Recitation 4: Continuous-Time Fourier Transform (CTFT)

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CTFT: Definition

Analysis (Forward Transform):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$e^{-\alpha t} u(t)$

Synthesis (Inverse Transform):

$$x(t) = \boxed{\frac{1}{2\pi}} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

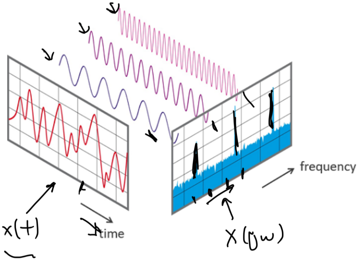
$\frac{\omega}{2\pi} = f$

Interpretation: Frequency-domain function $X(j\omega)$ describes how much of each complex exponential $e^{j\omega t}$ is in $x(t)$.

Time vs Frequency Domain (CTFT Intuition)

CTFT as a Bridge Between Two Views:

- **Time domain (left)**: signal behavior over time.
- **Frequency domain (right)**: reveals the spectral (frequency) content.
- Each spike in the frequency domain represents a sinusoidal component in time.
- CTFT breaks down a signal into continuous sinusoids.



CTFT: Common Transform Pairs

Time-Domain $x(t)$	**Frequency-Domain** $X(j\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{a+j\omega}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

CTFT: Key Properties

- **Linearity:** $ax_1(t) + bx_2(t) \leftrightarrow aX_1(j\omega) + bX_2(j\omega)$
- **Time Shift:** $x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$
- **Frequency Shift (Modulation):** $e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega - \omega_0))$
- **Time Scaling:** $x(at) \leftrightarrow \boxed{\frac{1}{|a|}}X\left(\frac{j\omega}{a}\right)$
- **Time Reversal:** $x(\underset{a=-1}{-t}) \leftrightarrow X(-j\omega)$
- **Differentiation in Time:** $\frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(j\omega)$
- **Multiplication in Time:** $t^n x(t) \leftrightarrow j^n \frac{d^n X(j\omega)}{d\omega^n}$
- **Convolution in Time:** $x(t) * h(t) \leftrightarrow X(j\omega)H(j\omega)$
- **Multiplication in Time $\leftrightarrow$ Convolution in Frequency:**

$$x(t)h(t) \leftrightarrow \frac{1}{2\pi}X(j\omega) * H(j\omega)$$

- **Parseval's Relation:**

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{Energy of } x(t)} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}_{\text{Energy of } x(t)}$$

Example: CTFT of Rectangular Pulse

Let:

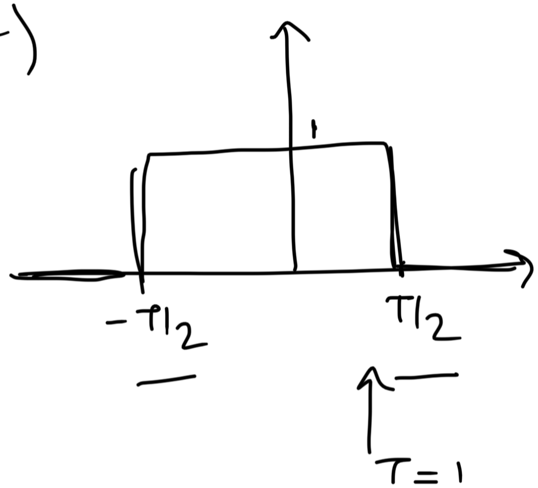
$$x(t) = \begin{cases} 1, & |t| \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

Then:

$$X(j\omega) = \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t} dt = \boxed{T \cdot \text{sinc}\left(\frac{\omega T}{2\pi}\right)}$$

$$\text{where } \boxed{\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}}$$

