CH 11 Suggested Problem Solutions

$\boxed{\text{P 11.4}}$

(a) $x(t) = u(t) - u(t-4)$ is a shifted pulse

$$= \underset{\nearrow}{\delta(t-2)} * \underbrace{[u(t+2) - u(t-2)]}$$

time-shift $\quad\quad$ $\hookrightarrow$ F.T. $= \dfrac{\sin(2\omega)}{\omega/2}$

$$X(j\omega) = e^{-j2\omega} \frac{\sin(2\omega)}{\omega/2}$$

(b) Each impulse in ω inverts to a complex exponential

$$S(j\omega) = 4\pi\delta(\omega) + 2\pi\delta(\omega - 10\pi) + 2\pi\delta(\omega + 10\pi)$$

$$s(t) = 2e^{j0} + e^{j10\pi t} + e^{-j10\pi t}$$

$$= 2 + 2\cos(10\pi t)$$

(c) $R(j\omega) = \frac{1}{2} - \frac{2}{4 + j2\omega} = \frac{1}{2} - \frac{1}{2 + j\omega}$

$$r(t) = \tfrac{1}{2}\delta(t) - e^{-2t}u(t)$$

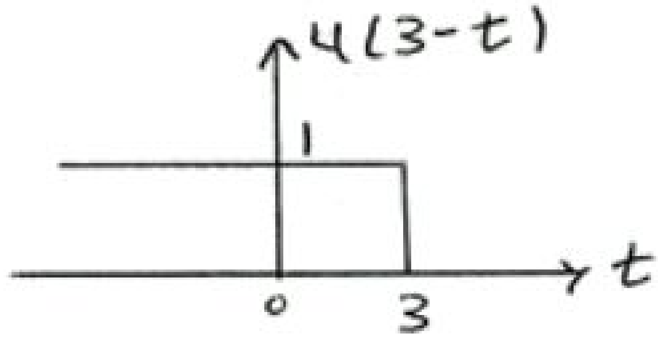
(d) $y(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$

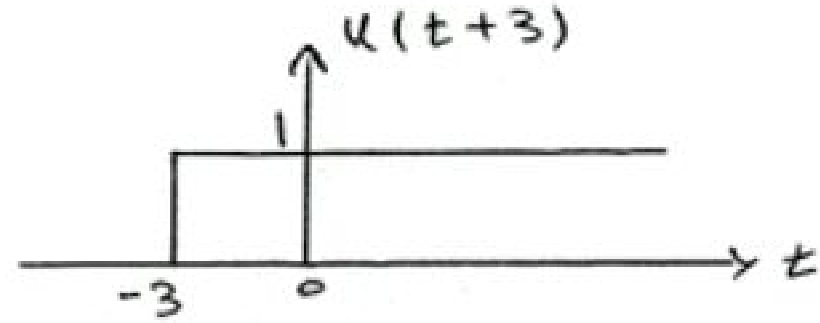
$$Y(j\omega) = e^{j\omega} + 2 + e^{-j\omega}$$

$$= 2 + 2\cos(\omega)$$

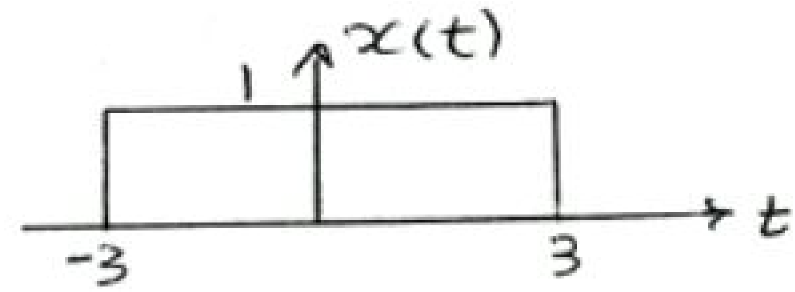
$\boxed{P.11.6}$

(a)





