j2\pi k} - 1}{\alpha - j2\pi k} = \frac{e^{\alpha} - 1}{\alpha - j2\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k2\pi)$$

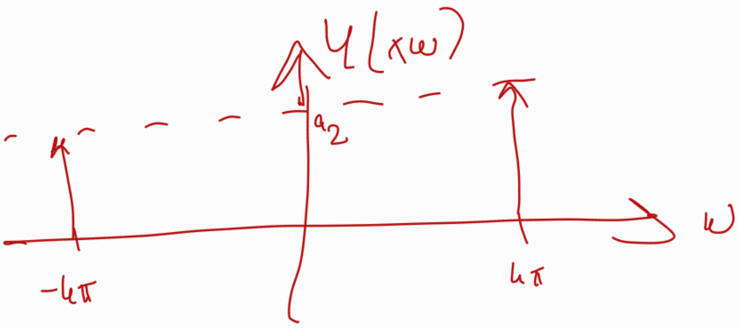


(b) $\alpha \to 0,\ e^{\alpha t}u(t) \to 1 \to$

$\to a_0 = 1,\ a_k = 0,\ k \neq 0$

$a_0 = \frac{e^{\alpha} - 1}{\alpha} \to$ L'hopital $\frac{e^{\alpha}}{1} = \frac{e^0}{1} = 1$ ✓ $\quad \frac{e^{\alpha} - 1}{\alpha - j2\pi k} = \frac{e^0 - 1}{-j2\pi k} = 0$ ✓

c)



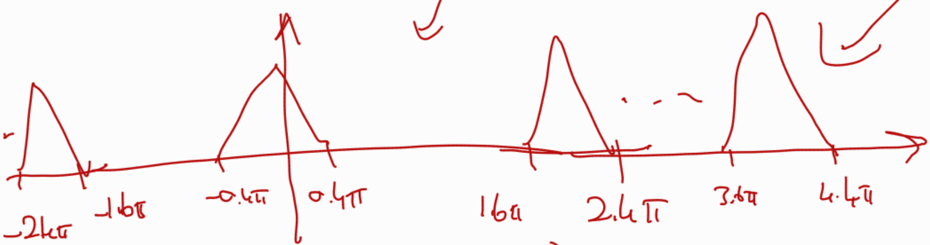
d)

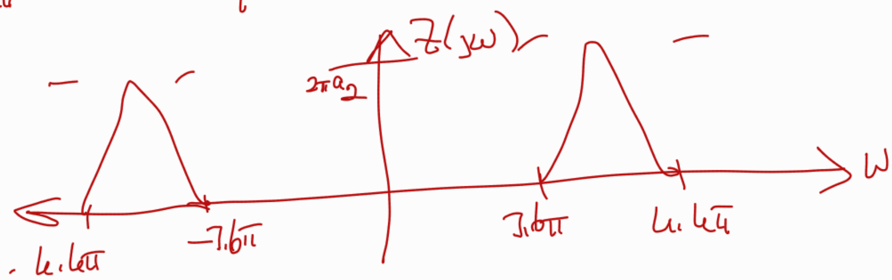
$$[x(t)\, m(t)] * h(t) = z(t)$$

$$Z(j\omega) = \underbrace{\left(X(j\omega) * M(j\omega)\right)} \cdot \underbrace{H(j\omega)}$$

$$\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k) * M(j\omega)$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} 2\pi a_k M(j(\omega - 2\pi k))}$$





Consider a continuous-time signal $x(t)$ whose Fourier Transform is given as follows:

$$X(j\omega) = \begin{cases} e^{3\omega} & if\ \omega < 0 \\ 0 & otherwise. \end{cases}$$

a) Determine $x(t)$.

b) Find the total energy of the signal $x(t)$, that is, $\int_{-\infty}^{\infty} |x(t)|^2 dt$.

c) Find the Fourier Transform of $z(t) = \frac{e^{-j3t}}{3+3jt}$. (Hint: you can use Fourier Transform properties.)

