HW #11

SOLUTIONS

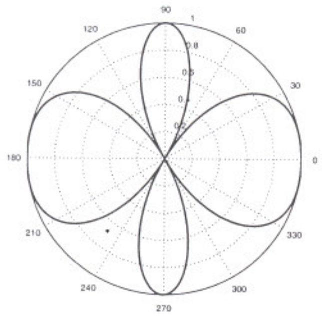
a)

$$AF=\sum_{n=1}^{N}\exp\left(j(n-1)\varphi\right)=1+\exp(j\varphi)=2\exp(-j\frac{\varphi}{\varphi})\cos\left(\frac{\varphi}{2}\right)$$

$$\rightarrow\left|AF\right|_{Normalized}=\cos\left(\frac{k2d\cos\theta+\alpha 2d}{2}\right)=\cos\left(kd\cos\theta+\alpha d\right)$$

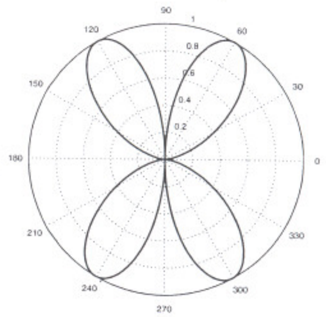
i) $\alpha d = 0$, $kd = \pi$

$$\left|AF\right|_{Normalized}=\left|\cos(\pi\cos\theta)\right|$$



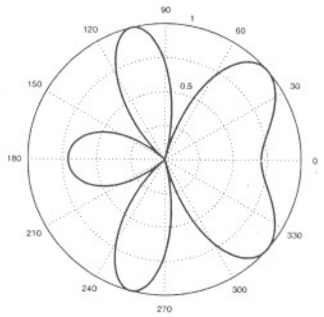
ii) $\alpha d = \pi/2$, $kd = \pi$

$$\left|AF\right|_{Normalized}=\left|\cos\left(\pi\cos\theta+\frac{\pi}{2}\right)\right|=\left|\sin(\pi\cos\theta)\right|$$



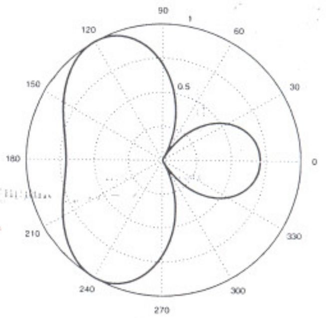
iii) $\alpha d = \pi/4$, $kd = \pi$

$$\left|AF\right|_{Normalized}=\left|\cos\left(\pi\cos\theta+\frac{\pi}{4}\right)\right|$$



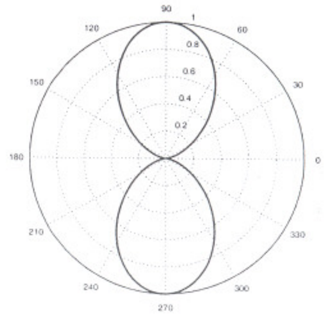
iv) $\alpha d = \pi/4$, $kd = \pi/2$

$$\left|AF\right|_{Normalized}=\left|\cos\left(\frac{\pi}{2}\cos\theta+\frac{\pi}{4}\right)\right|$$



v) $\alpha d = 0$, $kd = \pi/2$

$$\left|AF\right|_{Normalized}=\left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|$$



b) **$E_3 = 2E_0$:**

$$AF = \sum_{n=1}^{N} \exp\left(j(n-1)\varphi\right) = 1 + 2e^{j\varphi} + e^{j2\varphi} = e^{j\varphi}\left(e^{-j\varphi} + 2 + e^{j\varphi}\right)$$

$$\rightarrow \left|AF\right| = \left|e^{-j\varphi} + 2 + e^{j\varphi}\right| = \left|2 + 2\cos\varphi\right| = 4\cos^2\frac{\varphi}{2}$$

$$\rightarrow \left|AF\right|_{Normalized} = \cos^2\left(\frac{kd\cos\theta + \alpha d}{2}\right) = \cos^2\left(\frac{kd\cos\theta}{2}\right) = \cos^2\left(\frac{\pi}{4}\cos\theta\right)$$

$$Nulls:\ No\ null \qquad Maxima:\ \theta = \frac{\pi}{2},\ \frac{3\pi}{2}$$

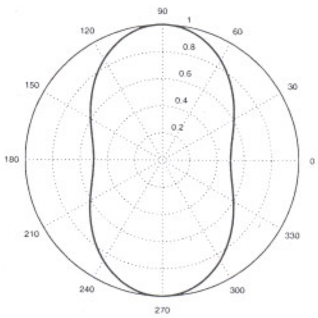
$E_3 = -2E_0$:

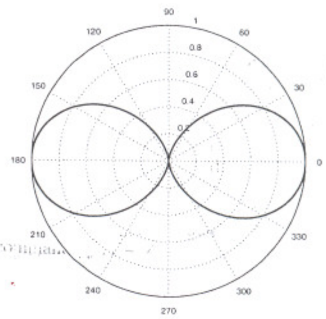
$$AF = \sum_{n=1}^{N} \exp\left(j(n-1)\varphi\right) = 1 - 2e^{j\varphi} + e^{j2\varphi} = e^{j\varphi}\left(e^{-j\varphi} - 2 + e^{j\varphi}\right)$$

$$\rightarrow \left|AF\right| = \left|e^{-j\varphi} - 2 + e^{j\varphi}\right| = \left|-2 + 2\cos\varphi\right| = 4\sin^2\frac{\varphi}{2}$$

