Signals and Systems
Problem-Solving Session

November 3, 2025

DTFS Q1 (§3.6) — Coefficients via symmetry

A discrete-time periodic signal has period $N = 6$ with one period

$$x[n] = \{\,2,\,1,\,0,\,-1,\,-2,\,0\,\}, \quad n = 0, \ldots, 5,$$

extended periodically.
(i) Decide if $x[n]$ is even/odd/neither (with respect to a suitable center).
(ii) Compute the DTFS coefficients a_k, $k = 0, \ldots, 5$, using any symmetry.

Solution to DTFS Q1 — Detailed

Step 1 (Definition).

$$a_k = \frac{1}{6}\sum_{n=0}^{5} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}kn} = \frac{1}{6}\big(2\mathrm{e}^{-\mathrm{j}0} + 1\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}k} + 0 + (-1)\mathrm{e}^{-\mathrm{j}\pi k} + (-2)\mathrm{e}^{-\mathrm{j}\frac{4\pi}{3}k} + 0\big).$$

Thus

$$a_k = \frac{1}{6}\Big(2 + \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}k} - (-1)^k - 2\mathrm{e}^{-\mathrm{j}\frac{2\pi}{3}k}\Big).$$

Step 2 (Evaluate per k).

$$k = 0 : a_0 = \tfrac{1}{6}(2 + 1 - 1 - 2) = 0.$$

$$k = 1 : \mathrm{e}^{-\mathrm{j}\pi/3} = \frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2},\ \mathrm{e}^{-\mathrm{j}2\pi/3} = -\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2},\ (-1)^1 = -1.$$

$$a_1 = \tfrac{1}{6}\Big(2 + \tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2} - (-1) - 2(-\tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2})\Big) = \boxed{\tfrac{3}{4} - \mathrm{j}\tfrac{\sqrt{3}}{4}}.$$

$$k = 2 : \mathrm{e}^{-\mathrm{j}2\pi/3} = -\tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2},\ \mathrm{e}^{-\mathrm{j}4\pi/3} = -\tfrac{1}{2} + \mathrm{j}\tfrac{\sqrt{3}}{2},\ (-1)^2 = 1.$$

$$a_2 = \tfrac{1}{6}\Big(2 - \tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2} - 1 - 2(-\tfrac{1}{2} + \mathrm{j}\tfrac{\sqrt{3}}{2})\Big) = \boxed{\tfrac{1}{4} + \mathrm{j}\tfrac{\sqrt{3}}{12}}.$$

DTFS Q2 (§3.7) — Compound operations on coefficients

Let $x[n] \leftrightarrow \{a_k\}$ with period N. Define

$$y[n] = (-1)^n \, x[n-2] + \frac{1}{2}x[n].$$

Express the DTFS coefficients b_k in terms of $\{a_k\}$.

Solution to DTFS Q2 — Detailed

Step 1 (Break operations). Linearity:

$$b_k = b_k^{(1)} + b_k^{(2)}, \quad y_1[n] = (-1)^n x[n-2], \quad y_2[n] = \tfrac{1}{2}x[n].$$

Step 2 (Time shift). $x[n-n_0] \;\Rightarrow\; a_k\,\mathrm{e}^{-\mathrm{j}2\pi k n_0/N}$. For $n_0 = 2$:

$$x[n-2] \;\Rightarrow\; a_k\,\mathrm{e}^{-\mathrm{j}4\pi k/N}.$$

Step 3 (Modulation). $(-1)^n = \mathrm{e}^{\mathrm{j}\pi n}$. For even N, this corresponds to frequency index shift $k \mapsto k - N/2$:

$$\mathrm{e}^{\mathrm{j}\pi n} x[n] \;\Rightarrow\; a_{k-N/2}.$$

Applied after shift:

$$y_1[n] = \mathrm{e}^{\mathrm{j}\pi n}\,x[n-2] \;\Rightarrow\; b_k^{(1)} = a_{k-N/2}\,\mathrm{e}^{-\mathrm{j}4\pi k/N}.$$

Step 4 (Scaling). $y_2[n] = \frac{1}{2}x[n] \Rightarrow b_k^{(2)} = \frac{1}{2}a_k$.

$$\boxed{b_k = a_{k-\frac{N}{2}}\,\mathrm{e}^{-\mathrm{j}\frac{4\pi}{N}k} + \frac{1}{2}\,a_k, \quad \text{(indices mod } N\text{)}}.$$

If N is odd, keep modulation as $b_k^{(1)} = \sum_m a_m\,\delta[k - (m - \frac{N}{2})]\mathrm{e}^{-\mathrm{j}4\pi k/N}$.