$x(t)$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} 1 .\, e^{-j\omega t}\, dt$$

$$= \left.\frac{e^{-j\omega t}}{j\omega}\right|_{T/2}^{-T/2}$$

$$= \boxed{\frac{2\left(e^{j\omega T/2} - e^{-j\omega T/2}\right)}{2j\omega}}$$

$$= \frac{2\sin(\omega T/2)}{\omega}$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$x = \frac{\omega T}{2\pi}$$

$$T\,\operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) = \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} = \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}$$

$$\boxed{= T\,\operatorname{sinc}\left(\frac{\omega T}{2\pi}\right)}$$

Practice: Fourier Transforms

Compute the Fourier Transform of each signal:

(a) $[e^{-\alpha t}\cos(\omega_0 t)]u(t), \quad \alpha > 0$

(b) $e^{t+2}u(-t+1)$

(d) $e^{-t}\,[u(t+2) - u(t-3)]$

a) $\underbrace{e^{-\alpha t} u(t)}_{x(t)} \cdot \underbrace{\cos(w_0 t)}_{h(t)} \longrightarrow \frac{1}{2\pi} X(jw) * H(jw)$

$$\frac{1}{2\cancel{\pi}} \left[\frac{1}{\alpha + jw} * \left[\cancel{\pi}\, \delta(w - w_0) + \cancel{\pi}\, \delta(w + w_0) \right] \right]$$

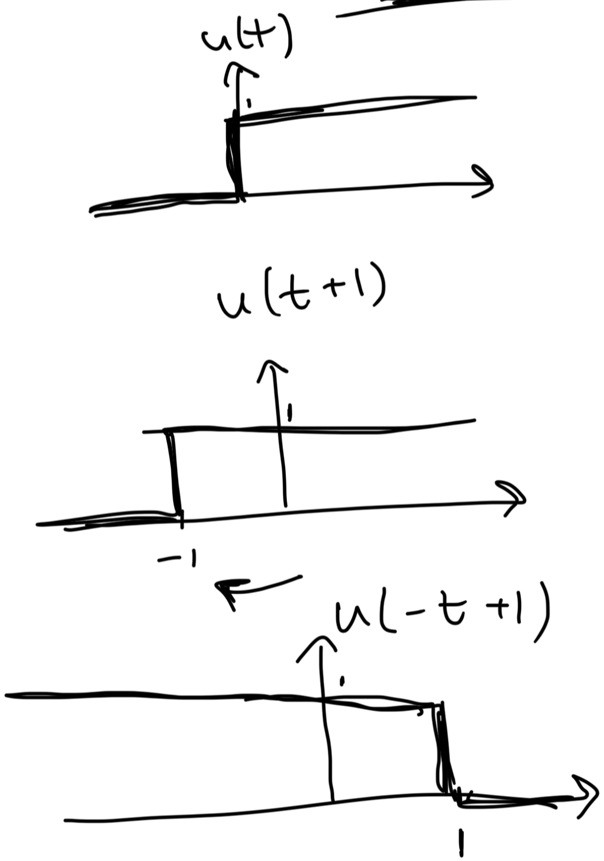
$$\frac{1}{2} \left[\frac{1}{\alpha + j(w - w_0)} + \frac{1}{\alpha + j(w + w_0)} \right]$$

$(\alpha + j(w + w_0))$ $(\alpha + j(w - w_0))$

$$\frac{1}{2} \left[\frac{\alpha + jw + \cancel{jw_0} + \alpha + jw - \cancel{jw_0}}{\alpha^2 + \alpha jw - \cancel{\alpha jw_0} + \alpha jw + \cancel{\alpha jw_0} + \underset{-1}{j^2}(w - w_0)(w + w_0)} \right]$$

$$\left[\frac{(\alpha+j\omega)}{\alpha^2+2\alpha j\omega-\omega^2+\omega_0^2}\right]$$

