Chapter 10

10.2: $H(j\omega) = \dfrac{3 - j\omega}{3 + j\omega} e^{-j\omega}$

(a) $|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = \dfrac{3 - j\omega}{3 + j\omega} e^{-j\omega} \dfrac{3 + j\omega}{3 - j\omega} e^{j\omega}$

$\Rightarrow |H(j\omega)|^2 = 1$ for all ω.

(b) $\angle H(j\omega) = \angle \text{Numerator} - \angle \text{Denominator} - \underset{\substack{\uparrow \\ \text{from } e^{-j\omega}}}{\omega}$

$= \tan^{-1}\left(\dfrac{-\omega}{3}\right) - \tan^{-1}\left(\dfrac{\omega}{3}\right) - \omega$

(c) $x(t) = 4 + \cos(3t)$.

There are two freqs in $x(t) = 0$ and 3 rad/s

$\Rightarrow$ Evaluate $H(j\omega)$ at $\omega = 0$ and $\omega = 3$.

$H(j0) = \dfrac{3 - j0}{3 + j0} \cdot e^{-j0} = 1$

$H(j3)$ has a magnitude of 1 (from part (a))

$\angle H(j3) = \tan^{-1}\left(\dfrac{-3}{3}\right) - \tan^{-1}\left(\dfrac{3}{3}\right) - 3$ (from part (b))

$= -\pi/4 - (\pi/4) - 3$.

$= -\pi/2 - 3 \simeq -4.571$

If we add 2π, the phase becomes $\angle H(j3) = 1.712$.

$y(t) = 4 \cdot H(j0) + |H(j3)| \cos(3t + \angle H(j3))$.

$= 4 + \cos(3t + 1.912)$

10.4: (a) $H(j\omega) = \int_{-\infty}^{\infty} \{ \delta(t) - 0.1 e^{-0.1t} u(t) \} e^{-j\omega t} \, dt$.

$$= \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} \, dt}_{e^{-j\omega(0)} = 1} - 0.1 \underbrace{\int_{-\infty}^{\infty} e^{-0.1t} u(t) e^{-j\omega t} \, dt}$$

$$\int_{0}^{\infty} e^{-0.1t} e^{-j\omega t} \, dt = \left. \frac{e^{-(0.1 + j\omega)t}}{-(0.1 + j\omega)} \right|_0^{\infty}$$

$$= 0 - \frac{1}{-(0.1 + j\omega)} = \frac{1}{0.1 + j\omega}.$$

Thus, $H(j\omega) = 1 - \frac{0.1}{0.1 + j\omega} = \frac{j\omega}{0.1 + j\omega}$

(b) $|H(j\omega)|^2 = \left(\frac{j\omega}{0.1 + j\omega} \right) \left(\frac{-j\omega}{0.1 - j\omega} \right) = \frac{\omega^2}{0.01 + j0.1\omega - j0.1\omega - (j\omega)^2}$

$$= \frac{\omega^2}{0.01 + \omega^2}$$

At $\omega = 0$, $|H(j\omega)|^2 = 0$

At $\omega = \infty$, $|H(j\infty)|^2 = \lim_{\omega \to \infty} \frac{\omega^2}{0.01 + \omega^2} = \lim_{\omega \to \infty} \frac{\omega^2}{\omega^2} = 1$.

At $\omega = 0.1$, $|H(j0.1)|^2 = \frac{0.01}{0.01 + 0.01} = \frac{1}{2}$.

$$\angle H(j\omega) = \angle j\omega - \angle(0.1 + j\omega) = \begin{cases} \pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega > 0 \\ -\pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega < 0 \end{cases}$$

(c) From the plot in part (b), the max value is one as $\omega \to \infty$. Also $|H(j\omega)|^2 = 1/2$ at $\omega = 0.1$ rad/s.

Why is it called a 3 dB points?

$$10 \log_{10} |H(j\omega)|^2_{\omega=0.1} = 10 \log_{10} \left(\frac{1}{2}\right) = 10(-0.301) = -3.01 \text{ dB}.$$

Notice that $10 \log_{10} |H(j\infty)|^2 = 10 \log_{10}(1) = 0$, so the decibel value at $\omega = 0.1$ rad/s is -3.01 dB down from the maximum dB value.