$x(t) = u(t+3)\cdot u(3-t) \longrightarrow$



$$T_0/2 = 3 \rightarrow X(j\omega) = \frac{\sin(3\omega)}{\omega/2}$$

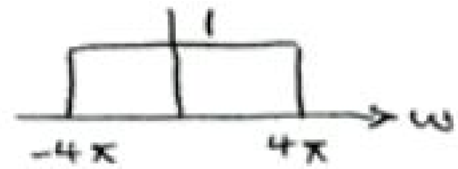
(b) Note

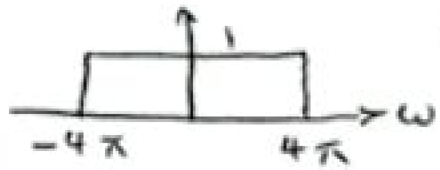
$$\underset{t\text{-domain}}{\sin 4\pi t} \longleftrightarrow \underset{\omega\text{-domain}}{\frac{\pi}{j}\delta(\omega-4\pi) - \frac{\pi}{j}\delta(\omega+4\pi)}$$

Convolution property of the Fourier Transform:

$$X(j\omega) = \frac{1}{2\pi}\left\{\frac{\pi}{j}\delta(\omega-4\pi) - \frac{\pi}{j}\delta(\omega+4\pi)\right\} * \left\{\frac{\pi}{j}\delta(\omega-50\pi) - \right.$$
$$\left. - \frac{\pi}{j}\delta(\omega+50\pi)\right\}$$

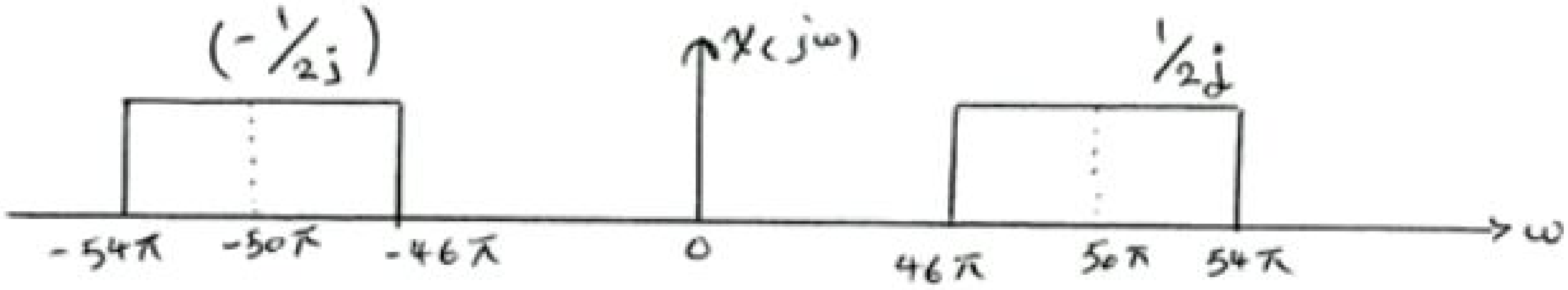
$$X(j\omega) = \frac{\pi}{2}\delta(\omega-46\pi) + \frac{\pi}{2}\delta(\omega+46\pi) - \frac{\pi}{2}\delta(\omega-54\pi)$$
$$- \frac{\pi}{2}\delta(\omega+54\pi)$$

(c) Note $\frac{\sin 4\pi t}{\pi t} \longleftrightarrow$ 

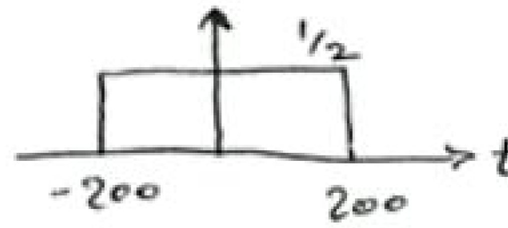
$X(j\omega) = \frac{1}{2\pi}\Big\{$ 

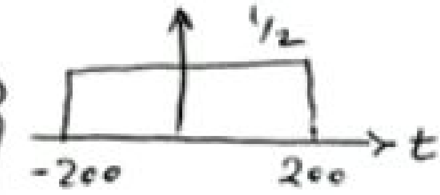
$\Big\} * \left\{ \frac{\pi}{j}\delta(\omega - 50\pi) - \frac{\pi}{j}\delta(\omega + 50\pi) \right\}$