b) $e^{t+2}\,\underline{u(-t+1)} \xrightarrow{FT} \,?$

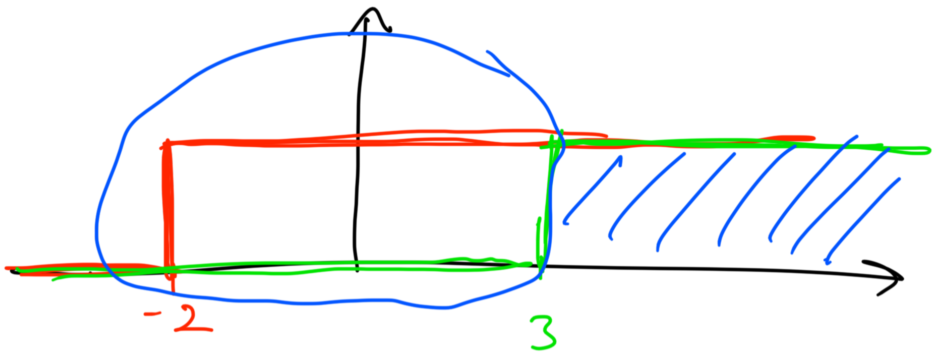


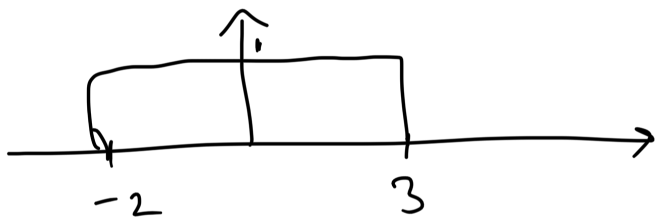
$$\int_{-\infty}^{\infty} e^{t+2}\,\underline{u(-t+1)}\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{1} 1 \cdot e^{t} \cdot e^{2} \cdot e^{-j\omega t}\,dt = e^2 \int_{-\infty}^{1} e^{t(1-j\omega)}\,dt$$

$$= e^2 \left. \frac{e^{t(1-j\omega)}}{1-j\omega} \right|_{-\infty}^{1} = e^2 \left[\frac{e^{1-j\omega}}{1-j\omega}\right]$$

d) $e^{-t}\left[u(t+2) - u(t-3)\right]$





$$\int_{-2}^{3} e^{-t} e^{-j\omega t}\, dt$$

$$= \int_{-2}^{3} e^{t(-1-j\omega)}\, dt$$

$$= \left.\frac{e^{t(-1-j\omega)}}{-1-j\omega}\right|_{-2}^{3}$$

$$= \frac{e^{3(-1-j\omega)} - e^{-2(-1-j\omega)}}{-1-j\omega}$$

System with impulse response $h(t) = e^{-2t}u(t)$

(a) Find the **frequency response** $H(j\omega)$ of the system and express it in **magnitude-phase form**.

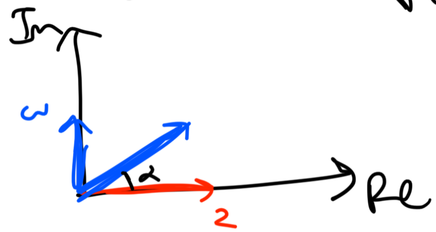
(b) Find the output $y(t)$ for the input $x(t) = e^{j(\pi t + \pi/2)}$

(c) Find the output $y(t)$ for the input $x(t) = \overbrace{\cos(\pi t + \pi/2)}^{x_1(t)} + \overbrace{\delta(t-1)}^{x_2(t)}$

$x(t) * h(t) = y(t)$

$X(j\omega) H(j\omega) = Y(j\omega) \rightarrow y(t)$

a) $H(j\omega) = \frac{1}{2+j\omega} = \frac{1}{\sqrt{\omega^2+4}\; e^{j\arctan(\omega/2)}} = \frac{1}{\sqrt{\omega^2+4}}\, e^{j\arctan(-\omega/2)}$



$$|H(j\omega)|\, e^{j\angle H(j\omega)}$$

$$\angle H(j\omega) = \arctan(-\omega/2)$$

b) $e^{j\pi t}\, \overbrace{e^{j\frac{\pi}{2}}}^{j} = j e^{j\pi t} \longrightarrow j2\pi\,\delta(\omega-\pi) = X(j\omega)$

$$y(j\omega) = X(j\omega)H(j\omega) = j2\pi\,\delta(\omega-\pi)\overset{a(j\omega)}{\frac{1}{2+j\omega}} = \frac{j2\pi}{2+j\pi}\,\delta(\omega-\pi)$$