DTFS Q3 (§3.8) — Output of an LTI via harmonics

Given $h[n] = \frac{1}{3}\{1, 1, 1\}$ and

$$x[n] = \cos\left(\frac{\pi n}{2}\right) + 2\sin\left(\frac{\pi n}{3}\right).$$

(i) Find a fundamental period N. (ii) Compute $y[n] = (h * x)[n]$ using DTFS (harmonic scaling) and simplify in time domain.

Solution to DTFS Q3 — Detailed

Step 1 (Periods). $\Omega_1 = \pi/2 \Rightarrow N_1 = 4$. $\Omega_2 = \pi/3 \Rightarrow N_2 = 6$. LCM$\Rightarrow N = 12$. **Step 2 (Freq response).**

$$H(\mathrm{e}^{\mathrm{j}\Omega}) = \tfrac{1}{3}\left(1 + \mathrm{e}^{-\mathrm{j}\Omega} + \mathrm{e}^{-\mathrm{j}2\Omega}\right) = \mathrm{e}^{-\mathrm{j}\Omega}\frac{\sin(3\Omega/2)}{3\sin(\Omega/2)}.$$

Step 3 (Evaluate at tones).

$$\Omega = \tfrac{\pi}{2}: \ \frac{\sin(3\pi/4)}{3\sin(\pi/4)} = \frac{\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2}} = \tfrac{1}{3}, \quad H = \tfrac{1}{3}\mathrm{e}^{-\mathrm{j}\pi/2} = -\tfrac{\mathrm{j}}{3}.$$

$$\Omega = \tfrac{\pi}{3}: \ \frac{\sin(\pi/2)}{3\sin(\pi/6)} = \frac{1}{3\cdot\frac{1}{2}} = \tfrac{2}{3}, \quad H = \tfrac{2}{3}\mathrm{e}^{-\mathrm{j}\pi/3} = \tfrac{2}{3}\left(\tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2}\right).$$

Step 4 (Apply to harmonics).

$$\cos(\Omega_1 n) \ \Rightarrow \ \mathrm{Re}\{H(\mathrm{e}^{\mathrm{j}\Omega_1})\mathrm{e}^{\mathrm{j}\Omega_1 n}\} = (-\tfrac{\mathrm{j}}{3})\mathrm{e}^{\mathrm{j}\Omega_1 n} \Rightarrow \tfrac{1}{3}\sin(\Omega_1 n).$$

$$2\sin(\Omega_2 n) \ \Rightarrow \ 2\ \mathrm{Im}\{H(\mathrm{e}^{\mathrm{j}\Omega_2})\mathrm{e}^{\mathrm{j}\Omega_2 n}\}.$$

Compute $H(\mathrm{e}^{\mathrm{j}\Omega_2}) = \frac{1}{3} - \mathrm{j}\frac{\sqrt{3}}{3}$. Thus scaling magnitude $= \frac{2}{3}$, phase shift $-\pi/3$. In time:

$$2\sin(\Omega_2 n) \ \rightarrow \ \tfrac{4}{3}\sin(\Omega_2 n).$$

DTFS Q4 (§3.9) — Parseval-based average power

Period $N = 6$, only nonzero

$$a_{\pm 1} = \frac{1}{3}, \quad a_{\pm 2} = \frac{1}{6}, \quad a_0 = a_3 = 0.$$

(i) Use Parseval to compute average power P_{avg}.
(ii) Can $x[n]$ be purely real and odd?

Solution to DTFS Q4 — Detailed

Step 1 (Parseval).

$$P_{\text{avg}} = \sum_{k=0}^{5} |a_k|^2 = 2\left(\left|\tfrac{1}{3}\right|^2 + \left|\tfrac{1}{6}\right|^2\right) = 2\big(\tfrac{1}{9} + \tfrac{1}{36}\big) = \tfrac{5}{18}.$$

Step 2 (Reality & oddness). Real $\Rightarrow a_{-k} = a_k^*$ (true here). Odd $\Rightarrow a_k$ purely imaginary (since time-domain oddness imposes $x[-n] = -x[n]$). Given real $a_{\pm 1}, a_{\pm 2}$, oddness fails. $\Rightarrow$ Real but not purely odd.

CTFT Q5 — Triangular pulse

Let $x(t) = \mathrm{tri}\left(\frac{t}{T}\right)$ (height 1 at $t = 0$, base $2T$). Derive $X(\omega)$ via time-domain convolution or known pairs.