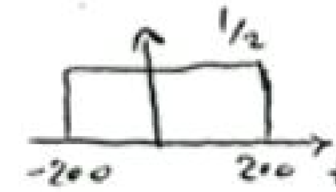
convolution

Thus 

$$X(j\omega) = \begin{cases} \frac{1}{2j} & 46\pi \le \omega \le 54\pi \\ -\frac{1}{2j} & -54\pi \le \omega \le -46\pi \\ 0 & \text{else} \end{cases}$$

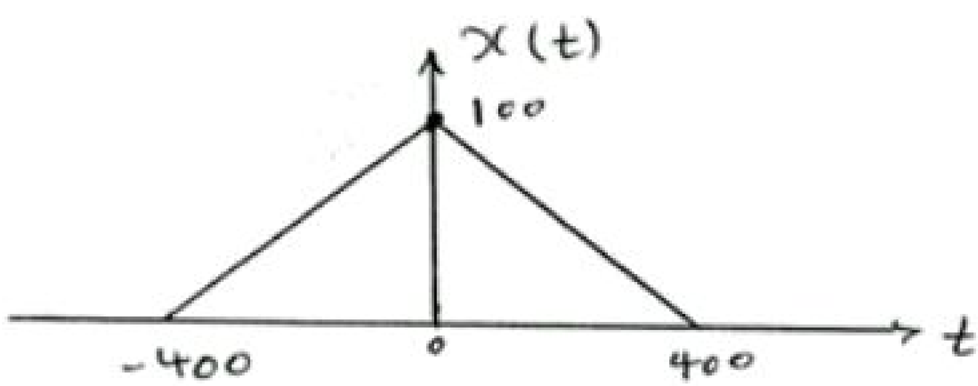
(d) Note: $\frac{1}{2}\frac{\sin(200\omega)}{\omega/2} \longleftrightarrow$ 

$\frac{\sin^2(200\omega)}{\omega^2} \longleftrightarrow \Big\{$ 

$*$ 

$\Big\}$

convolution

Thus 

(e) $\cos\omega \longleftrightarrow \frac{1}{2}\{\delta(t-1) + \delta(t+1)\}$

$\cos^2\omega \longleftrightarrow \frac{1}{2}\{\delta(t-1) + \delta(t+1)\} * \frac{1}{2}\{\delta(t-1) + \delta(t+1)\}$

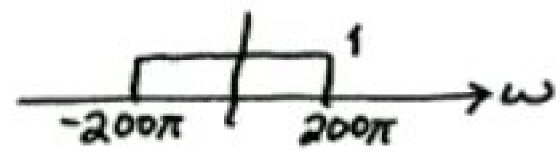
$x(t) = \frac{1}{4}\{\delta(t-2) + \delta(t+2) + 2\delta(t)\}$

P11.7

(a)

Use derivative property: $\frac{d}{dt}x(t) \longrightarrow j\omega X(j\omega)$

F.T. of $\frac{\sin(200\pi t)}{\pi t}$ is a rectangle



Thus

$$X(j\omega) = \begin{cases} j10\omega & \text{if } |\omega| \le 200\pi \\ 0 & \text{if } |\omega| > 200\pi \end{cases}$$

or,

$$X(j\omega) = j10\omega\left[u(\omega+200\pi) - u(\omega-200\pi)\right]$$

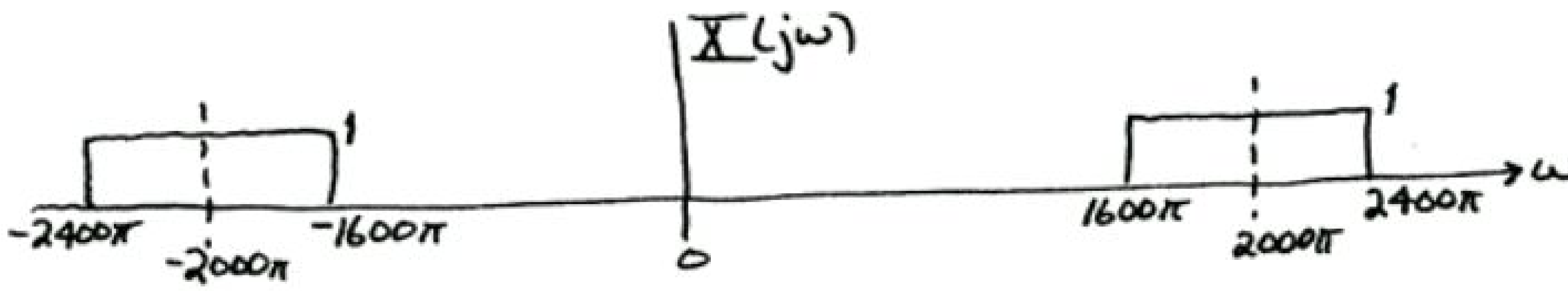
(b) Multiply by cosine $\Rightarrow$ frequency shifting