Sifting Property:

$$\delta(t-t_0)\,x(t) = x(t_0)\,\delta(t-t_0)$$

$$\delta(\omega-\pi)\,a(j\omega) = \delta(\omega-\pi)\,\frac{1}{2+j\pi}$$

$$y(t) = \frac{j}{2+j\pi}\, e^{j\pi t}$$

c) $x(t) = \underbrace{\cos(\pi t + \pi/2)}_{x_1(t)} + \underbrace{\delta(t-1)}_{x_2(t)}$

$x_2(t) * h(t)$

$x(t) * h(t) = \underbrace{x_1(t) * h(t)}_{y_1(t)} + \underbrace{x_2(t) * h(t)}_{y_2(t)}$

$x_1(t) = \frac{1}{2} e^{j\pi t} \underbrace{e^{j\frac{\pi}{2}}}_{j} + \frac{1}{2} e^{-j\pi t} \underbrace{e^{-j\frac{\pi}{2}}}_{-j} = \frac{j}{2} e^{j\pi t} - \frac{j}{2} e^{-j\pi t}$

$X_1(j\omega) = \frac{j}{\cancel{2}} \cancel{2}\pi \delta(\omega - \pi) - \frac{j}{\cancel{2}} \cancel{2}\pi \delta(\omega + \pi)$

$H(j\omega) = \frac{1}{2 + j\omega}$

$Y_1(j\omega) = \frac{j\pi}{2 + j\omega} \left[\delta(\omega - \pi) - \delta(\omega + \pi) \right]$

$\pi, -\pi$

$= \frac{j\pi 2}{(2 + j\pi)2} \delta(\omega - \pi) - \frac{j\pi 2}{(2 - j\pi)2} \delta(\omega + \pi)$

$$y_1(t) = \frac{j}{4+j2\pi} e^{j\pi t} - \frac{j}{4-2j\pi} e^{-j\pi t}$$

$$x_2(t) = \delta(t-1) \quad \Rightarrow \quad x_2(t) * h(t) = e^{-2(t-1)} u(t-1)$$

$$h(t) = e^{-2t} u(t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$= \frac{j}{4+j2\pi} e^{j\pi t} - \frac{j}{4-j2\pi} e^{-j\pi t} + e^{-2(t-1)} u(t-1)$$

Transform Properties: Time Scaling & Shifting

Given the Fourier transform $X(j\omega)$ of a signal $x(t)$, find the Fourier transform $Y(j\omega)$ of the signal $y(t)$ defined as:

(a) $y(t) = x\left(\frac{t-5}{2}\right)$

(b) $y(t) = t \cdot x\left(\frac{t}{2} - 3\right)$

(c) $y(t) = \frac{d^2}{dt^2}x(t)$

a) $x(t) \longrightarrow X(jw)$

$x\left(\frac{t}{2}\right) \Longrightarrow 2X(j2w)$

$x\left(\frac{t-5}{2}\right) \longrightarrow 2e^{-jw5}X(j2w)$

b) $x(t) \longrightarrow X(jw)$ $\qquad x(t-3) \longrightarrow e^{-jw3}X(jw)$

$\qquad\qquad 2w \qquad\qquad 2w$

$x\left(\frac{t}{2}-3\right) \longrightarrow 2e^{-j6w}X(j2w)$

$tx\left(\frac{t}{2}-3\right) \longrightarrow j\frac{d\left(2e^{-j6w}X(j2w)\right)}{dw}$

$= j2\left[\underbrace{-6je^{-j6w}}_{-12j^2}X(j2w) + \underbrace{2je^{-j6w}}_{4j^2}X'(j2w)\right]$