Also consider a stable LTI system with input $x(t)$ and output $y(t)$ defined by

$$\frac{d^2y(t)}{dt^2} - \alpha^2 y(t) = \alpha x(t) - \frac{dx(t)}{dt}$$

d) Find the frequency response of the system where α is a positive real number.

e) When the input $x(t)$ in part (a) is fed into the LTI system in part (d), find the Fourier Transform of the output, $Y(j\omega)$. Also determine the magnitude $|Y(j\omega)|$ in terms of α.

$X(at) \to \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{3\omega} e^{j\omega t} d\omega = \frac{e^{3\omega + j\omega t}}{2\pi(3+jt)}\Big|_{-\infty}^{0} = \frac{1}{2\pi(3+jt)}$

b) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{6\omega} d\omega = \frac{e^{6\omega}}{2\pi 6}\Big|_{-\infty}^{0} = \frac{1}{12\pi}$

$X(j\omega) = e^{3\omega}$

$X(j\omega/3) = \frac{1}{3} e^{\omega}$

$2\pi X(j\omega/3) = \frac{2\pi}{3} e^{\omega}$

c) $z(t) = \frac{e^{-j3t}}{3+3jt} \to Z(j\omega) = \frac{2\pi}{3} e^{\omega+3}$

d) $Y(j\omega)(-\omega^2 - \alpha^2) = X(j\omega)(\alpha - j\omega)$

$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\alpha - j\omega}{-\omega^2 - \alpha^2}$

e) $\frac{\alpha - j\omega}{-\omega^2 - \alpha^2} e^{3\omega} \leftarrow$ output freq response

Mag = α

$\frac{\sqrt{\alpha^2 + \omega^2}}{\alpha^2 + \omega^2} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$

a. Consider a filter whose frequency response $H\left(e^{j\Omega}\right)$ is specified over the frequency interval $0 \le \Omega < 2\pi$ as follows:

$$H\left(e^{j\Omega}\right)=\begin{cases}5 & \frac{3\pi}{4}\le\Omega\le\frac{5\pi}{4}\\ 0 & \text{otherwise}\end{cases}$$

i. Find and plot $H\left(e^{j\Omega}\right)$ in the interval $(-\pi,\pi]$. Specify the type of this filter (e.g. low-pass, high-pass, band-pass, band-stop).

ii. Calculate the output signal $y[n]$ of this filter when the input signal is given as

$$x[n]=10+\underbrace{\cos(88\pi n)}_{=\cos(0)=1}+5\underbrace{\sin(11\pi n)}_{=\sin(\pi n)=0}-5\cos\left(\frac{23\pi n}{8}\right). = 11-5\cos\left(\frac{7\pi n}{8}\right)$$

freq = 0

iii. Calculate the output signal $y[n]$ of this filter when the input signal is given as $x[n] = 2\,(-1)^n$.

b. Let $H\left(e^{j\Omega}\right)$ be the frequency response of an LTI system. It is given that

- The system is **causal.** → $h[n]=0,\ n<0.$
- The impulse response $h[n]$ of the system is **real-valued**.
- $\mathrm{DTFT}\left(h[n]-h[-n]\right)=4j\sin(2\Omega)-2j\sin(3\Omega)+10j\sin(5\Omega)$.

  (Note that this expression implies that $h[n]$ has **finite duration**.)
- $\int_{4\pi}^{6\pi} H\left(e^{j\Omega}\right)H^*\left(e^{j\Omega}\right)d\Omega = 64\pi$.
- $h[0] > 0$.

$h[n] \rightarrow H(e^{j\Omega})$

$-h[n] \rightarrow -H(e^{j\Omega})$

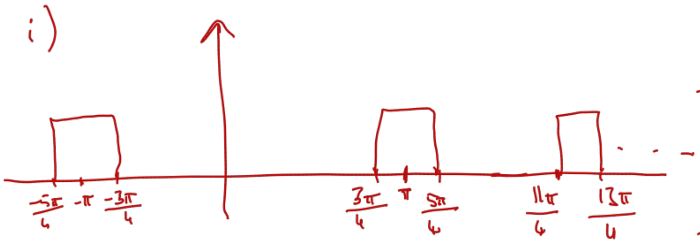
$h[-n] \rightarrow H^*(e^{j\Omega})$

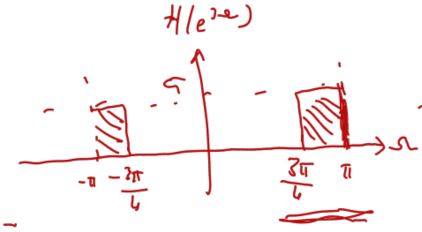
$-h[-n] \rightarrow -H^*(e^{j\Omega})$

Evaluate and plot $h[n]$.