$$x(t) = 2\,\frac{\sin(400\pi t)}{\pi t}\left\{\frac{1}{2}e^{j2000\pi t} + \frac{1}{2}e^{-j2000\pi t}\right\}.$$

F.T. is a rectangle

shift to $\omega = 2000\pi$

shift to $\omega = -2000\pi$



(c)

The Fourier Transform of an impulse train in time is a (different) impulse train in frequency.

$T = 10$ secs from the definition of $x(t)$.

$$\Rightarrow \quad X(j\omega) = \frac{2\pi}{10}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{10}\right)$$

spacing is $\frac{\pi}{5}$ rads.

P 11.8

(a) $X(j\omega) = e^{-j3\omega}\left(\frac{1}{2+j\omega}\right)$ FT^{-1} is $e^{-2t}u(t)$

$$x(t) = e^{-2(t-3)}u(t-3)$$

(b) $X(j\omega) = j\omega\left(\frac{1}{2+j\omega}\right)$ use derivative property

$$x(t) = \frac{d}{dt}\left\{e^{-2t}u(t)\right\} = \underbrace{e^{-2t}\delta(t)}_{\text{eval @ } t=0} - 2e^{-2t}u(t)$$

$$x(t) = \delta(t) - 2e^{-2t}u(t)$$

(c) $X(j\omega) = e^{-j3\omega}\left(\frac{j\omega}{2+j\omega}\right)$ use time-shift on the result of (b)

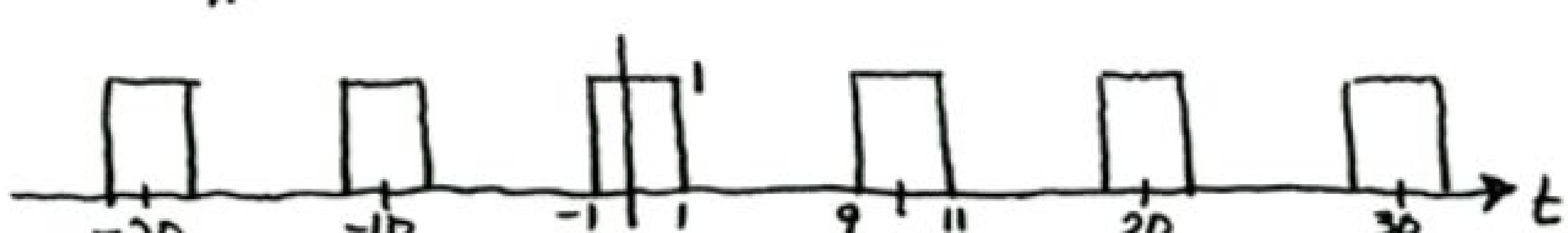
$$x(t) = \delta(t-3) - 2e^{-2(t-3)}u(t-3)$$

(d) $$\frac{2\sin(\omega)}{\omega} = \frac{\sin(\omega)}{\omega/2} \xrightarrow{FT^{-1}} u(t+1) - u(t-1)$$

$$\frac{2\pi}{10}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi}{10}k\right) \xrightarrow{FT^{-1}} \sum_{n=-\infty}^{\infty}\delta(t-10n)$$

Convolve: $\left[u(t+1)-u(t-1)\right] * \sum_{n=-\infty}^{\infty}\delta(t-10n)$

$$= \sum_{n=-\infty}^{\infty}\left[u(t+1-10n) - u(t-1-10n)\right]$$



P11.13

(a) Since the system is LINEAR, the two inputs can be treated separately and then combined.

