Sampling – Practice Problems

Problem 1

A continuous-time signal $x(t)$ is bandlimited to $B = 4$ kHz.

(a) What is the minimum sampling frequency that avoids aliasing?

(b) If the signal is sampled at $f_s = 6$ kHz, will aliasing occur?

Solution 1

(a) Nyquist rate:

$$f_s \geq 2B = 8 \text{ kHz}$$

Minimum sampling frequency:

$$\boxed{8 \text{ kHz}}$$

(b) Since $6 < 8$ kHz, aliasing occurs:

$$\boxed{\text{Aliasing occurs.}}$$

Problem 2

A signal with Fourier transform $X(\omega)$ is ideally sampled with period T. Find the Fourier transform of the sampled signal:

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Solution 2

Using properties of Fourier transforms:

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s), \quad \omega_s = \frac{2\pi}{T}.$$

$$\boxed{X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)}$$

Problem 3

A signal has maximum frequency component $\omega_{\text{max}} = 4000\pi$ rad/s.
Find all sampling periods T that avoid aliasing.

Solution 3

Nyquist condition:

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_{\text{max}} = 8000\pi.$$

Then:

$$T \leq \frac{1}{4000}.$$

Thus,

$$\boxed{T \leq 0.00025 \text{ s}}$$

Problem 4

Show that sampling with an impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

produces:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).$$

Solution 4

Start with:

$$x_s(t) = x(t) \sum_n \delta(t - nT).$$

Using the sifting property:

$$x(t)\delta(t - nT) = x(nT)\delta(t - nT).$$

Therefore:

$$\boxed{x_s(t) = \sum_n x(nT)\delta(t - nT)}$$

Problem 5

A bandlimited signal ($B = 3$ kHz) is sampled at $f_s = 10$ kHz.
Write the reconstruction formula assuming ideal sinc interpolation.

Solution 5

Since $f_s > 2B$, reconstruction is possible:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right), \quad T = \frac{1}{10\,000}.$$

$$\boxed{x(t) = \sum_{n} x(nT)\,\mathrm{sinc}(10^4(t-nT))}$$

Problem 6

A signal bandlimited to $|\omega| < 2000$ rad/s is sampled with $T = 1$ ms.

(a) Find ω_s.

(b) Determine the spectral replica centers.

Solution 6

$$\omega_s = \frac{2\pi}{T} = 2000\pi \approx 6283 \text{ rad/s}.$$

Replica centers:

$$\omega = k\omega_s = 0, \pm 6283, \pm 12566, \ldots$$

$$\boxed{\text{Replicas at } k\omega_s}$$

Problem 7

A bandlimited signal ($B = 5$ kHz) is sampled at $f_s = 7$ kHz. Find the aliasing overlap range.

Solution 7

Nyquist: $2B = 10$ kHz (not satisfied). Aliasing region:

$$|f| > \frac{f_s}{2} = 3.5 \text{ kHz}.$$

Overlap:

$$5 - 3.5 = 1.5 \text{ kHz}.$$

$$\boxed{\text{Aliasing overlap } = 1.5 \text{ kHz}}$$

Problem 8

Explain how increasing the sampling rate affects the spacing of spectral replicas.

Solution 8

Replica spacing equals:

$$\omega_s = \frac{2\pi}{T}.$$

Increasing sampling rate increases ω_s, moving replicas farther apart.

$$\boxed{\text{Higher } f_s \Rightarrow \text{wider spacing and less aliasing}}$$

Problem 9

For ideal low-pass reconstruction with cutoff $\omega_c = \omega_s/2$, write the filter frequency response $H(\omega)$.

Solution 9

$$H(\omega) = \begin{cases} T, & |\omega| < \omega_s/2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{H(\omega) = T \operatorname{rect}\left(\frac{\omega}{\omega_s}\right)}$$

Problem 10

Show that ideal LPF reconstruction yields:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t - nT}{T}\right).$$

Solution 10

Reconstruction:

$$x(t) = x_s(t) * h(t)$$

with

$$h(t) = \mathrm{sinc}\left(\frac{t}{T}\right).$$

Using:

$$x_s(t) = \sum_n x(nT)\delta(t - nT)$$

$$x(t) = \sum_n x(nT)h(t - nT)$$

$$\boxed{x(t) = \sum_n x(nT)\mathrm{sinc}\left(\frac{t - nT}{T}\right)}$$

Problem 11

A continuous-time signal has Fourier transform

$$X(\omega) = \begin{cases} 1, & |\omega| < 3000 \\ 0, & \text{otherwise.} \end{cases}$$

The signal is sampled at $\omega_s = 4000$ rad/s.

(a) Will aliasing occur?

(b) If yes, determine the aliasing frequency range.

Solution 11

(a) Nyquist requirement:

$$\omega_s \geq 2\omega_{\text{max}} = 6000.$$

Since $4000 < 6000$, aliasing occurs:

$$\boxed{\text{Aliasing occurs}}$$

(b) Aliasing overlap:

$$\omega_{\text{max}} - \frac{\omega_s}{2} = 3000 - 2000 = 1000 \text{ rad/s}.$$

Thus aliasing in:

$$\boxed{|\omega| > 2000 \text{ rad/s with } 1000 \text{ rad/s overlap}}$$

Problem 12

A bandlimited signal is sampled using an impulse train:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).$$

Assume it is passed through an ideal LPF with cutoff

$$\omega_c = \frac{\omega_s}{2}.$$

Show that the spectrum of the output is exactly $X(\omega)$ in the non-aliased case.

Solution 12

Spectrum of sampled signal:

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

Ideal LPF:

$$H(\omega) = \begin{cases} T, & |\omega| < \omega_s/2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus output spectrum:

$$Y(\omega) = X_s(\omega)H(\omega) = X(\omega), \quad |\omega| < \frac{\omega_s}{2}.$$

If no aliasing exists, $X(\omega)$ is untouched:

$$\boxed{Y(\omega) = X(\omega)}$$

Problem 13

A signal $x(t)$ is bandlimited to $|\omega| < 1500$ rad/s.

$$X(\omega) = 0 \quad \text{for } |\omega| > 1500.$$

The signal is sampled with period $T = 2$ ms.

(a) Compute the sampling frequency ω_s.

(b) Determine whether perfect reconstruction is possible.

Solution 13

(a) Sampling frequency:

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.002} = 1000\pi \approx 3141.59 \text{ rad/s.}$$

(b) Nyquist requirement:

$$\omega_s \geq 2\omega_{\text{max}} = 3000.$$

Since:

$$3141.59 > 3000,$$

perfect reconstruction **is possible.**

$$\boxed{\text{No aliasing; reconstruction is achievable}}$$