a)

i)





→ high-pass filter

$e^{j\pi} = -1$

ii)

$x[n] = 11 - 5\cos\left(\frac{7\pi n}{8}\right)$

$y[n] = -25\cos\left(\frac{7\pi n}{8}\right)$

iii) $x[n] = 2(-1)^n = 2e^{j\pi n} \rightarrow \omega = \pi$

$y[n] = 10e^{j\pi n} = 10(-1)^n$

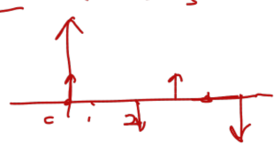
b) $4j\sin(2\Omega) = 4j\,\frac{e^{j2\Omega}-e^{-j2\Omega}}{2j}$

$-h[-n] \rightarrow 2e^{j2\Omega} + (-e^{j3\Omega}) + (5e^{j5\Omega})$

$h[n] \rightarrow -2e^{-j2\Omega} + e^{-j3\Omega} + (-5e^{-j5\Omega})$

$h[n] = -2\delta[n-2] + \delta[n-3] - 5\delta[n-5] + a\delta[n]$

$\frac{1}{2\pi}\int_{4\pi}^{6\pi}|H(e^{j\Omega})|^2 d\Omega = 32 = \sum_{n=-\infty}^{\infty}(h[n])^2 = \underbrace{4+1+25}_{30}+a^2 = 32 \quad a^2 = 2 \quad a = \sqrt{2}$

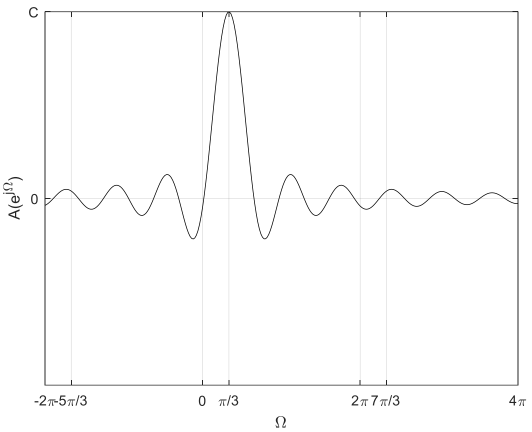


Part (c) of this question is independent of parts (a) and (b).

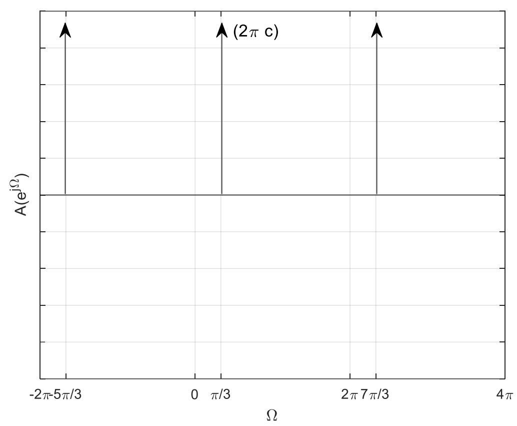
Consider a DT signal defined as $a[n] = \begin{cases} e^{j\frac{\pi}{3}n} & , |n| \le L. \\ 0 & , o.w. \end{cases}$

a) Determine the DT Fourier transform $A(e^{j\Omega})$ of $a[n]$. (Hint: $\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha}$)

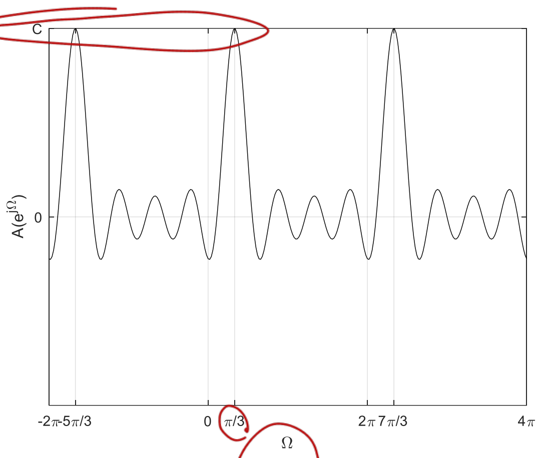
b) Simplify your answer in part (a) as much as you can. Pick which of the following functions corresponds to $A(e^{j\Omega})$. What is the value of c indicated in the plot?



