# Recitation 3: LTI Systems and Continuous-Time Fourier Series

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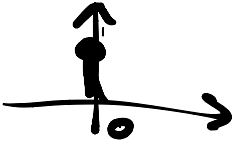
October 15, 2025

# System Properties from the Impulse Response




**Reminders:**

- Memoryless: $h(t) = C\delta(t)$ or $h[n] = C\delta[n]$
- Causal: $h(t) = 0$ for $t < 0$, $h[n] = 0$ for $n < 0$
- Stable: $\int |h(\tau)| d\tau < \infty$ or $\sum |h[k]| < \infty$

finite

# Example 1: $h[n] = (0.8)^n u[n]$

$h[1] = 0.8$
$h[2] = 0.64$
$h[0] = 1$
$h[-1] = 0$

For $n<0$, $h[n] = 0$

$h[n] = K\,\delta[n]$

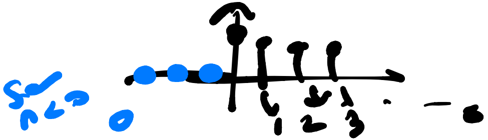

**Check Properties:**

- Memoryless? No ($h[1]$, $h[2]$, ... nonzero)
- Causal? Yes ($h[n] = 0$ for $n < 0$)
- Stable? Yes, $\sum (0.8)^n = 5 < \infty$

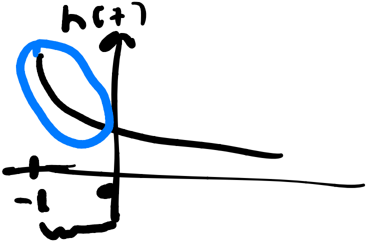
$$\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=-\infty}^{\infty} 0.8^n u[n] = \sum_{n=0}^{\infty} 0.8^n$$

$$= \frac{1}{1-0.8} = 5 < \infty$$

finite



# Example 2a: $h(t) = e^{-3t}u(t+1)$

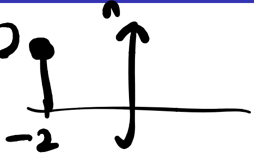



- Memoryless? No
- Causal? No
- Stable? Yes

$h(t) \neq 0$ for $-1 < t < 0$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-1}^{\infty} e^{-3t}\, dt = \frac{e^{3}}{3} < \infty$$

$h[n] = \delta[n+2]$




**2b**

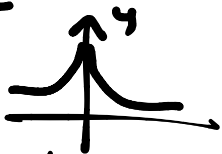
- Memoryless? No
- Causal? No
- Stable? 1

$$\sum_{-\infty}^{\infty} h[n] = 1$$

**2c**

No $h(t) = e^{-2|t|}$

- Memoryless?
- Causal? No
- Stable? Yes




$$\int_{-\infty}^{\infty} e^{-2|t|}dt = 2\int_{0}^{\infty} e^{-2t}dt = 2 \cdot \frac{1}{2} = 1 < \infty$$

# Systems Described by LCCDEs

## What are they?

A fundamental way to describe a huge class of LTI systems is with a Linear Constant-Coefficient Differential (or Difference) Equation. They represent the relationship between the input $x$ and output $y$.

## Continuous-Time

$y(t)+\cdots = x(t)+x'(t)+x''\cdots$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

- The left side (y-terms) describes the **system's characteristics**.
- The right side (x-terms) describes how the **input drives** the system.

## Discrete-Time

$2y[n]+3y[n-1] = x[n]+4x[n-1]$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- The left side is the **recursive** part (feedback).
- The right side is the **non-recursive** part (feed-forward).

The system is at rest initially

$y[-1] = 0$

**Discrete-Time Example:**

$n = 0$

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$y[0] = \frac{1}{2}y[-1] + \underbrace{\delta[0]}_{1}$

- Impulse Response:
- Stable? Yes
- Signal Flow Graph:

$y[0] = 1$

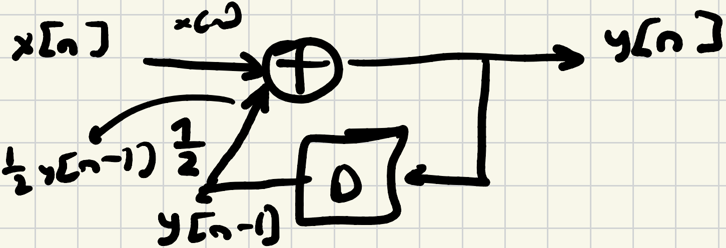
$n = 1$ $\quad y[1] = \frac{1}{2}y[0] + \cancel{\delta[1]}^{0}$