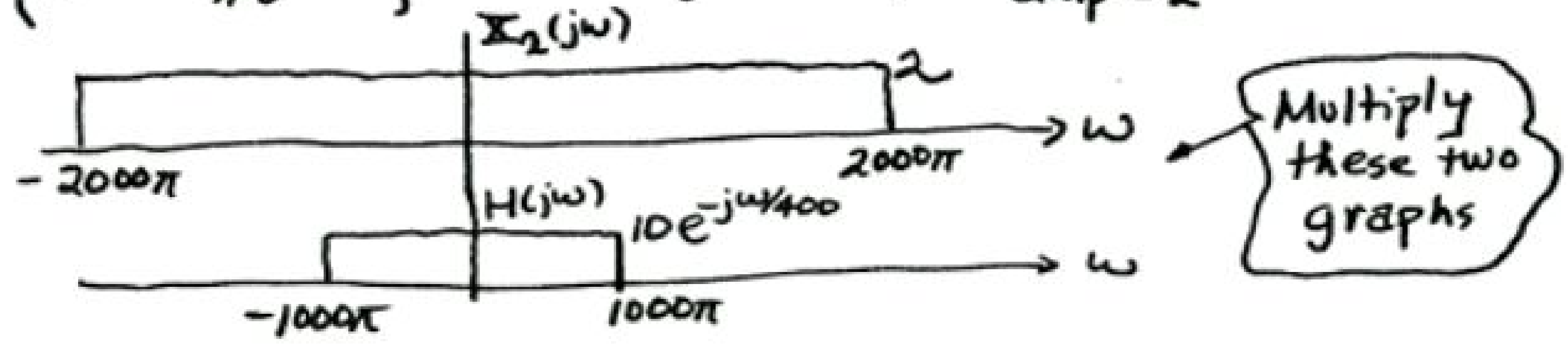
- For the cosine input, the output will be a cosine with a new magnitude and phase. We evaluate $H(j\omega)$ at the input frequency: $\omega = 200\pi$ rad/s.

$$H(j200\pi) = 10e^{-j(200\pi)(0.0025)} = 10e^{-j0.5\pi}$$

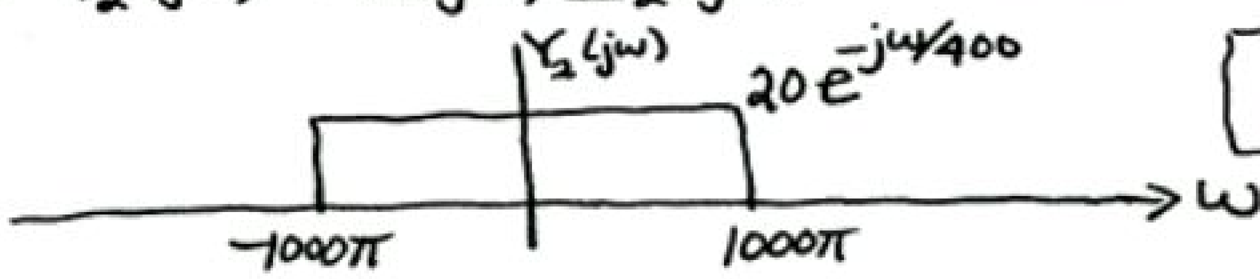
Call this output $y_1(t)$: $\quad y_1(t) = 10\cos(200\pi t - \pi/2)$

- For the "sinc" input, take the F.T. of the input, then multiply by $H(j\omega)$ and then inverse transform.

$$\text{F.T.}\left\{2\frac{\sin(2000\pi t)}{\pi t}\right\} = \text{Rectangular Shape: width of } 4000\pi \text{ rad/s. amp} = 2$$



Thus $Y_2(j\omega) = H(j\omega)X_2(j\omega)$ is also a rectangle



$\boxed{\frac{1}{400} = 0.0025\text{s}}$

The inverse F.T. of this rectangle is a SHIFTED "sinc"

$$y_2(t) = 20\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

Finally, the total output is the sum of $y_1(t)$ & $y_2(t)$

$$y(t) = 10\cos(200\pi t - \pi/2) + 20\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

We have used SUPERPOSITION to do this part.

(b) Use SUPERPOSITION again. Two of the inputs are the same, so we don't have to rework them. We only need to consider the input $x_3(t) = \cos(3000\pi t)$. For a cosine input, we must evaluate $H(j\omega)$ at the input frequency; in this case, at $\omega = 3000\pi$.

