EEE 321: Signals and Systems

Recitation 2: Linear Time-Invariant Systems

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Discrete-Time LTI Systems & Convolution

Representation of DT Signals

Any discrete-time signal $x[n]$ can be represented as a sum of scaled and shifted unit impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The Convolution Sum

The output $y[n]$ of a discrete-time LTI system is the convolution of the input signal $x[n]$ with the system's impulse response $h[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Example: Discrete-Time Convolution

Question: Let $x[n] = u[n]$ and $h[n] = a^n u[n]$ with $|a| < 1$. Find the output $y[n] = x[n] * h[n]$.

Solution: Discrete-Time Convolution

We use the convolution sum formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Substitute $x[k] = u[k]$ and $h[n-k] = a^{n-k}u[n-k]$:

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]a^{n-k}u[n-k]$$

The term $u[k]$ is non-zero only for $k \geq 0$. The term $u[n-k]$ is non-zero only for $n - k \geq 0 \implies k \leq n$. So, for $n \geq 0$, the sum becomes:

$$y[n] = \sum_{k=0}^{n} a^{n-k} = a^n \sum_{k=0}^{n} (a^{-1})^k = a^n \frac{(a^{-1})^{n+1} - 1}{a^{-1} - 1}$$

$$= a^n \frac{a^{-n-1} - 1}{(1-a)/a} = a^n \frac{(1-a^{n+1})/a^{n+1}}{(1-a)/a} = \frac{1-a^{n+1}}{1-a}$$

For $n < 0$, the sum. range is empty, so $y[n] = 0$. Then the final result is:

$$y[n] = \left(\frac{1-a^{n+1}}{1-a}\right) u[n]$$

Continuous-Time LTI Systems & Convolution

Representation of CT Signals

Any continuous-time signal $x(t)$ can be represented as an integral of scaled and shifted unit impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The Convolution Integral

The output $y(t)$ of a continuous-time LTI system is the convolution of the input signal $x(t)$ with the system's impulse response $h(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Example: Continuous-Time Convolution

Question (based on Example 3): A continuous-time LTI system has an impulse response $h(t) = e^{-at}u(t)$ for $a > 0$. Find the system's output $y(t)$ for the input signal $x(t) = u(t) - u(t - T)$.

Solution: Continuous-Time Convolution

We use the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} [u(\tau) - u(\tau - T)]e^{-a(t-\tau)}u(t-\tau)d\tau$$

The integrand is non-zero for $0 < \tau < T$ and $t - \tau > 0 \implies \tau < t$.
Case 1: $t < 0$ The condition $\tau < t$ and $0 < \tau < T$ cannot be simultaneously met. So, $y(t) = 0$.
Case 2: $0 \leq t \leq T$ The integration is from $\tau = 0$ to $\tau = t$.

$$y(t) = \int_0^t e^{-a(t-\tau)}d\tau = e^{-at}\int_0^t e^{a\tau}d\tau = e^{-at}\left[\frac{e^{a\tau}}{a}\right]_0^t = \frac{1-e^{-at}}{a}$$

Case 3: $t > T$ The integration is from $\tau = 0$ to $\tau = T$.

$$y(t) = \int_0^T e^{-a(t-\tau)}d\tau = e^{-at}\int_0^T e^{a\tau}d\tau = e^{-at}\left[\frac{e^{a\tau}}{a}\right]_0^T = \frac{e^{aT}-1}{a}e^{-at}$$

Properties of LTI Systems

- **Linearity:** A system is linear if it satisfies the superposition principle.

$$T(ax_1(t) + bx_2(t)) = aT(x_1(t)) + bT(x_2(t))$$

- **Time-Invariance:** A system is time-invariant if a time shift in the input signal causes the same time shift in the output signal.