I



II



III

ENCIRCLE YOUR CHOICE:

I II III

VALUE OF c = $\boxed{}$

c) [Independent of previous parts] Consider a system described by $y[n] = x[n]e^{j\frac{\pi}{3}n}$. Check this system's linearity, time-invariance, stability properties.

a) $A(e^{j\Omega}) = \sum_{n=-L}^{L} e^{j\frac{\pi}{3}n} e^{-j\Omega n} = \sum_{n=0}^{2L} e^{j\frac{\pi}{3}(n-L)} e^{-j\Omega(n-L)} = e^{jL(-\frac{\pi}{3}+\Omega)} \sum_{n=0}^{2L} e^{j\frac{\pi}{3}n - j\Omega n}$

$= e^{jL(-\frac{\pi}{3}+\Omega)} \left(\frac{1 - e^{j\frac{\pi}{3}(2L+1)} e^{-j\Omega(2L+1)}}{1 - e^{j\frac{\pi}{3}} e^{-j\Omega}} \right)$

b) $A(e^{j\Omega}) = \frac{e^{jL(-\frac{\pi}{3}+\Omega)} - e^{j(\frac{\pi}{3}-\Omega)(L+1)}}{1 - e^{j\frac{\pi}{3}} e^{-j\Omega}} = \frac{\cancel{e^{j(\frac{\pi}{3}-\Omega)\frac{1}{2}}} \left(e^{-j(\frac{\pi}{3}-\Omega)(L+\frac{1}{2})} - e^{j(\frac{\pi}{3}-\Omega)(L+\frac{1}{2})}\right)}{\cancel{e^{j(\frac{\pi}{3}-\Omega)\frac{1}{2}}} \left(e^{-j(\frac{\pi}{3}-\Omega)\frac{1}{2}} - e^{j(\frac{\pi}{3}-\Omega)\frac{1}{2}}\right)}$

$= \frac{\sin\left((\frac{\pi}{3}-\Omega)(L+\frac{1}{2})\right)}{\sin\left((\frac{\pi}{3}-\Omega)\frac{1}{2}\right)} \rightarrow$ III

$\frac{L+\frac{1}{2}}{\frac{1}{2}} = c$

$\boxed{2L+1=c}$

$$y[n] = \underbrace{x[n]\, e^{j\frac{\pi}{3}n}}_{} \to \text{mag.} = 1$$

$$x_1[n] \to y_1[n]$$
$$x_2[n] \to y_2[n]$$

$$\left.\begin{aligned} \alpha x_1[n] &\to \alpha y_1[n] \\ \beta x_2[n] &\to \beta y_2[n] \end{aligned}\right)$$

$$\underbrace{\alpha x_1[n] + \beta x_2[n]} = \alpha y_1[n] + \beta y_2[n] \to \text{Linear}$$

$$x_1[n-n_0] \to x_1[n-n_0]\, e^{j\frac{\pi}{3}n} \to \text{Time-Variant}$$
$$y_1[n-n_0] = x_1[n-n_0]\, e^{j\frac{\pi}{3}(n-n_0)}$$

$\cancel{\text{LTI}}$

BIBO stable ✓

$$|x[n]| \le B \qquad |y[n]| \le B$$

Consider a continuous time (CT) signal defined as $x(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$,

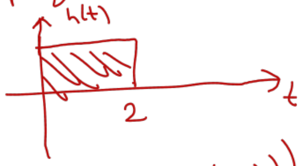
a) The signal $x(t)$ is given as an input to the following CT system: $y(t) = \int_{t-2}^{t} x(\tau)d\tau$.