$y[1] = \frac{1}{2}y[0] = \frac{1}{2}$

$h[n] = y[n] = \left(\frac{1}{2}\right)^n u[n].$

$\sum_{-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n] = 2 < \infty$

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$



# LCCDEs and Difference Equations

The system is at rest

For $n = 0$

$$y[0] = \underbrace{\tfrac{1}{4}y[-1]}_{0} + \overbrace{\delta[0]}^{1} + 2\delta[-1]$$

$$y[0] = 1$$

**Discrete-Time Example:**

$n = 1$

$$y[n] - \frac{1}{4}y[n-1] = x[n] + 2x[n-1]$$

$$y[1] = \frac{1}{4}y[0] + \delta[1] + 2\delta[0] = \frac{9}{4}$$

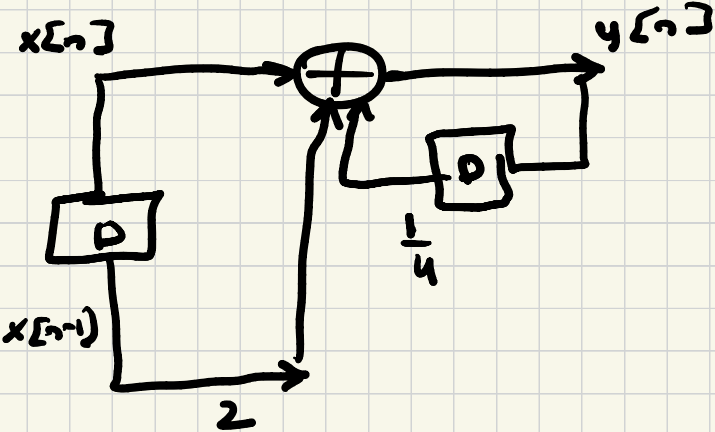
$$h[n] = \delta[n] + \frac{9}{4}\delta[n-1] + \frac{9}{4}\left(\frac{1}{4}\right)^{n-1}u[n-2]$$

- Impulse Response:
- Stable? Yes
- Signal Flow Graph:

$$y[2] = \frac{1}{4}y[1] + \delta[2] + 2\delta[1]$$

$$y[3] = \frac{1}{4}y[2]$$

$$y[n] = x[n] + 2x[n-1] + \frac{1}{4}y[n-1]$$



## Continuous-Time Example:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

**Impulse Response:**

$$h'(t) + 2h(t) = \delta(t)$$

For $t \neq 0$, $h'(t) + 2h(t) = 0 \Rightarrow h(t) = Ce^{-2t}$. Integrating around $t = 0$:

$$h(0^+) - h(0^-) = 1$$

Causality gives $h(0^-) = 0 \Rightarrow h(0^+) = 1 \Rightarrow C = 1$. Hence,

$$\boxed{h(t) = e^{-2t}u(t)}$$

**Stable?** Yes, since $\int_0^\infty |e^{-2t}|dt = \frac{1}{2} < \infty$

# Fourier Series: The Core

## The Main Idea

Any "well-behaved" periodic signal can be decomposed into a sum of harmonically related sinusoids or complex exponentials.

## The Complex Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{(Synthesis)}$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \quad \text{(Analysis)}$$

**Key Components:**

- $T$: Fundamental Period of the signal.
- $\omega_0 = 2\pi/T$: Fundamental Frequency
- $k$: An integer, called the harmonic index.
- $a_k$: Fourier coefficient for the $k$-th harmonic.

# Fourier Series: Key Properties

## The DC Component (k=0)

The $a_0$ coefficient is special. It represents the **average (DC) value** of the signal over one period.