$$y(t - t_0) \text{ is the output for } x(t - t_0)$$

- **Causality:** A system is causal if the output at any time depends only on present and past values of the input. For an LTI system, this means:

$$h(t) = 0 \text{ for } t < 0 \quad \text{and} \quad h[n] = 0 \text{ for } n < 0$$

- **Stability (BIBO):** A system is Bounded-Input, Bounded-Output (BIBO) stable if every bounded input produces a bounded output. For an LTI system, this means the impulse response must be absolutely integrable/summable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Example Question 1 (Fall 21 MT)

Consider the system defined by

$$y(t) = (t+1)x(t) + t^2 x(t-1)$$

(a) Is this system continuous-time or discrete-time?
(b) Is this system linear? Justify your answer.
(c) Is this system time-invariant? Justify your answer.
(d) Is this system causal? Justify your answer.

Solution for Example 1

$$y(t) = (t+1)x(t) + t^2 x(t-1)$$

(a) Continuous-time or discrete-time? The signals are functions of a continuous variable t, so the system is **continuous-time**.

(b) Linearity? Let $x(t) = ax_1(t) + bx_2(t)$.

$$\begin{aligned} y(t) &= (t+1)[ax_1(t) + bx_2(t)] + t^2[ax_1(t-1) + bx_2(t-1)] \\ &= a[(t+1)x_1(t) + t^2 x_1(t-1)] + b[(t+1)x_2(t) + t^2 x_2(t-1)] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

The system satisfies superposition. Thus, it is **linear**.

Solution for Example 1 (Cont.)

(c) Time-Invariance? Let the output for a shifted input $x(t - t_0)$ be $y_{t_0}(t)$.

$$y_{t_0}(t) = (t + 1)x(t - t_0) + t^2 x(t - t_0 - 1)$$

Now, let's shift the original output $y(t)$ by t_0:

$$y(t - t_0) = (t - t_0 + 1)x(t - t_0) + (t - t_0)^2 x(t - t_0 - 1)$$

Since $y_{t_0}(t) \neq y(t - t_0)$, the system is **not time-invariant**.

(d) Causality? The output $y(t)$ at time t depends on the input $x(t)$ (present) and $x(t - 1)$ (past). It does not depend on future values of the input. Thus, the system is **causal**.

LTI Systems Described by Difference Equations

A general Nth-order linear constant-coefficient difference equation is:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

These systems are causal and LTI. They can be solved recursively.

Example Question 2

Consider the following causal LTI discrete-time system:

$$y[n] - ay[n-1] = x[n] - bx[n-1]$$

(a) Find the impulse response $h[n]$ and state the constraints on a and b for this system to be stable.
(b) For the stable system, if $a = 1/b$, find the output $y[n]$ for $x[n] = e^{j\Omega_1 n} + e^{j\Omega_2 n}$.

Solution for Example 2(a): Impulse Response

To find the impulse response $h[n]$, we set $x[n] = \delta[n]$, so $y[n] = h[n]$.

$$h[n] - ah[n-1] = \delta[n] - b\delta[n-1]$$

For $n = 0$: $h[0] - ah[-1] = \delta[0] - b\delta[-1]$. With causality, $h[-1] = 0$, so $h[0] = 1$. For $n = 1$: $h[1] - ah[0] = \delta[1] - b\delta[0] \implies h[1] - a(1) = -b \implies h[1] = a - b$. For $n \geq 1$, the equation is homogeneous: $h[n] - ah[n-1] = 0$. The solution is of the form $h[n] = Ca^n$. Using $h[1] = a - b$, we get $Ca^1 = a - b \implies C = 1 - b/a$. So for $n \geq 1$, $h[n] = (1 - b/a)a^n = a^n - ba^{n-1}$. Let's check for $n = 1$: $h[1] = a - b$. Correct. Let's combine results. A simpler way: The impulse response is the sum of the responses to $\delta[n]$ and $-b\delta[n-1]$. The response to $\delta[n]$ is $a^n u[n]$. The response to $-b\delta[n-1]$ is $-ba^{n-1}u[n-1]$. So, $h[n] = a^n u[n] - ba^{n-1}u[n-1]$.

