# Signals and Systems – Question Set (Q1–Q6)

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## Question 1

A causal and stable LTI system $S$ has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{10 - \omega^2 + j6\omega}.$$

1. Determine a differential equation relating the input $x(t)$ and output $y(t)$ of the system.
2. Determine the impulse response $h(t)$ of the system.

$-w^2 = (jw)^2$

1)

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw+4}{10-w^2+jbw}$$

$$F^{-1} \quad Y(jw)\boxed{[10+(jw)^2+j6w]} = X(jw)\boxed{(jw+4)}$$

differentiation property

$$x(t) \xleftrightarrow{F} X(jw) \Leftrightarrow \frac{d}{dt}x(t) \xleftrightarrow{F} jw\,X(jw)$$

$$10y(t) + \frac{d^2y(t)}{dt^2} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

D.E.

2) $(10+(jw)^2+j6w = s^2+6s+10$

$s_1 = -3+j$

$s_2 = -3-j$

$\boxed{s = jw}$

$$\Delta = \frac{36-40}{2} = -2$$

$$H(jw) = \frac{A}{jw+3+j} + \frac{B}{jw+3-j}$$

$$A = \left.\frac{jw+4}{jw+3-j}\right|_{jw=-3+j} = \frac{1+j}{2} \qquad B = \left.\frac{jw+4}{jw+3+j}\right|_{jw=-3-j} = \frac{1-j}{2}$$

$$e^{-at}u(t) \overset{F}{\longleftrightarrow} \frac{1}{j\omega + a}$$

$$h(t) = \frac{1+j}{2} e^{-(3+j)t} u(t) + \frac{1-j}{2} e^{-(3-j)t} u(t)$$

$$= e^{-3t} u(t) \left[ \frac{1}{2} (\underbrace{e^{jt} + e^{-jt}}_{2\cos t}) + \frac{j}{2} (\underbrace{e^{jt} - e^{-jt}}_{-2j \sin t}) \right]$$

$$= e^{-3t} [\cos t + \sin t] u(t)$$

## Question 2

Consider the Fourier transform pair:

$$e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

1. Use the appropriate Fourier transform property to find the Fourier transform of $te^{-|t|}$.
2. Use the result from part (a) and the duality property to find the Fourier transform of

$$g(t) = \frac{4t}{(1+t^2)^2}.$$

3. Simplify your results and sketch the magnitude spectrum.

1)

$$\boxed{F\{t\,x(t)\} = j\frac{d}{dw}X(jw)}$$

$$F\{t e^{-|t|}\} = j\frac{d}{dw}\frac{2}{1+w^2} = j\frac{-4w}{(1+w^2)^2}$$

2)

$$\boxed{\begin{array}{l} x(t) \overset{F}{\longleftrightarrow} X(jw) \\ X(t) \overset{F}{\longleftrightarrow} 2\pi x(-w) \end{array}}$$

Duality property

$$x_1(t) = \frac{-j\,4t}{(1+t^2)^2}$$

$$\boxed{g(t) = j\,x_1(t)}$$

$$F\{x_1(t)\} = 2\pi x_1(-w) = 2\pi\, -w\, e^{-|w|}$$

$$t e^{-|t|} \longleftrightarrow j\frac{-4w}{(1+w^2)^2} \quad X(jw)$$

$$j\left(\frac{-j\,4t}{(1+t^2)^2}\right) \longleftrightarrow j\,2\pi\, -w\, e^{-|w|}$$