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

average

## Property for Real Signals: Conjugate Symmetry

If $x(t)$ is a real-valued signal, its coefficients are always conjugate symmetric:

$$a_k = a^*_{-k}$$

This implies:

- Magnitudes are even: $|a_k| = |a_{-k}|$
- Phases are odd: $\angle a_k = -\angle a_{-k}$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$a_{-k} = \frac{1}{T} \int_T x(t) e^{jk\omega_0 t} dt$$

$$a_{-k}^* = \frac{1}{T} \int_T x^*(t) e^{-jk\omega_0 t} dt$$

$$a_k = a_{-k}^*$$

# Fourier Series: Key Properties

## Differentiation Property

If $y(t) = \frac{dx(t)}{dt}$, their coefficients are related by:

$$\boxed{b_k = (jk\omega_0)a_k}$$

This is a powerful tool to simplify calculations.

## Parseval's Relation (Power)

This relates time-domain power to frequency-domain power. Average Power $P$:

$$P = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$x(t) = \sum_{k=-\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k \frac{d e^{jk\omega_0 t}}{dt}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k jk\omega_0}_{b_k} e^{jk\omega_0 t}$$

$$b_k = a_k jk\omega_0$$

# Periodicity and FS Representation

**Given:** $x(t) = 2\cos(\pi t - \pi/3) - \sin(\frac{5\pi}{3}t + \pi/4)$

$x(t) = x(t+T)$ for all $t$, $T$ period

**Check periodicity:**

$$x(t) = 2\frac{1}{2}\left[e^{j3\frac{2\pi}{6}t - j\frac{\pi}{3}} + e^{-j3\frac{2\pi}{6}t + j\frac{\pi}{3}}\right] - \frac{1}{2j}\left[e^{j5\frac{2\pi}{6}t + j\frac{\pi}{4}} - e^{-j5\frac{2\pi}{6}t - j\frac{\pi}{4}}\right]$$

$$x(t) = \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j5\frac{2\pi}{6}t} + e^{j\frac{\pi}{3}}e^{-j3\frac{2\pi}{6}t} + e^{-j\frac{\pi}{3}}e^{j3\frac{2\pi}{6}t} - \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j5\frac{2\pi}{6}t} \qquad (2)$$

$$\boxed{\begin{aligned} a_3 &= e^{-j\pi/3}, & a_{-3} &= e^{j\pi/3}, \\ a_5 &= \frac{e^{j3\pi/4}}{2}, & a_{-5} &= \frac{e^{-j3\pi/4}}{2}. \end{aligned}}$$

conjugate symmetric

Euler formula

$\cos(t) = \dfrac{e^{jt} + e^{-jt}}{2}$ $\qquad$ $\sin(t) = \dfrac{e^{jt} - e^{-jt}}{2j}$

$$x(t+T) = 2\cos\left(\pi(t+T) - \frac{\pi}{3}\right) + \sin\left(\frac{5\pi}{3}(t+T) + \frac{\pi}{4}\right)$$

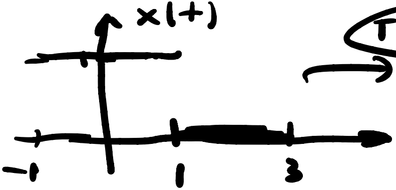
$$\pi T = 2k\pi, \quad \frac{5\pi}{3} T = 2r\pi$$

$$k, r \in \mathbb{Z}$$

$$\frac{k}{r} = \frac{3}{5} \qquad k = 3, r = 5, \quad T = 6$$

fundamental period

# Square Wave ($T = 4$)




$T = 4$

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$x(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & 1 < t < 3 \end{cases}$$

- $\omega_0 = \pi/2$
- $a_0 = 1/2$
- $a_k = \frac{\sin(k\pi/2)}{k\pi}$

$$a_0 = \frac{1}{2}$$

For $k \neq 0$,

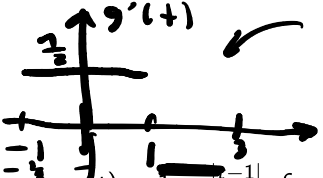
$$a_k = \frac{1}{T} \int_{-1}^{3} x(t) e^{-jk\frac{\pi}{2}t} dt$$