Solution to CTFT Q5 — Detailed

Method (rect * rect). Let $r(t) = \text{rect}(t/T)$. Then $x(t) = (r * r)(t)$.
Fourier domain: $R(\omega) = T\ \text{sinc}\left(\frac{\omega T}{2}\right) \Rightarrow X(\omega) = R^2(\omega)$.

$$\boxed{X(\omega) = T^2\ \text{sinc}^2\left(\frac{\omega T}{2}\right).}$$

Sanity check: $x(0) = 1$ and $\frac{1}{2\pi}\int |X(\omega)|\, d\omega$ finite (energy signal).

CTFT Q6 — Time-multiplication property

Given $x(t) = \mathrm{rect}(t/T)$ with $X(\omega) = T\ \mathrm{sinc}\big(\frac{\omega T}{2}\big)$. Find $\mathcal{F}\{t\,x(t)\}$ using $t\,x(t) \iff \mathrm{j}\,\frac{d}{d\omega}X(\omega)$.

Solution to CTFT Q6 — Detailed

Step 1. Property: $\mathcal{F}\{t\,x(t)\} = \mathrm{j}\,\frac{d}{d\omega}X(\omega)$.

$$\frac{d}{d\omega}\left[T\ \mathrm{sinc}\Big(\frac{\omega T}{2}\Big)\right] = T \cdot \frac{T}{2}\ \mathrm{sinc}'\Big(\frac{\omega T}{2}\Big).$$

Step 2 (Explicit derivative). For $\mathrm{sinc}\,z = \frac{\sin z}{z}$:

$$\mathrm{sinc}'(z) = \frac{z\cos z - \sin z}{z^2}.$$

Result.

$$\boxed{\mathcal{F}\{t\ \mathrm{rect}(t/T)\} = \mathrm{j}\,\frac{T^2}{2}\,\frac{\big(\frac{\omega T}{2}\big)\cos\big(\frac{\omega T}{2}\big) - \sin\big(\frac{\omega T}{2}\big)}{\big(\frac{\omega T}{2}\big)^2}}.$$

CTFT Q7 — Damped sinusoid

For $a > 0$, $x(t) = \mathrm{e}^{-at}u(t)\cos(\omega_0 t)$. Find $X(\omega)$.

Solution to CTFT Q7 — Detailed

Step 1 (Split cosine).

$$x(t) = \tfrac{1}{2}\mathrm{e}^{-at}u(t)\mathrm{e}^{\mathrm{j}\omega_0 t} + \tfrac{1}{2}\mathrm{e}^{-at}u(t)\mathrm{e}^{-\mathrm{j}\omega_0 t}.$$

Step 2 (Known pair). $\mathcal{F}\{\mathrm{e}^{-at}u(t)\} = \frac{1}{a+\mathrm{j}\omega}$. Frequency shift gives:

$$\mathcal{F}\{\mathrm{e}^{-at}u(t)\mathrm{e}^{\mathrm{j}\omega_0 t}\} = \frac{1}{a+\mathrm{j}(\omega-\omega_0)}.$$

$$\mathcal{F}\{\mathrm{e}^{-at}u(t)\mathrm{e}^{-\mathrm{j}\omega_0 t}\} = \frac{1}{a+\mathrm{j}(\omega+\omega_0)}.$$

Combine.

$$\boxed{X(\omega) = \tfrac{1}{2}\left(\frac{1}{a+\mathrm{j}(\omega-\omega_0)} + \frac{1}{a+\mathrm{j}(\omega+\omega_0)}\right).}$$

CTFT Q8 — Modulated triangle (AM spectrum)

Let $x(t) = \text{tri}(t/T)$ and $y(t) = x(t)\cos(\omega_0 t)$. Express $Y(\omega)$ and describe its features.

Solution to CTFT Q8 — Detailed

Step 1 (Modulation). $\cos(\omega_0 t) = \frac{1}{2}(\mathrm{e}^{\mathrm{j}\omega_0 t} + \mathrm{e}^{-\mathrm{j}\omega_0 t}) \Rightarrow$ spectral shifts:

$$Y(\omega) = \tfrac{1}{2}X(\omega - \omega_0) + \tfrac{1}{2}X(\omega + \omega_0).$$

Step 2 (Use Q5). $X(\omega) = T^2 \operatorname{sinc}^2\left(\frac{\omega T}{2}\right)$.

$$\boxed{Y(\omega) = \frac{T^2}{2}\left[\operatorname{sinc}^2\left(\frac{(\omega-\omega_0)T}{2}\right) + \operatorname{sinc}^2\left(\frac{(\omega+\omega_0)T}{2}\right)\right].}$$

Two identical sidebands centered at $\pm\omega_0$, each scaled by $1/2$.