$= 12e^{-j6w}X(j2w) - 4e^{-j6w}X'(2jw)$

c) $x(t) \rightarrow X(j\omega)$

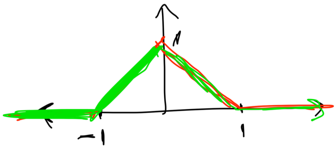
$$\frac{d^2 x(t)}{dt^2} \rightarrow (j\omega)^2 X(j\omega) = -\omega^2 X(j\omega)$$

Fourier Transform: Triangular Function

Compute the Continuous-Time Fourier Transform of the triangular function:

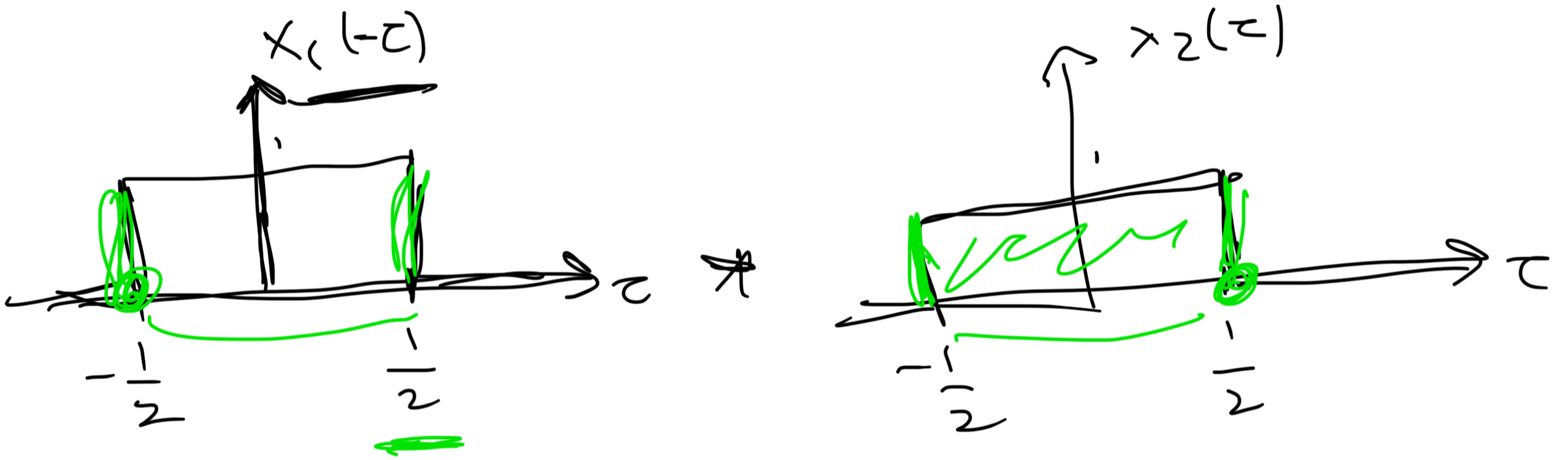
$$x(t) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Derive $X(j\omega)$.