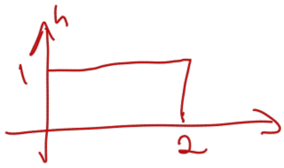
   i. State and justify whether the system is linear-time-invariant (LTI) or not. Then, find and plot the output signal $y(t)$.

   ii. Find the Fourier Transform (FT) of $y(t)$, namely $Y(j\omega)$, in terms of $X(j\omega)$.

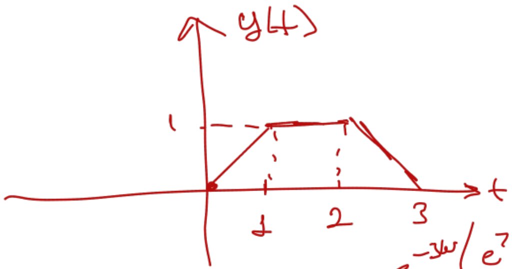
a) $y(t) = \int_{t-2}^{t} x(\tau)d\tau \searrow \alpha x_1 + \beta x_2 \quad a\int x_1 + b\int x_2 \rightarrow$ Linear

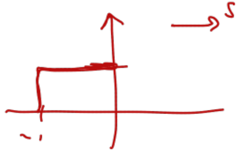
$\rightarrow$ ~~time~~-invariant

$h(t) = \int_{t-2}^{t} \delta(\tau)d\tau =$  $= u(t) - u(t-2)$

 $*$ 

$x(t) * h(t) = x(t) * (u(t) - u(t-2))$



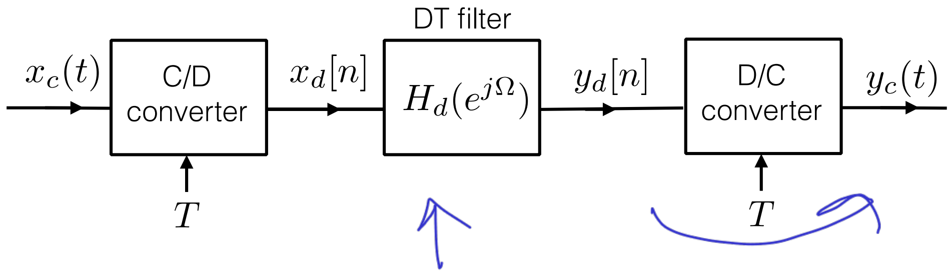
$\searrow -t$  $\rightarrow$ shift $\rightarrow$ multiply integral

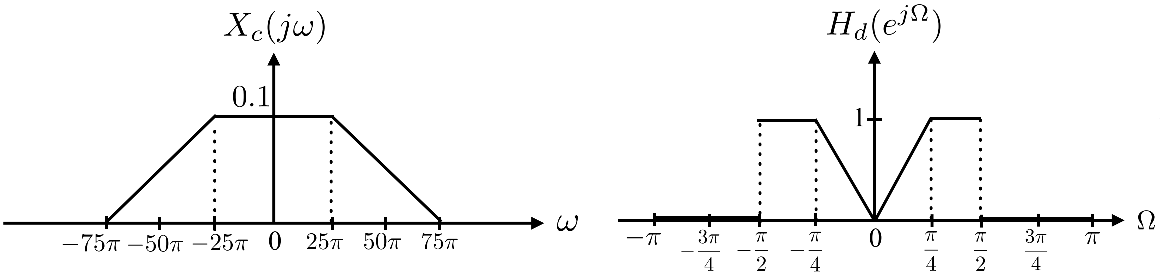
$Y(j\omega) = H(j\omega)\,X(j\omega)$

$h(t) = u(t) - u(t-2) \rightarrow H(j\omega) = \underbrace{\frac{1}{j\omega} - \frac{e^{-j2\omega}}{j\omega}}_{e^{-j\omega}(e^{j\omega} - e^{-j\omega})} = \frac{2e^{-j\omega}}{\omega}\sin(\omega)$

$Y(j\omega) = \frac{2e^{-j\omega}}{\omega}\sin(\omega)\,X(j\omega)$

a) (10 pts) A continuous-time signal $x_c(t)$ with spectrum $X_c(j\omega)$ is sampled with sampling period $T = 0.01$ seconds and the sampled signal $x_d[n] = x_c(nT)$ is processed with the discrete-time filter with frequency response $H_d(e^{j\Omega})$ as illustrated below. The filter output $y_d[n]$ is finally converted back to continuous-time to obtain the signal $y_c(t)$ (via conversion to impulse train and ideal low-pass filtering) such that $y_d[n] = y_c(nT)$.