$$G(jw) = -j\,2\pi\, w e^{-|w|}$$

$$G(j\omega) = -j2\pi\, \omega e^{-|\omega|}$$

$$|G(j\omega)| = 2\pi|\omega| e^{-|\omega|}$$

even function of $\omega$

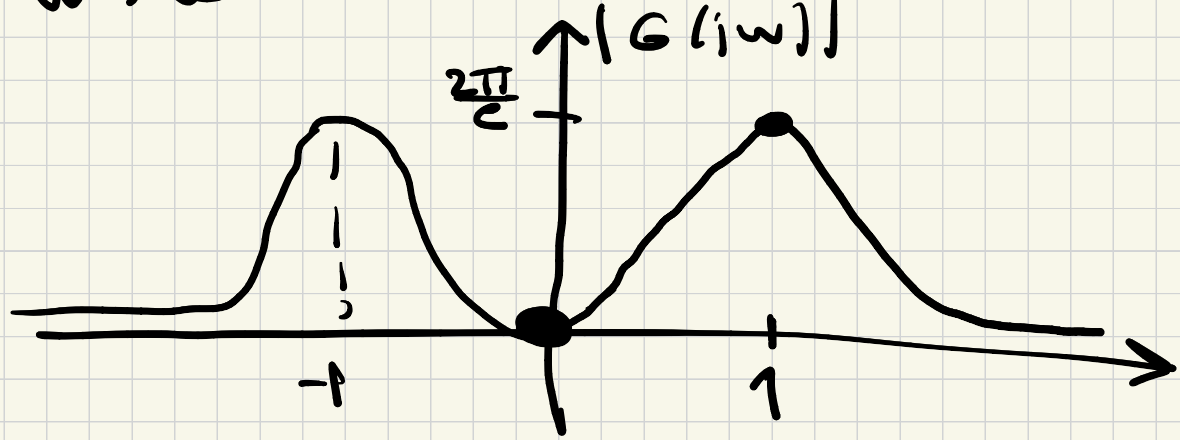
$\omega = 0 \Rightarrow |G(j\omega)| = 0$

$\omega > 0$

$$\frac{d\,|G(j\omega)|}{d\omega} = 2\pi e^{-\omega} - 2\pi \omega e^{-\omega} = 0$$

$$\omega = 1 \Rightarrow |G(j\omega)| = 2\pi\,|1|\, e^{-|1|} = \frac{2\pi}{e}$$

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$



## Question 3

A periodically charged capacitor generates the following output signal with period $T_c$:

$$x(t) = e^{-t}, \quad 0 \le t \le T_c.$$

1. Find the Fourier series coefficients $a_k$ of $x(t)$.
2. Given an LTI system with the impulse response $h(t) = \delta(t) - \delta(t - T_1)$, evaluate the Fourier series representation of the output when $x(t)$ is applied to the system.
3. There is a voltage subtractor device which can produce $x(t) - x(t - \tau)$, where $\tau$ can be adjusted. How would you use this device to find $T_c$? Justify your answer both in time and frequency domains.

$x(t) = e^{-t}$ periodic with period $T_c$

$w_0 = \frac{2\pi}{T_c}$ → period

1) $a_k = \frac{1}{T_c} \int_0^{T_c} e^{-t} e^{-jk\frac{2\pi}{T_c}t} dt$

$= \frac{1}{T_c} \frac{e^{-(1+jk\frac{2\pi}{T_c})}}{-(1+jk\frac{2\pi}{T_c})} \Big|_0^{T_c} = \frac{1-e^{-T_c}}{T_c(1+jk\frac{2\pi}{T_c})}$

2) $h(t) = \delta(t) - \delta(t-T_1)$

$x(t)$ periodic $\Leftrightarrow$ $y(t)$ periodic with the same period $T_c$

LTI

$x(t)$ → $h(t)$ / $H(jw)$ → $y(t)$

$a_k$ $\qquad$ $b_k$

$\boxed{b_k = a_k H(jkw_0)}$

$H(jw) = 1 - e^{-jwT_1}$

$H(jkw_0) = 1 - e^{-jk\frac{2\pi}{T_C}T_1}$

$b_k = \frac{1-e^{-T_c}}{T_c(1+jk\frac{2\pi}{T_c})}\left(1 - e^{-jk\frac{2\pi}{T_c}T_1}\right)$

3) Time domain

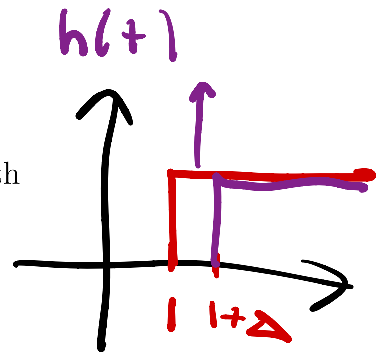
$x(t) - x(t-\tau) = 0$ for all $t$ is $\tau = m T_c$, $m \in \mathbb{Z}$

Then, I can start with $\tau = 0$ ↑ $\tau =$ first time, I see 0 output this is $\tau$

Frequency domain:

This time $a_k$ will be $0$ for $\gamma = mT_c$
$m \in \mathbb{Z}$

## Question 4




Assume a CT signal $x(t) = e^{-3t}u(t)$ is input to an LTI system with

$$h(t) = \frac{1}{\Delta}\left[u(t-1) - u(t-1-\Delta)\right],$$

where $\Delta > 0$.