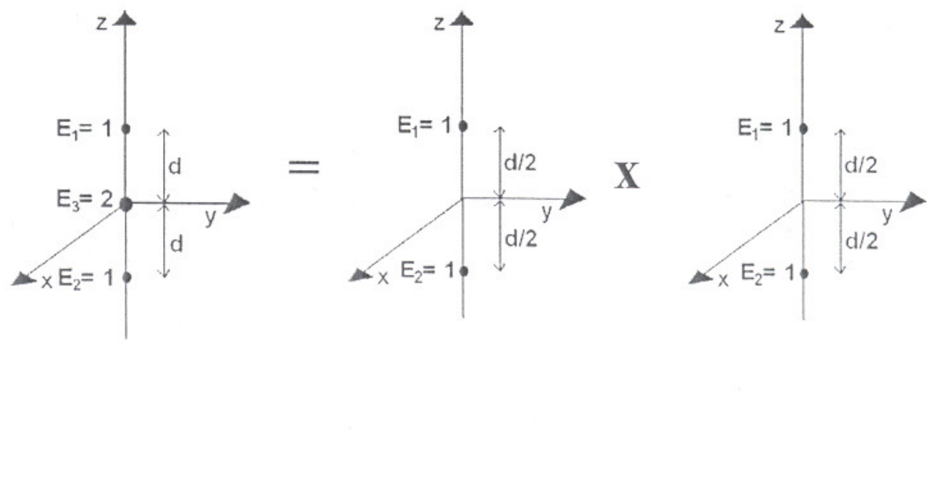
$$\rightarrow \left|AF\right|_{Normalized} = \sin^2\left(\frac{kd\cos\theta + \alpha d}{2}\right) = \sin^2\left(\frac{kd\cos\theta}{2}\right) = \sin^2\left(\frac{\pi}{4}\cos\theta\right)$$

$$Nulls:\ \theta = \frac{\pi}{2},\ \frac{3\pi}{2} \qquad Maxima:\ \theta = 0,\ \pi$$





c)



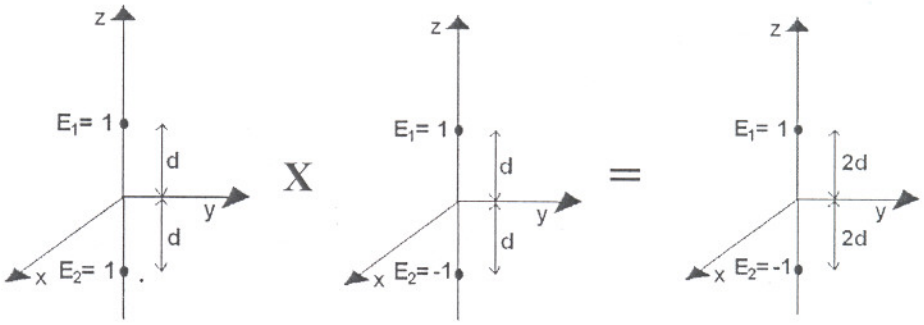
As shown in the figure, the 3-element array is the array of two arrays that are similar to (scaled) one in (v). Therefore the array factor of the 3-element array is

$$\left|AF\right|_{Normalized} = \left|AF_V\left(d/2\right)\right|\left|AF_V\left(d/2\right)\right|$$
$$= \left|\cos\left(\frac{\pi}{4}\cos\theta\right)\right|\left|\cos\left(\frac{\pi}{4}\cos\theta\right)\right| = \cos^2\left(\frac{\pi}{4}\cos\theta\right)$$

as we found in (b).

d) $$\left|AF\right|_{Normalized} = 2\left|\cos(\pi\cos\theta)\right|\left|\sin(\pi\cos\theta)\right| = \left|\sin(2\pi\cos\theta)\right|$$

What we do schematically is



If we calculate the array factor of the resulting array directly

$$\left|AF\right|_{Normalized} = \left|\cos\left(2\pi\cos\theta + \frac{\pi}{2}\right)\right| = \left|\sin(2\pi\cos\theta)\right|$$

as expected.

e) $$AF = \sum_{n=1}^{N}\exp\left(j(n-1)\varphi\right) = 1 + \exp(j\varphi) = 1 + \exp\left(jkd\cos\theta + j\beta\right)$$

$$AF = 1 + \exp\left(j\frac{\pi}{2}\cos\theta + j\beta\right) \rightarrow \left|AF\right| = \left|\cos\left(\frac{\pi}{4}\cos\theta + \frac{\beta}{2}\right)\right|$$

$$\theta = \frac{\pi}{2} \rightarrow \left|AF\right| = \left|\cos\left(\frac{\pi}{4} + \frac{\beta}{2}\right)\right|$$

For maximum array factor, $\beta = -\pi/2 \pm 2\pi k$