EEE 321: Signals and Systems

The Discrete-Time Fourier Transform

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30 November 2025

Outline

Recall: DTFT Formulas

The Discrete-Time Fourier Transform (DTFT) pair is defined as follows:

Analysis Equation (Transform)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Synthesis Equation (Inverse Transform)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Important Note: $X(e^{j\omega})$ is periodic with respect to ω with period 2π.

Question 1: Basic DTFT Calculation

Problem: Determine the DTFT of the following causal signal:

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Where $u[n]$ is the unit step function.

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Solution: Using the definition equation:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

Since $u[n]$ is non-zero only for $n \geq 0$:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

Question 1: Solution Continued

This is an infinite geometric series:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad \text{if } |r| < 1$$

Here, $r = ae^{-j\omega}$. The convergence condition implies $|ae^{-j\omega}| = |a| < 1$, which is given.

Result

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

This is a fundamental pair often used in system analysis.

Question 2: Time Shifting and Linearity

Problem: Find the DTFT of the following signal using properties (avoid direct summation).

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Strategy: Instead of calculating the sum from scratch, we use the result from Q1 and the *Time Shifting Property*.

$$x[n - n_0] \longleftrightarrow e^{-j\omega \mathbf{n_0}} X(e^{j\omega})$$

Question 2: Solution

Let us define a standard signal $y[n] = (1/2)^n u[n]$.
From Question 1, we know (with $a = 1/2$):

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

The given signal $x[n]$ is simply $y[n]$ shifted to the right by 1 unit:

$$x[n] = y[n-1]$$

Applying the Time Shifting property ($n_0 = 1$):

$$X(e^{j\omega}) = e^{-j\omega \cdot 1} Y(e^{j\omega})$$

Result

$$X(e^{j\omega}) = \frac{e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

Question 3: Inverse DTFT

Problem: Determine the signal $x[n]$ corresponding to the following frequency spectrum (periodic with 2π):

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0), \quad |\omega| < \pi$$

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Solution: Use the Synthesis Equation (Inverse Transform).

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega$$

Question 3: Solution Continued

The Dirac delta function inside the integral utilizes the *Sifting Property*, evaluating the function only at $\omega = \omega_0$.

$$x[n] = \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} d\omega$$

$$x[n] = e^{j\omega_0 n}$$

Interpretation

This result demonstrates the duality: A complex exponential in the time domain ($e^{j\omega_0 n}$) corresponds to a single impulse in the frequency domain. This is a key concept in Fourier analysis.

Question 4: Difference Equations

Problem: An LTI system is characterized by the following Linear Constant-Coefficient Difference Equation (LCCDE):

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

a) Find the Frequency Response, $H(e^{j\omega})$.
b) Find the Impulse Response, $h[n]$.

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a) Find the Frequency Response, $H(e^{j\omega})$.
b) Find the Impulse Response, $h[n]$.

Solution (a): Take the DTFT of both sides (using Linearity and Time Shifting):

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

Question 4: Frequency Response

Factor out $Y(e^{j\omega})$:

$$Y(e^{j\omega})\left[1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right]=X(e^{j\omega})$$

Since the frequency response is defined as $H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}$:

$$H(e^{j\omega})=\frac{1}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$$

Question 4: Impulse Response (b)

To find $h[n]$, we need the inverse DTFT of $H(e^{j\omega})$. We factor the denominator. Let $z^{-1} = e^{-j\omega}$ for simplicity.

$$1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = \left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)$$

Using Partial Fraction Expansion:

$$H(e^{j\omega}) = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}}$$

Solving for coefficients yields: $A = 2, \quad B = -1.$

Question 4: Final Result

$$H(e^{j\omega}) = \frac{2}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

Recall the transform pair from Question 1: $\frac{1}{1-ae^{-j\omega}} \leftrightarrow a^n u[n]$.
Applying the inverse transform term by term:

Result

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Summary & Next Steps

Key Takeaways:

- Analyzing signals using the DTFT analysis equation.
- Simplifying calculations using the Time Shifting property.
- Deriving the Frequency Response from LCCDEs.
- Calculating Inverse DTFT via Partial Fraction Expansion.

Recommended Reading: Review Section 5.4 (The Convolution Property) and Parseval's Relation (Energy Conservation) in the textbook.