1. Find the output $y(t)$ for this input in terms of $\Delta$.
2. Repeat part (a) for $h_0(t) = \lim_{\Delta \to 0} h(t)$.
3. Find the continuous-time Fourier transforms (CTFTs) of $x(t)$ and $h_0(t)$.
4. Find the inverse CTFT of $X(j\omega)H_0(j\omega)$.

1) $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$h(t-\tau) = \frac{1}{\Delta}\left[u(t-\tau-1) - u(t-\tau-1-\Delta)\right]$

$x(\tau) \neq 0$ for $\tau \geq 0$

$h(t-\tau) \neq 0$ for $1 < t-\tau \leq 1+\Delta$

$t-1-\Delta < \tau \leq t-1$

Case 1: $t-1 \leq 0$ $\quad t \leq 1$ $\quad y(t) = 0$

Case 2: $t-1-\Delta < 0 < t-1$ $\quad \tau \in [0, t-1]$

$$y(t) = \int_0^{t-1} e^{-3\tau}\, 1\, d\tau = \frac{1 - e^{-3(t-1)}}{3\Delta}$$

Case 3

$t-1-\Delta \geq 0$ $\quad \tau \in [t-1-\Delta, t-1]$

$$y(t) = e^{-3(t-1)}\,\frac{e^{3\Delta} - 1}{3\Delta}$$

2) $\lim_{\Delta \to 0} h(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[ u(t-1) - u(t-1-\Delta) \right]$

$= \delta(t-1)$

$\lim_{\Delta \to 0} \frac{1}{\Delta} \left[ u(t) - u(t-\Delta) \right] = \delta(t)$

$y^*(t) = \lim_{\Delta \to 0} h(t) * x(t) = \delta(t-1) * x(t)$

$= x(t-1)$

$= e^{-3(t-1)} u(t-1)$

*

3) $x(t) = e^{-3t} u(t)$

$X(jw) = \frac{1}{jw+3}$

$h_0(t) = \delta(t-1) \Leftrightarrow H_0(jw) = e^{-jw}$

4) $Y_0(jw) = X(jw) H_0(jw) = \frac{e^{-jw}}{\underline{jw+3}} = e^{-3(t-1)} u(t-1)$

**

* and ** are equal

## Question 5

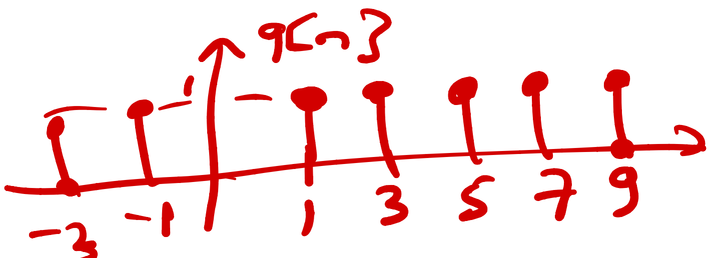
Let $x[n]$ be a periodic discrete-time sequence with period $N$ (even and positive). The Fourier Series coefficients of $x[n]$ are $a_k$. Another sequence is defined as

$$y[n] = g[n]x[n], \quad \text{where} \quad g[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k - 1].$$

1. Find the Fourier series coefficients of $g[n]$ and express $g[n]$ as a sum of complex exponentials.
2. Obtain the Fourier series coefficients of $x[n]e^{jM(2\pi/N)n}$ in terms of $a_k$.
3. Determine whether $y[n]$ is periodic, find its period, and express its FS coefficients $b_k$ in terms of $a_k$.