$H(j3000\pi) = 0 \Rightarrow$ NO OUTPUT, i.e. $y_3(t) = 0$.

So, the answer is the same as part (a) !

(c) Again, use SUPERPOSITION. We already know the output for $x_1(t) = \cos(200\pi t)$

$$y_1(t) = 10\cos(200\pi t - \pi/2)$$

We need to find the output for $x_4(t) = 2\delta(t)$.

Thus we need the impulse response

But this is just the inverse F.T. of $H(j\omega)$

And we already know that is a shifted "sinc"

$$y_4(t) = 2h(t) = 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

Finally,

$$y(t) = y_1(t) + y_4(t)$$

$$= 10\cos(200\pi t - \pi/2) + 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

It is very interesting to see that all three parts have the same answer. Why? Because the filter is an ideal LPF, so only the part of the input signal between -1000π and $+1000\pi$ matters. For example, in part (c) the F.T. of $2\delta(t)$ is $X_4(j\omega) = 2$ for all ω, but only the part for $|\omega| < 1000\pi$ rad/s matters. Over that range the "sinc" input of part (a) is the same

(d) Superposition simplifies the work.

P11.14

The periodic input to the above system is defined by the equation:

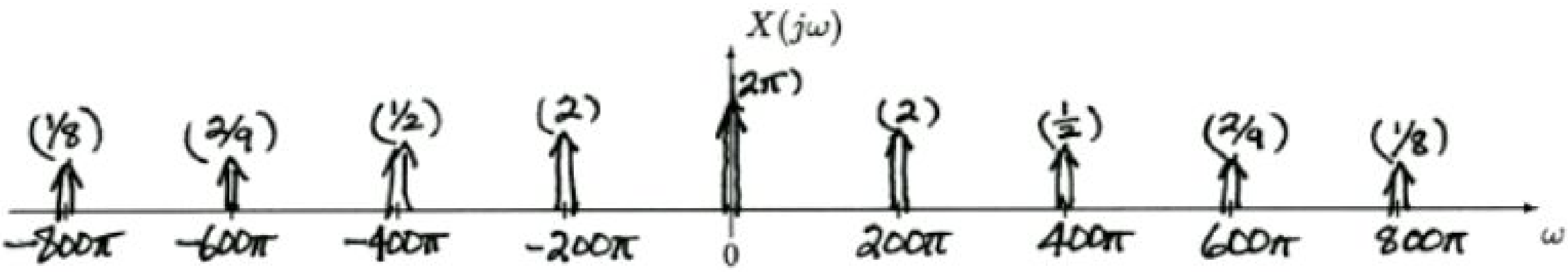
$$x(t) = \sum_{k=-4}^{4} a_k e^{j200\pi kt}, \quad \text{where} \quad a_k = \begin{cases} \dfrac{1}{\pi|k|^2} & k \neq 0 \\ 1 & k = 0 \end{cases}$$

$a_1 = 1/\pi$
$a_2 = 1/4\pi$
$a_3 = 1/9\pi$
$a_4 = 1/16\pi$

(a) Determine the Fourier transform of the periodic signal $x(t)$. Give a formula and then plot it on the graph below.