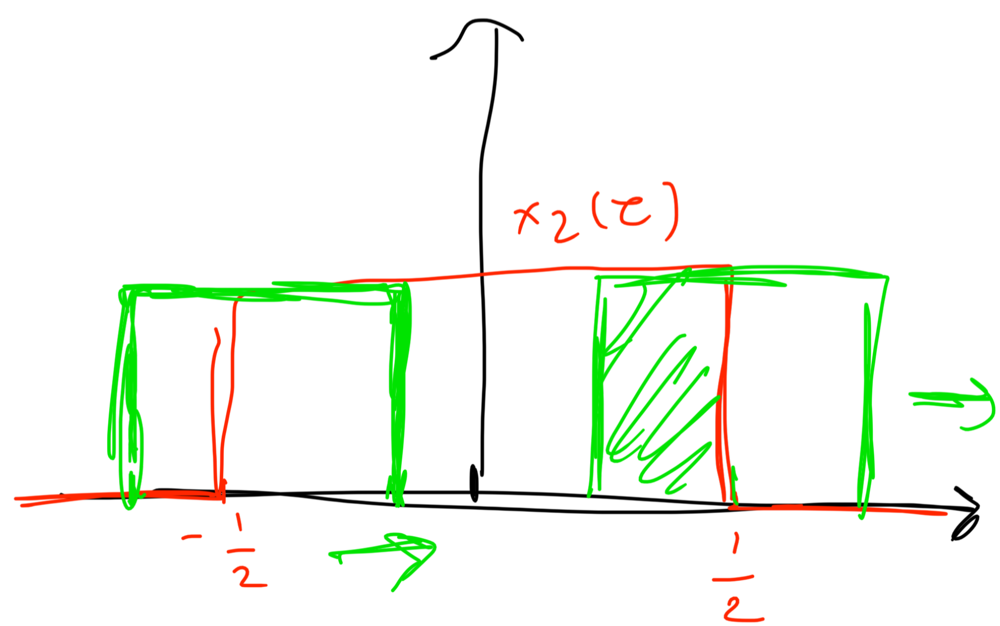


$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-1}^{0} (1+t)\, e^{-j\omega t}\, dt + \int_{0}^{1} (1-t)\, e^{-j\omega t}\, dt$$

it requires integration by parts



$$\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau =$$



$$\left\{ \begin{array}{ll} 0 \, , & t < -1 \\ 1+t \, , & 0 > t > -1 \\ 1-t \, , & 1 > t > 0 \\ 0 \, , & t > 1 \end{array} \right\}$$

$$a(t) = \begin{cases} 1, & |t| \leq 1/2 \\ 0, & \text{o.w.} \end{cases}$$

$$a(t) * a(t) = x(t)$$

$$A(jw)A(jw) = X(jw)$$

$$T\,\text{sinc}\left(\frac{wT}{2\pi}\right) \rightarrow T=1$$

$$A(jw) = \text{sinc}\left(\frac{w}{2\pi}\right)$$

$$X(jw) = \left[\text{sinc}\left(\frac{w}{2\pi}\right)\right]^2$$

Let $x(t)$ be a periodic signal with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

Fourier Series Representation:

$k\omega_0 \quad \omega_0, 2\omega_0, -\omega_0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,dt$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

$\leftarrow$ synthesis

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$c_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_0 t}\,dt$$

$\leftarrow$ analysis

Fourier Transform of a Periodic Signal:

$$e^{jk\omega_0 t} \downarrow 2\pi\,\delta(\omega - k\omega_0)$$

$$\boxed{X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k\delta(\omega - k\omega_0)}$$

Key Insight

The Fourier Transform of a periodic signal is a **train of impulses** in the frequency domain, located at harmonic frequencies $k\omega_0$, with amplitudes scaled by the Fourier Series coefficients c_k.

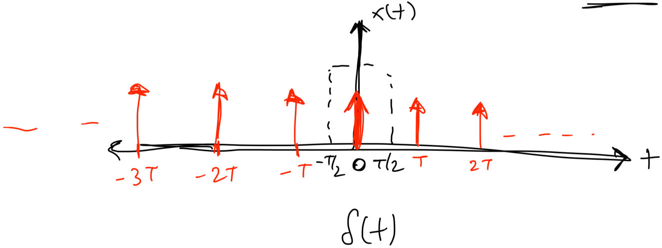
Question: CTFT of Impulse Train

Question:
Let $x(t)$ be a periodic impulse train defined as:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$n=-1$ $\delta(t+T)$
$n=0$ $\delta(t)$
$n=1$ $\delta(t-T)$

Find the Continuous-Time Fourier Transform (CTFT) of $x(t)$.



$$c_k = \frac{1}{T} \int_{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, \underbrace{e^{-jk\omega_0 \underset{0}{t}}}_{1}\, dt = \overbrace{\frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, dt}^{1}$$

$$= \frac{1}{T}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

$$X(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Question: CTFT of Impulse Train

Question:
Let $x(t)$ be a periodic impulse train defined as:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Find the Continuous-Time Fourier Transform (CTFT) of $x(t)$.

Answer (shown on click):

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \quad \text{where } \omega_0 = \frac{2\pi}{T}$$