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The Discrete-Time Fourier Transform - 2

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Question 5: Problem Statement

Consider a filter whose frequency response $H(e^{j\Omega})$ is specified over the frequency interval $0 \leq \Omega < 2\pi$ as follows:

$$H(e^{j\Omega}) = \begin{cases} 5, & \frac{3\pi}{4} \leq \Omega \leq \frac{5\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

i. Find and plot $H(e^{j\Omega})$ in the interval $(-\pi, \pi]$. Specify the type of this filter (e.g. low-pass, high-pass, band-pass, band-stop).
ii. Calculate the output signal $y[n]$ of this filter when the input signal is given as:

$$x[n] = 10 + \cos(88\pi n) + 5\sin(11\pi n) - 5\cos\left(\frac{23\pi n}{8}\right)$$

iii. Calculate the output signal $y[n]$ of this filter when the input signal is given as:

$$x[n] = 2(-1)^n$$

Question 5(i): Filter Identification

Find and plot $H(e^{j\Omega})$ in the interval $(-\pi, \pi]$. Specify the type of this filter.

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Answer: Range $[\frac{3\pi}{4}, \frac{5\pi}{4}]$ maps to high frequencies near π (due to 2π periodicity).

- The active region in $(-\pi, \pi]$ is $[-\pi, -\frac{3\pi}{4}] \cup [\frac{3\pi}{4}, \pi]$.
- Filter Type: **High-Pass Filter**.

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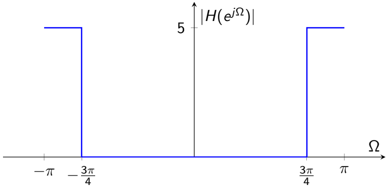
- Filter Type: **High-Pass Filter**.

Visual Solution:



Question 5(ii): Output Calculation 1

Problem: Calculate the output $y[n]$ for input:

$$x[n] = 10 + \cos(88\pi n) + 5\sin(11\pi n) - 5\cos\left(\frac{23\pi n}{8}\right)$$

Question 5(ii): Output Calculation 1

Problem: Calculate the output $y[n]$ for input:

$$x[n] = 10 + \cos(88\pi n) + 5\sin(11\pi n) - 5\cos\left(\frac{23\pi n}{8}\right)$$

Solution:

1. **DC (10):** $\Omega = 0$ (Stopband) $\rightarrow 0$.
2. $\cos(88\pi\text{n})$: $\Omega \equiv 0$ (Stopband) $\rightarrow 0$.
3. $5\sin(11\pi\text{n})$: $\Omega \equiv \pi$ (Passband). Gain $= 5$.
4. $-5\cos(\frac{23\pi\text{n}}{8})$: $\Omega = \frac{7\pi}{8}$. Since $\frac{7\pi}{8} > \frac{6\pi}{8}$, it is in Passband. Gain $= 5$.

$$y[n] = 25\sin(11\pi n) - 25\cos\left(\frac{23\pi n}{8}\right)$$

Question 5(iii): Output Calculation 2

Problem: Calculate the output $y[n]$ for input:

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Problem: Calculate the output $y[n]$ for input:

$$x[n] = 2(-1)^n$$

Solution:

$$x[n] = 2\cos(\pi n) \implies \Omega = \pi$$

π is in the center of the passband ($\frac{3\pi}{4} \leq \pi \leq \frac{5\pi}{4}$). The gain is 5.

$$y[n] = 5 \cdot x[n] = 10(-1)^n$$

Question 6: Problem Statement

Let $H(e^{j\Omega})$ be the frequency response of an LTI system. It is given that:

- The system is **causal**.
- The impulse response $h[n]$ of the system is **real-valued**.
- $DTFT(h[n] - h[-n]) = 4j\sin(2\Omega) - 2j\sin(3\Omega) + 10j\sin(5\Omega)$.
- $\int_{4\pi}^{6\pi} H(e^{j\Omega})H^*(e^{j\Omega})d\Omega = 64\pi$.
- $h[0] > 0$.

Task: Evaluate and plot $h[n]$.

Question 6: Step 1 - Time Domain Analysis

$$\mathcal{F}\{h[n] - h[-n]\} = 4j\sin(2\Omega) - 2j\sin(3\Omega) + 10j\sin(5\Omega)$$

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Using $2j\sin(k\Omega) \leftrightarrow \delta[n+k] - \delta[n-k]$:

$$h[n] - h[-n] = 2(\delta[n+2] - \dots) - (\delta[n+3] - \dots) + 5(\delta[n+5] - \dots)$$

Since causal ($h[-k] = 0$ for $k > 0$):

$$h[2] = -2, \quad h[3] = 1, \quad h[5] = -5$$

Question 6: Step 2 - Parseval's Relation

Condition: $\int_{2\pi} |H(e^{j\Omega})|^2 d\Omega = 64\pi$.

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$$64\pi = 2\pi \sum |h[n]|^2 \implies 32 = \sum |h[n]|^2$$

$$32 = h[0]^2 + (-2)^2 + (1)^2 + (-5)^2$$

$$32 = h[0]^2 + 30 \implies h[0] = \sqrt{2}$$

Question 6: Final Result

$$h[n] = \sqrt{2}\delta[n] - 2\delta[n-2] + \delta[n-3] - 5\delta[n-5]$$