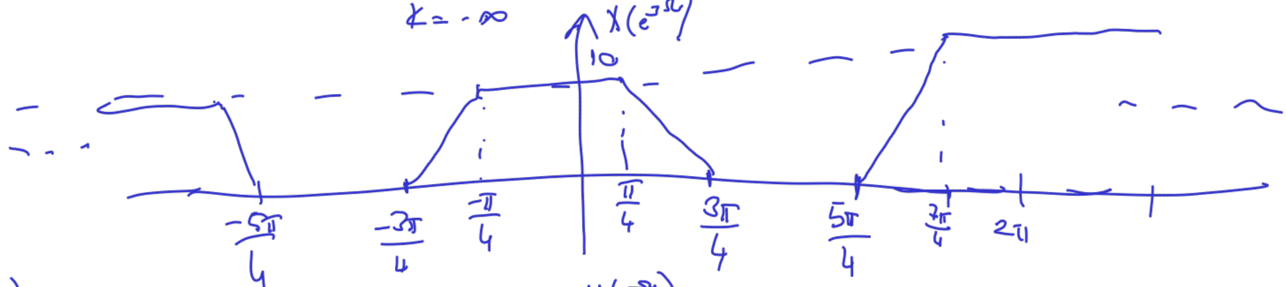
Find and plot:

i. The spectrum $X_d(e^{j\Omega})$ of $x_d[n]$ for $\Omega \in [-2\pi, 2\pi]$
ii. The spectrum $Y_d(e^{j\Omega})$ of $y_d[n]$ for $\Omega \in [-2\pi, 2\pi]$
iii. The spectrum $Y_c(j\omega)$ of the output signal $y_c(t)$

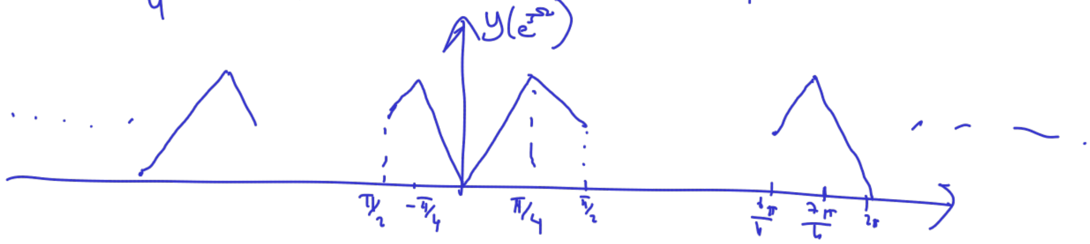
i) $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$ $\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.01}$

$$X_d(e^{j\Omega}) = X_p(j\omega)\Big|_{\omega=\frac{\Omega}{T}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega - k2\pi}{T}\right)\right)$$

$$= 100\sum_{k=-\infty}^{\infty} X_c(j100(\Omega - k2\pi))$$



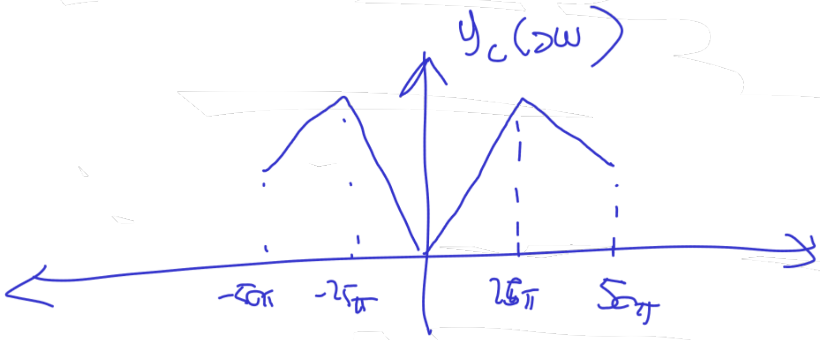
ii)



iii)

$$Y_c(jw) = T\,Y_d(e^{j\Omega}) \quad \text{for} \quad w \in \left[-\frac{w_s}{2}, \frac{w_s}{2}\right]$$