(d) Use SUPERPOSITION to do each input separately and then add them together.

$$x(t) = \underset{\substack{\uparrow \\ x_1(t)}}{10} + \underset{\substack{\uparrow \\ x_2(t)}}{20\cos(0.1t)} + \underset{\substack{\uparrow \\ x_3(t)}}{\delta(t-0.2)}.$$

① $x_1(t)$ is a sinusoid whose frequency is zero

Thus. we need $H(j\omega)$ at $\omega = 0$

$$H(j0) = \frac{j0}{0.1 + j0} = 0$$

$$\Rightarrow y_1(t) = 0$$

(2) $x_2(t)$ is a sinusoid with $\omega = 0.1$ rad/s

$H(j\omega)$ at $\omega = 0.1$ is $H(j0.1) = \dfrac{j0.1}{0.1 + j0.1} = \dfrac{j}{1+j}$.

we need $H(j0.1)$ in power form

$$H(j0.1) = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{j+1}{2} = \frac{\sqrt{2}}{2} e^{j\pi/4}$$

$$\Rightarrow y_2(t) = \left(\frac{\sqrt{2}}{2}\right) 20\cos(0.1t + \pi/4) = 10\sqrt{2} \cos(0.1t + \pi/4).$$

(3) For $x_3(t)$ we have a shifted impulse, so use $h(t)$

$$y_3(t) = h(t-0.2) = \delta(t-0.2) - 0.1\, e^{-0.1(t-0.2)} u(t-0.2)$$

Now, add them together.

$$y(t) = y_1(t) + y_2(t) + y_3(t).$$

$$y(t) = 10\sqrt{2}\cos(0.1t + \pi/4) + \delta(t-0.2) - 0.1 e^{-0.1(t-0.2)} u(t-0.2)$$

<u>10.5</u>: (a) The period is $T_0 = 8$, so $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}$ rad/s.

$$a_k = \frac{1}{8}\int_{-1}^{1} 10\, e^{-j\pi/4\, kt}\, dt.$$

The limits on the integrals are NOT -4 to 4 because $x(t)$ is ZERO for $-4 \le t \le -1$ and $1 \le t \le 4$

(b) To plot the spectrum, we need the values of a_k for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

At $k=0$ use L'Hopital's rule or take $\lim_{k\to 0}$

$$a_0 = \frac{10(\pi k/4)}{\pi k} = \frac{10}{4} = 2.5$$

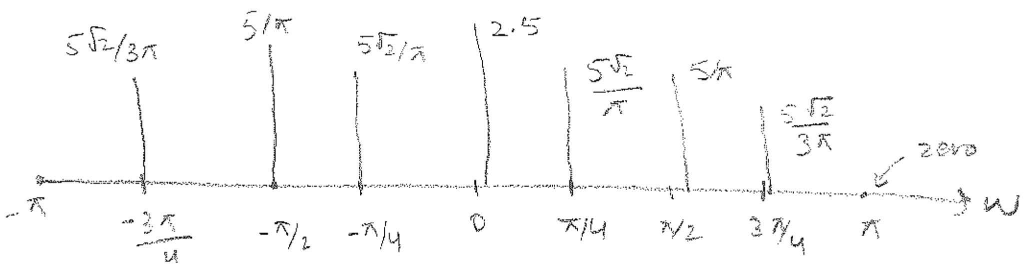
$$a_1 = \frac{10\sin(\pi/4)}{\pi} = \frac{10 \cdot \sqrt{2}/2}{\pi} = \frac{5\sqrt{2}}{\pi}$$

NOTE: $a_{-1} = a_1$ and generally we have $a_k = a_{-k}$.

$$a_2 = \frac{10\sin(\pi/2)}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi} = a_{-2}$$

$$a_3 = \frac{10\sin(3\pi/4)}{3\pi} = \frac{10\sqrt{2}/2}{3\pi} = \frac{5\sqrt{2}}{3\pi} = a_{-3}.$$