FS $a_k \leftrightarrow x[n]$

$$y[n] = \begin{cases} x[n] & \text{is } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$




g[n] period 2

$$\omega_g = \frac{2\pi}{2} = \pi$$

$$c_k = \frac{1}{2}\sum_{n=0}^{1} g[n]\, e^{-jk\pi n}$$

$$= \frac{1}{2}\left[g(0) + g(1)e^{-jk\pi}\right]$$

$$= \frac{1}{2} e^{-jk\pi}$$

$$g[n] = \sum_{k=0}^{1} c_k e^{jk\pi n} = c_0 e^0 + c_1 e^{j\pi n} = \frac{1}{2}\left[1-(-1)^n\right]$$

2) Frequency shifting

$$x[n]\, e^{jM\left(\frac{2\pi}{N}\right)n} \leftrightarrow a_{k-M}$$

$$d_k = a_{k-M}$$

3) $T_y = LCM(T_g, T_x) = N$

$T_g = 2$

$T_x = N$ (even)

$$e^{j\pi n} = e^{j\frac{N}{2}\frac{2\pi}{N}n} = (-1)^n$$

$$y[n] = \frac{1}{2}x[n] - \frac{1}{2}e^{j\frac{N}{2}\frac{2\pi}{N}n}$$

$$b_k \overset{FS}{\longleftrightarrow} \frac{1}{2}a_k - \frac{1}{2}a_{k-\frac{N}{2}}$$

## Question 6: CTFT Properties – Find the Signal

Let the signals listed below be real-valued functions of continuous time $t$. Their continuous-time Fourier transforms are denoted by $X(j\omega)$.

### Signal List

**(A)** $x(t) = e^{-t^2}$ Even (Gaussian pulse)

**(B)** $x(t) = te^{-t^2}$ odd

**(C)** $x(t) = e^{-(t-2)^2}$ neither even neither odd (shifted Gaussian pulse)

**(D)** $x(t) = \cos(3t)$ even

**(E)** $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$ even (impulse train)

**(F)** $x(t) = \underbrace{e^{-|t|}}_{\text{even}} \underbrace{\sin(t)}_{\text{odd}}$ Odd

**Instructions:** Determine which, if any, of the signals (A)–(F) have Fourier transforms that satisfy each of the following conditions. A condition may be satisfied by one, more than one, or no signals.

section 4 $x(t)$ real and odd signal

1. $\text{Re}\{X(j\omega)\} = 0$ B, F
2. $\text{Im}\{X(j\omega)\} = 0$ real and even A, D, E
3. There exists a real $\alpha$ such that $\underbrace{e^{j\alpha\omega}X(j\omega)}_{Y(jw)}$ is real even
4. $\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 0$
5. $\int_{-\infty}^{\infty} \omega X(j\omega)\, d\omega = 0$
6. $X(j\omega)$ is periodic

$Y(jw) = e^{j\alpha w} X(jw)$

$y(t) = x(t+\alpha)$

$\alpha = 0$ A, D, E automatically

$x(t+2) = e^{-t^2} \to$ even

therefore, C also applies

4) $\int_{-\infty}^{\infty} X(jw)\, dw = 0$

$\int_{-\infty}^{\infty} X(jw) e^{jwt}\, dt = x(t)$ B, F

$w = 0$ $\int_{-\infty}^{\infty} X(jw)\, dw' = x(0) = 0$

5) $\int_{-\infty}^{\infty} w X(jw) dw = 0$

$$\underbrace{jw X(jw)}_{Y(jw)} = F\left\{ \frac{dx(t)}{dt} \right\}$$

$$Y(jw) = F\left\{ -j \frac{dx(t)}{dt} \right\}$$

$$\int_{-\infty}^{\infty} Y(jw) dw = 0 \iff y(0) = 0$$

$$y(t) = \frac{dx(t)}{dt} -j$$

$$y(0) = 0 \iff \frac{dx(t)}{d(t)} = x'(t) = 0 \qquad \text{A, D}$$

6) $X(jw)$ periodic $\iff$ $x(t)$ discrete
aperiodic $\iff$ continuous

contradictory

Therefore any signal applies here but we can say that fourier transform of E is periodic

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$