$\Omega = Tw$



Problem

A real-valued message signal m(t) is band-limited to 10 kHz and satisfies $|m(t)| \le 1$. The message is transmitted using sinusoidal amplitude modulation with carrier frequency $f_c = 500$ kHz. The transmitted signal is $s(t) = A_c[1 + \alpha\, m(t)] \cos(2\pi f_c t)$, where $A_c > 0$ and $\alpha > 0$.

(a) Express s(t) using complex exponentials and identify the frequency components.

(b) Find the minimum transmission bandwidth of s(t).

(c) Assume synchronous demodulation by multiplying s(t) with $2\cos(2\pi f_c t)$ followed by an ideal low-pass filter. Find the output signal.

(d) For envelope detection:
1) State the condition on α for distortionless demodulation.
2) Explain what happens if the condition is violated.

(e) Three independent message signals, each band-limited to 10 kHz, are transmitted using AM and frequency-division multiplexing. 1) What is the minimum carrier spacing?
2) If the lowest carrier is 500 kHz, find all carrier frequencies.

(a) $\cos(2\pi f_c t) = \dfrac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}$ m(t) bandlimited to 10kHz = f

frequency components: $f_c \pm f$ = 490 kHz, 510 kHz

(b) $B = 2 \times M_f = 2 \times 10\,kHz = 20\,kHz$ bandwidth

(c) $A_c(1 + \alpha M(t))$ ⟸ is left.
multiply s(t) by $2\cos(2\pi f_c t)$, write in exponential form, delete high freqs.

(d) $1 + \alpha m(t) > 0 \Rightarrow \alpha < 1$, if $\alpha > 1$ over-modulation → introduces distortion
$\uparrow \le 1$

(e) for 10kHz, we need 20kHz bandwidth = spacing we need
500 kHz → 520 kHz → 540 kHz

The system function $H(z)$ of a DT LTI system is given below.

$$H(z) = \frac{1 - z^{-1}}{1 - 2z^{-1}}$$

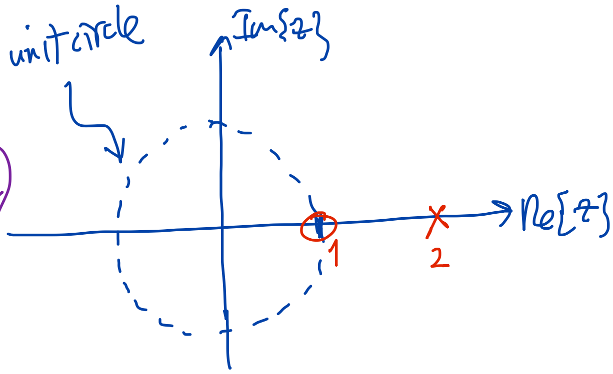
(a) Answer the following.

i. Plot the pole-zero diagram of $H(z)$.

ii. What are the possible Regions of Convergence (ROCs) that can be associated with this $H(z)$? State them mathematically.

iii. For each possible ROC, indicate whether the system is causal and stable.

$$H(z) = \frac{\frac{z-1}{z}}{\frac{z-2}{z}} = \frac{z-1}{z-2} \Rightarrow \text{pole}: z=2 \quad \text{zero}: z=1$$



+2

There're 2 possible ROCs.