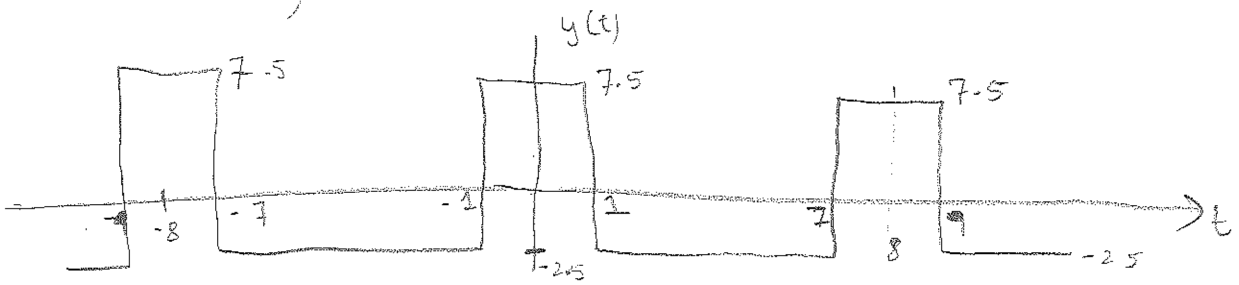
$$a_4 = \frac{10\sin(\pi)}{\pi k} = 0$$



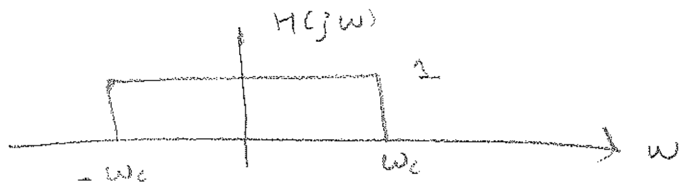
(c) The frequency response of the filter will MULTIPLY the spectrum of the input. Thus the spectrum of the output will be everything except the line at DC. Thus $y(t)$ has a fourier serie that is identical to the FS for $x(t)$ except the a_0 term is missing

$\Rightarrow y(t) = x(t) - a_0 = x(t) - 2.5$

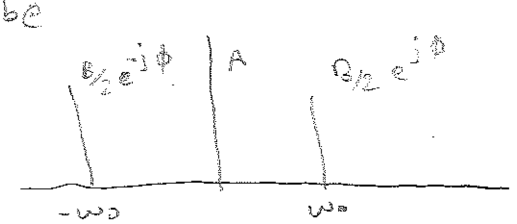
subtracting a constant will shift the plot down



(d)

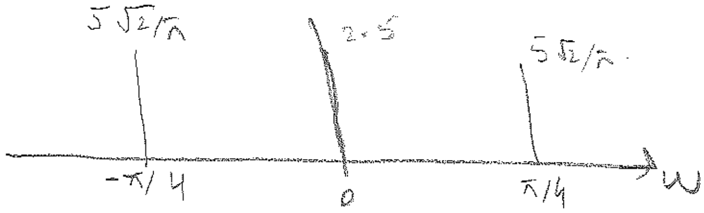


Again we note that $H(jw)$ will MULTIPLY the spectrum of $x(t)$. We want the spectrum of the output to be



since $w_0 = \pi/4$, we need $w_c > w_0$. But we also need $w_c < 2w_0$. Thus $\frac{\pi}{4} < w_c < \pi/2$.

With this w_c, the spectrum of $y(t)$ will be



$\Rightarrow \quad y(t) = 2.5 + \frac{10\sqrt{2}}{\pi} \cos\left(\frac{\pi}{4} t\right)$

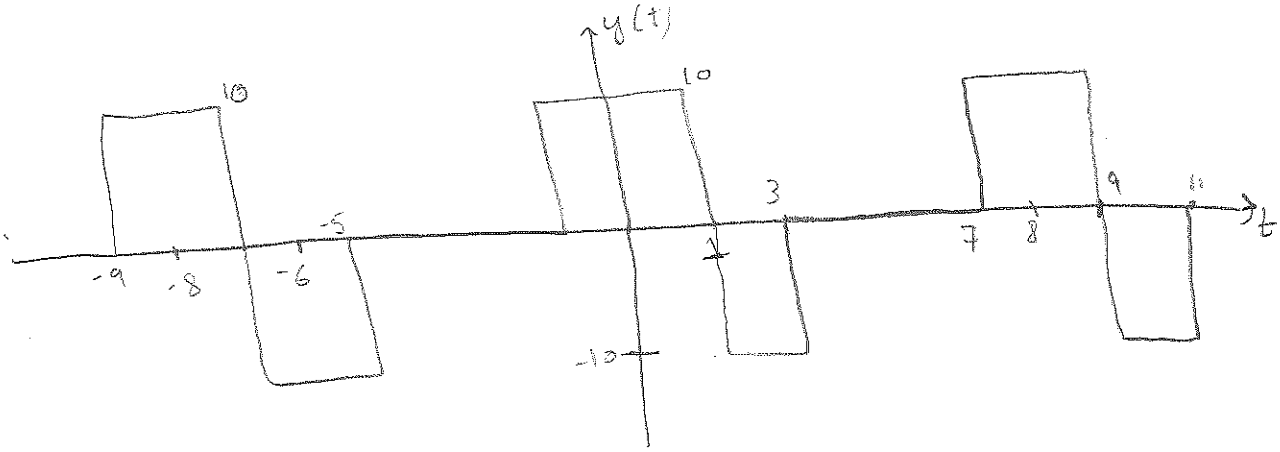
(e) If $H(jw) = 1 - e^{-j2w}$ we can find $h(t)$ by doing an inverse fourier transform.