For BIBO stability, $\sum |h[n]| < \infty$. This requires $|a| < 1$. There is no constraint on b.

Solution for Example 2(b): Output

We know that for an LTI system, the response to an complex exponential input $e^{j\Omega n}$ is $H(e^{j\Omega})e^{j\Omega n}$, where $H(e^{j\Omega})$ is the frequency response.

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

From the difference equation:

$$Y(e^{j\Omega}) - ae^{-j\Omega}Y(e^{j\Omega}) = X(e^{j\Omega}) - be^{-j\Omega}X(e^{j\Omega})$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1 - be^{-j\Omega}}{1 - ae^{-j\Omega}}$$

Given $a = 1/b$, then $b = 1/a$.

$$H(e^{j\Omega}) = \frac{1 - (1/a)e^{-j\Omega}}{1 - ae^{-j\Omega}} = \frac{e^{-j\Omega}(e^{j\Omega} - 1/a)}{1 - ae^{-j\Omega}} = -\frac{1}{a}e^{-j\Omega}$$

The input is $x[n] = e^{j\Omega_1 n} + e^{j\Omega_2 n}$. By linearity, the output is the sum of the responses to each term.

$$\begin{aligned} y[n] &= H(e^{j\Omega_1})e^{j\Omega_1 n} + H(e^{j\Omega_2})e^{j\Omega_2 n} \\ &= -\frac{1}{a}e^{-j\Omega_1}e^{j\Omega_1 n} - \frac{1}{a}e^{-j\Omega_2}e^{j\Omega_2 n} \end{aligned}$$

LTI Systems Described by Differential Equations

A general Nth-order linear constant-coefficient differential equation is:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

These systems are causal and LTI. Solving them requires finding both the homogeneous and particular solutions.

Example: Differential Equation

Question: Consider a causal LTI system described by the differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

(a) Find the impulse response $h(t)$ of the system.
(b) Find the output $y(t)$ if the input is $x(t) = e^{-t}u(t)$ and the system is initially at rest.

Solution: Differential Equation

(a) Impulse Response We need to solve $\frac{dh(t)}{dt} + 2h(t) = \delta(t)$. For $t > 0$, this is a homogeneous equation: $\frac{dh(t)}{dt} + 2h(t) = 0$. The solution is of the form $h(t) = Ce^{-2t}$. To find C, we integrate the equation from 0^- to 0^+:

$$\int_{0^-}^{0^+} \frac{dh(\tau)}{d\tau} d\tau + \int_{0^-}^{0^+} 2h(\tau) d\tau = \int_{0^-}^{0^+} \delta(\tau) d\tau$$

$$[h(0^+) - h(0^-)] + 0 = 1$$

Assuming causality and initial rest, $h(0^-) = 0$, so $h(0^+) = 1$. Using $h(t) = Ce^{-2t}$, we get $h(0^+) = Ce^0 = C = 1$. So, the impulse response is $h(t) = e^{-2t}u(t)$.

Solution: Differential Equation (Cont.)

(b) Output for $x(t) = e^{-t}u(t)$ Since the system is LTI, we can find the output by convolving the input with the impulse response.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

For $t > 0$:

$$\begin{aligned}
y(t) &= \int_0^t e^{-\tau}u(\tau) \cdot e^{-2(t-\tau)}u(t-\tau)d\tau \\
&= \int_0^t e^{-\tau}e^{-2t}e^{2\tau}d\tau \\
&= e^{-2t}\int_0^t e^{\tau}d\tau \\
&= e^{-2t}[e^{\tau}]_0^t \\
&= e^{-2t}(e^t - 1) \\
&= e^{-t} - e^{-2t}
\end{aligned}$$

So, $y(t) = (e^{-t} - e^{-2t})u(t)$.