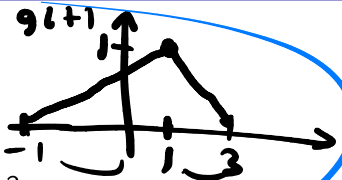
$$= \frac{1}{4} \int_{-1}^{1} e^{-jk\frac{\pi}{2}t} dt = \frac{1}{4} \left. \frac{e^{-jk\frac{\pi}{2}t}}{-jk\frac{\pi}{2}} \right|_{-1}^{1}$$

$$\frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right)$$

$$= \frac{1}{4} \frac{e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}}{-jk\frac{\pi}{2}} = \frac{1}{4} \frac{-2j\sin\left(k\frac{\pi}{2}\right)}{-jk\frac{\pi}{2}} = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}$$

# Triangular Wave $g(t)$




- $g(t) = 1 - \frac{|t-1|}{2}$, for $-1 < t < 3$
- Derivative $y(t) = g'(t)$ is a square wave.
- $b_k = \frac{\sin(k\pi/2)}{2k\pi}$
- $c_k = \frac{\sin(k\pi/2)}{jk^2\pi^2}$
- $c_0 = 1/2$

$\boxed{g'(t) = x(t) - \frac{1}{2}}$

$b_k = a_k$ for all $k$ other than 0

$b_0 = a_0 - \frac{1}{2}$

$\frac{4 \times 1}{2} = 2$

$\frac{\pi}{2}$

$c_0$ average $\frac{2}{4} = \frac{1}{2}$

$g(t) \rightarrow c_k$

$b_k = c_k jk\omega_0$

$c_k = \frac{b_k}{jk\frac{\pi}{2}} \Rightarrow c_k = \frac{\sin(\frac{k\pi}{2})}{2k\pi \, jk\frac{\pi}{2}}$

# A Fourier Series Puzzle

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$.

$$w_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

# A Fourier Series Puzzle

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$.
- Its Fourier series coefficients, $a_k$, are zero for all $|k| > 1$.

$a_{-1}, a_0, a_1$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{\pi}{3}$$

$$= a_{-1} e^{-j\frac{\pi}{3}t} + a_0 + a_1 e^{j\frac{\pi}{3}t}$$

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$.
- Its Fourier series coefficients, $a_k$, are zero for all $|k| > 1$.
- The derivative of the signal, $y(t) = x'(t)$, is an odd signal.

$x(t)$ real $\rightarrow$ $y(t)$ real

$y(t)$ real and odd $\longrightarrow$ $b_k$ purely imaginary and odd

$b_k = jA \quad b_k = -b_{-k}$

$$\boxed{b_k = j\underbrace{k\frac{\pi}{3}a_k}_{\text{real}}}$$

$a_k$ should be real

$a_1 = a_{-1} = A$

# A Fourier Series Puzzle

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$.
- Its Fourier series coefficients, $a_k$, are zero for all $|k| > 1$.
- The derivative of the signal, $y(t) = x'(t)$, is an odd signal.
- The total average power of the signal is 3.

$$a_{-1}^2 + a_0^2 + a_1^2 = 3$$

$$a_{-1}^2 = a_1^2 = A^2$$

$$\boxed{A^2 + a_0^2 = 3} \quad (1)$$

# A Fourier Series Puzzle

$x(t) = \underbrace{e^{-j\frac{2\pi}{6}t} + e^{j\frac{2\pi}{6}t}}_{2\cos\left(\frac{\pi}{3}t\right)} + 1 = \underbrace{2\cos\left(\frac{\pi}{3}t\right) + 1}$

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$. ✓
- Its Fourier series coefficients, $a_k$, are zero for all $|k| > 1$. ✓
- The derivative of the signal, $y(t) = x'(t)$, is an odd signal. ✓
- The total average power of the signal is 3. ✓
- The value of the signal at time $t = 0$ is $x(0) = 3$. ✓

$x(t) = a_{-1} e^{-j\frac{2\pi}{6}t} + a_1 e^{j\frac{2\pi}{6}t} + a_0$

$x(0) = a_{-1} + a_1 + a_0 = 2A + a_0 = 3$

we can solve, we have 2 unknowns (2) and 2 equation

$A = a_0 = 1$

# A Fourier Series Puzzle

**A real periodic signal, $x(t)$, has the following properties:**

- The fundamental period is $T = 6$.
- Its Fourier series coefficients, $a_k$, are zero for all $|k| > 1$.
- The derivative of the signal, $y(t) = x'(t)$, is an odd signal.
- The total average power of the signal is 3.
- The value of the signal at time $t = 0$ is $x(0) = 3$.

**Find the signal $x(t)$.**