$$h(t) = \delta(t) - \delta(t-2)$$

Then $y(t) = x(t) * h(t) = x(t) * \delta(t) - x(t) * \delta(t-2)$

$$= x(t) - x(t-2)$$

So we must shift $x(t)$ by 2 and then subtract



Problem 10.8:

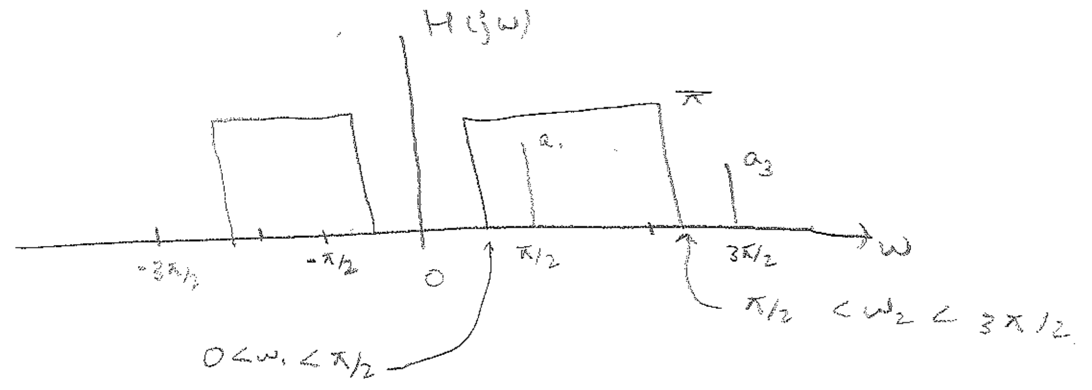
(a) $\omega_0 = 2\pi/T_0 = 2\pi/4 = \pi/2$ rad/s

(b) The Fourier series coefficients for the 50% duty cycle square wave were derived in chapter 3

$$a_k = \begin{cases} 1/2 & k=0 \\ 0 & k=\pm 2, \pm 4, \pm 6, \ldots \\ \frac{\sin(\pi k/2)}{\pi k} & k=\pm 1, \pm 3, \pm 5, \ldots \end{cases}$$

(c) $y(t) = 2\cos\left(\frac{2\pi t}{4}\right) = 2\cos\left(\frac{\pi}{2}t\right)$

since the frequency of $y(t)$ is $\pi/2$ which is ω_0 the filter just needs to pass a_1 & a_{-1}. Also the gain of the BPF needs to be π because $|a_1| = 1/\pi$.



$$H(j\omega) = \begin{cases} 0 & |\omega| < \omega_1 \\ \pi & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \omega_2 \le |\omega| \end{cases}$$

(d) $y(t) = 2\cos\left(\frac{2\pi}{3} t\right)$

The frequency of $y(t)$ is $\frac{2\pi}{3}$ rad/s which is <u>NOT</u> an integer multiple of $\omega_0 = \pi/2$. Hence, there is no LTI system that will have $y(t)$ as its output when the square wave $x(t)$ is the input

<u>Problem 10.9:</u>

For each filter (1 through 7), determine the output and then do the matching.

1. $H(j\omega)$ is a highpass filter. All components except DC are passed, so the output is

$$y(t) = x(t) - a_0$$
$$= x(t) - 1/2$$

2. $H(j\omega) = e^{-j\omega/2}$ corresponds to a pure delay of $1/2$

$$y(t) = x(t - 1/2)$$

3. Since the input signal only contains the discrete frequencies, $\omega_k = k\omega_0$, we evaluate $H(j\omega)$ at $\omega = k\omega_0$.

$$H(jk\omega_0) = \frac{1}{2}\left(1 + \cos(k\omega_0 T_0)\right)$$
$$= \frac{1}{2}\left(1 + \cos(2\pi k)\right)$$
$$= \frac{1}{2}(1 + 1) = 1$$

$$\Rightarrow y(t) = x(t)$$

5. This LPF passes DC only $\Rightarrow y(t) = 1/2$

4. This LPF pass DC and the lines at $\omega = \pm\omega_0$

$$y(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$

$$= \frac{1}{2} + \frac{1}{\pi} e^{j\omega_0 t} + \frac{1}{\pi} e^{-j\omega_0 t}$$

$$= \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 t)$$

6. This LPF has a delay of $1/2$ & passes $\omega = 0, \pm\omega_0$

$$\Rightarrow y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 (t - 1/2))$$

7. This BPF passes only the lines at $\omega = \pm\omega_0$

$$\Rightarrow y(t) = \frac{2}{\pi} \cos(\omega_0 t)$$

Now do the matching

(a) 5 (b) 6 (c) 7

(d) 1 (e) 2.