CTFT Q9 — Convolution of unequal rectangles

Let $x(t) = \text{rect}(t/T_1)$, $h(t) = \text{rect}(t/T_2)$ with $T_2 > T_1 > 0$. (i) Find $y(t) = (x * h)(t)$. (ii) Find $Y(\omega)$.

Solution to CTFT Q9 — Detailed

Step 1 (Time-domain geometry). Convolution of widths $T_1 < T_2$ gives trapezoid:

$$y(t) = \begin{cases} 0, & |t| \geq \frac{T_1+T_2}{2}, \\ \text{linear rise}, & -\frac{T_1+T_2}{2} < t < -\frac{T_2-T_1}{2}, \\ T_1, & |t| \leq \frac{T_2-T_1}{2}, \\ \text{linear fall}, & \frac{T_2-T_1}{2} < t < \frac{T_1+T_2}{2}. \end{cases}$$

Step 2 (Frequency). $X(\omega) = T_1 \operatorname{sinc}(\frac{\omega T_1}{2})$, $H(\omega) = T_2 \operatorname{sinc}(\frac{\omega T_2}{2})$.

$$\boxed{Y(\omega) = X(\omega)H(\omega) = T_1 T_2 \operatorname{sinc}\left(\frac{\omega T_1}{2}\right) \operatorname{sinc}\left(\frac{\omega T_2}{2}\right).}$$

CTFT Q10 — RC lowpass response to a pulse

Given $H(\omega) = \frac{1}{1+\mathrm{j}\omega RC}$ and $x(t) = \mathrm{rect}(t/T)$. (i) Find $Y(\omega)$. (ii) Obtain $y(t)$ as difference of step responses.

Solution to CTFT Q10 — Detailed

Step 1 (Spectrum). $X(\omega) = T\,\text{sinc}(\frac{\omega T}{2}) \Rightarrow Y(\omega) = \frac{T\,\text{sinc}(\omega T/2)}{1+\mathrm{j}\omega RC}$.

$$\boxed{Y(\omega) = \frac{T\ \text{sinc}\left(\frac{\omega T}{2}\right)}{1+\mathrm{j}\omega RC}.}$$

Step 2 (Time-domain construction). Write

$$x(t) = u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right).$$

Causal step response $s(t) = (1-\mathrm{e}^{-t/(RC)})u(t)$. Then

$$y(t) = s\left(t+\frac{T}{2}\right) - s\left(t-\frac{T}{2}\right) = \left(1-\mathrm{e}^{-\frac{t+T/2}{RC}}\right)u\left(t+\frac{T}{2}\right) - \left(1-\mathrm{e}^{-\frac{t-T/2}{RC}}\right)u\left(t-\frac{T}{2}\right).$$

Piecewise flat-rise-hold-fall shape as expected.

CTFT Q11 — Duality & normalization check

Using $\mathrm{rect}(t/T) \Longleftrightarrow T\ \mathrm{sinc}(\omega T/2)$, show that

$$x(t) = \frac{1}{\pi}\,\mathrm{sinc}(t) \quad \Longleftrightarrow \quad X(\omega) = \mathrm{rect}\left(\frac{\omega}{2}\right),$$

and verify $x(0) = \frac{1}{\pi} = \frac{1}{2\pi}\int_{-1}^{1} 1\, d\omega$.

Solution to CTFT Q11 — Detailed

Step 1 (Start from rect–sinc). With $T = 2$:

$$\text{rect}\left(\tfrac{t}{2}\right) \Longleftrightarrow 2\ \text{sinc}\left(\tfrac{\omega}{1}\right).$$

Step 2 (Duality). Swap $t \leftrightarrow \omega$ and include 2π-convention:

$$\frac{1}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{j}\omega t}\, d\omega = \frac{\sin t}{\pi t} = \frac{1}{\pi}\,\text{sinc}(t) \Longleftrightarrow X(\omega) = \text{rect}\left(\tfrac{\omega}{2}\right).$$

Step 3 (Normalization at $t = 0$).

$$x(0) = \lim_{t\to 0} \frac{\sin t}{\pi t} = \frac{1}{\pi} = \frac{1}{2\pi}\int_{-1}^{1} 1\, d\omega.$$