ROC_1: $|z| < 2 \rightarrow$ System not causal, stable

ROC_2: $|z| > 2 \rightarrow$ System causal, not stable

+3

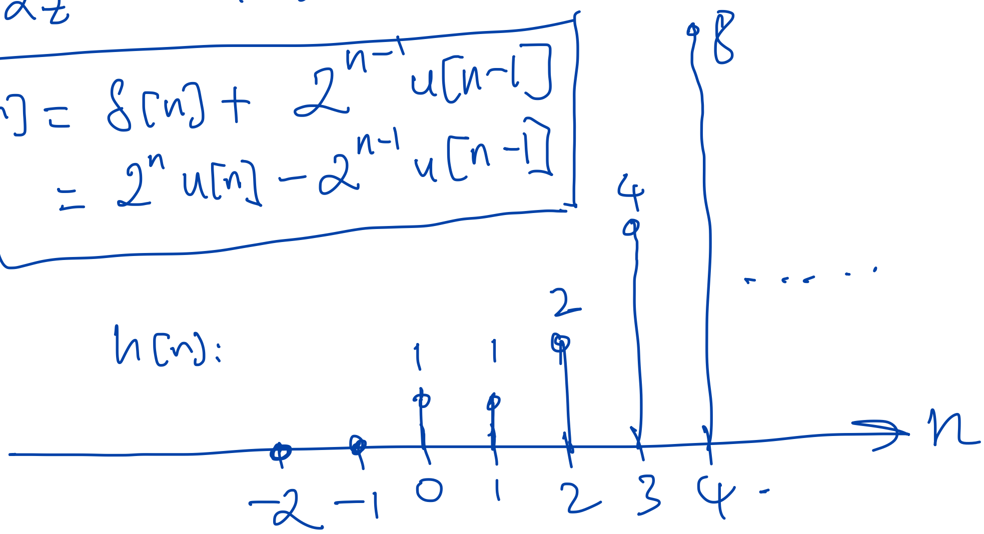
(b) Assume the system is causal in this part. Find the impulse response $h[n]$ of this system. Plot $h[n]$, labeling its values for at least $n = -2, -1, 0, 1, 2$.

Causal system assumption $\Rightarrow$ ROC = ROC_2 : $|z| > 2$.

$$H(z) = \frac{1 - z^{-1}}{1 - 2z^{-1}} = \frac{1 - 2z^{-1} + z^{-1}}{1 - 2z^{-1}} = 1 + z^{-1} \cdot \frac{1}{1 - 2z^{-1}}$$

$$\Rightarrow \boxed{\begin{aligned} h[n] &= \delta[n] + 2^{n-1} u[n-1] \\ &= 2^n u[n] - 2^{n-1} u[n-1] \end{aligned}}$$

$h[n]$:



(c) Assume again the system is causal in this part. Find a difference equation and associated initial conditions that can be used to find the impulse response $h[n]$ of the system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 2z^{-1}} \Rightarrow Y(z)[1 - 2z^{-1}] = X(z)[1 - z^{-1}]$$

$$\Rightarrow \boxed{y[n] - 2 \cdot y[n-1] = x[n] - x[n-1].}$$

$$\boxed{\text{Init. conditions: } y[-1] = (h[-1] =) 0}$$

(d) Assume the frequency response $H(e^{j\Omega})$ of the system exists in this part.

i. Find $H(e^{j\Omega})$.

ii. Find the step response $s[n]$ of the system. (Without using convolution with $h[n]$.)

i. $H(e^{j\Omega})$ exists $\Longrightarrow$ $ROC = ROC_1 : |z| < 2$ (includes unit circle)

+1

$$\Rightarrow \boxed{H(e^{j\Omega}) = H(z)\Big|_{z = e^{j\Omega}} = \frac{1 - e^{-j\Omega}}{1 - 2e^{-j\Omega}} = \frac{e^{j\Omega} - 1}{e^{j\Omega} - 2}}$$

ii. $\xrightarrow{u[n]}$ $\boxed{h[n]}$ $\xrightarrow{s[n]}$

+3

$U(z)$, $|z| > 1$ $\qquad$ $H(z)$, $|z| < 2$

$S(z) = U(z) H(z)$, ROC: at least $\{|z| > 1\} \cap \{|z| < 2\} = 1 < |z| < 2$

$$\Rightarrow S(z) = \frac{1}{\cancel{1 - z^{-1}}} \cdot \frac{\cancel{1 - z^{-1}}}{1 - 2z^{-1}} = \frac{1}{1 - 2z^{-1}}, \quad |z| < 2$$

(pole-zero cancell. occured)

$$\Rightarrow \boxed{s[n] = -2^n u[-n-1].}$$