$$X(j\omega) = \sum_{k=-4}^{4} 2\pi a_k \delta(\omega - 200\pi k)$$

$\omega_0 = 200\pi$

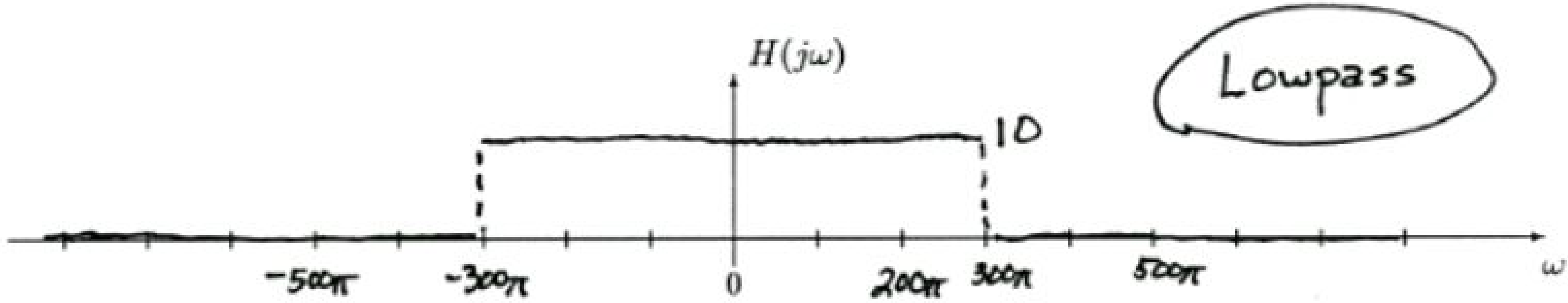


(b) The frequency response of the LTI system is given by the following equation:

$$H(j\omega) = \begin{cases} 10 & |\omega| \leq 300\pi \\ 0 & |\omega| > 300\pi \end{cases}.$$

Plot this function on the graph below using the same frequency scale as the plot in part (a). *Note carefully what type of filter (i.e., lowpass, bandpass, highpass) this is.*

Lowpass



(c) Write an equation for $y(t)$.

$$Y(j\omega) = H(j\omega)X(j\omega) = 20\pi\delta(\omega) + 20\delta(\omega - 200\pi) + 20\delta(\omega + 200\pi)$$

Invert: $y(t) = 10 + \underbrace{\frac{10}{\pi} e^{j200\pi t} + \frac{10}{\pi} e^{-j200\pi t}}_{\text{use Euler's inverse formula}}$

$$y(t) = 10 + \frac{20}{\pi}\cos(200\pi t)$$

$\boxed{\text{P11.1b}}$

$x(t)$ is real $\Rightarrow$ $x^*(t) = x(t)$

If $x(t) \longrightarrow X(j\omega)$, then $x^*(t) \longrightarrow X^*(-j\omega)$

$\Rightarrow$ $X^*(-j\omega) = X(j\omega)$

Express $X(j\omega)$ in terms of its real and imaginary parts:

$$X(j\omega) = A(\omega) + j B(\omega)$$

Then $X^*(-j\omega) = A(-\omega) - j B(-\omega)$

$\Rightarrow$ $A(\omega) = A(-\omega)$ and $-B(\omega) = B(-\omega)$

a) The magnitude is even:

$$|X(-j\omega)| = \sqrt{A^2(-\omega) + B^2(-\omega)}$$

$$= \sqrt{A^2(\omega) + B^2(\omega)} = |X(j\omega)|$$

b) The phase is odd:

$$\angle X(-j\omega) = \mathrm{Tan}^{-1}\left\{\frac{B(-\omega)}{A(-\omega)}\right\}$$

$$= \mathrm{Tan}^{-1}\left\{\frac{-B(\omega)}{A(\omega)}\right\}$$

$$= -\mathrm{Tan}^{-1}\left\{\frac{B(\omega)}{A(\omega)}\right\} = -\angle X(j\omega)$$

Recall that the tangent function is an <u>ODD</u> function.

$\boxed{\text{P11.17}}$

Define $s(t) = u(t) - \frac{1}{2}$

$$\Rightarrow \quad S(j\omega) = U(j\omega) - \pi\delta(\omega)$$

Since $s(t)$ is an odd function: $s(-t) = -s(t)$

and $s(-t) \xrightarrow{FT} S(-j\omega)$

$$S(-j\omega) = -S(j\omega)$$

$$\Rightarrow \quad U(-j\omega) \underbrace{-\pi\delta(-\omega)}_{=\pi\delta(\omega)} = -U(j\omega) + \pi\delta(\omega)$$

$$U(-j\omega) = -U(j\omega) + 2\pi\delta(\omega)$$

Thus, if we assume $U(j\omega) = \frac{1}{j\omega} + K\delta(\omega)$

$$\frac{1}{-j\omega} + K\delta(-\omega) = -\frac{1}{j\omega} - K\delta(\omega) + 2\pi\delta(\omega)$$

$$\Rightarrow \quad 2K\delta(\omega) = 2\pi\delta(\omega)$$

$$\Rightarrow \quad K = \pi$$

Note: $\delta(-\omega) = \delta(\omega)$, i.e. $\delta(\,)$ is <u>even</u>