EEE 452/552

Solutions to HW # 13

① β in Balanis corresponds to α in our notes.

(a) $\beta = 0$ rad. for a broadside array

(b) $kd\cos\theta + \beta \Big|_{\theta=0^\circ} = 0 \Rightarrow \beta = -\frac{2\pi}{\lambda}\frac{\lambda}{4} = -\frac{\pi}{2}$ rad.

(c) $kd\cos\theta + \beta \Big|_{\theta=180^\circ} = 0 \Rightarrow \beta = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$ rad.

(d) $\beta = -kd\cos\theta_0 \Big|_{\theta_0=30^\circ} \Rightarrow \beta = -\frac{2\pi}{\lambda}\frac{\lambda}{4}\frac{\sqrt{3}}{2} \Rightarrow \beta = -\frac{\sqrt{3}}{4}\pi$ rad.

(e) $\alpha = \beta \simeq -\left(kd + \frac{\pi}{N}\right)$ for $\theta_0 = 0 \Rightarrow \beta \simeq -\left(\frac{2\pi}{\lambda}\frac{\lambda}{4} + \frac{\pi}{20}\right)$

$\Rightarrow \beta \simeq -\frac{\pi}{2} - \frac{\pi}{20} = -\frac{11\pi}{20}$ rad.

(f) $\alpha = \beta \simeq +\left(kd + \frac{\pi}{N}\right)$ for $\theta_0 = 180^\circ \Rightarrow \beta = \frac{11\pi}{20}$ rad

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② $AF_n \propto \cos\theta$; $0^\circ \le \theta \le 90^\circ$; $0^\circ \le \phi \le 360^\circ$

(a) Replace $\cos\theta$ by $\sin\theta\sin\phi$ (i.e., $\hat{a}_z\cdot\hat{a}_r$ is replaced by $\hat{a}_y\cdot\hat{a}_r$)

$\Rightarrow AF_n \simeq \sin\theta\sin\phi$; $0^\circ \le \theta \le 180^\circ$, $0^\circ \le \phi \le 180^\circ$

(b) (i) xy-plane ($\theta = 90^\circ$) $\Rightarrow AF_n = \sin\phi$

$\sin\phi_h = 0.707 \Rightarrow \phi_h = 45^\circ \Rightarrow \underbrace{\Phi_H = 2(90^\circ - 45^\circ) = 90^\circ}_{max.}$

(ii) yz-plane ($\phi = 90^\circ$) $\Rightarrow AF_n \simeq \sin\theta$

$\sin\theta_h = 0.707 \Rightarrow \theta_h = 45^\circ \Rightarrow \Theta_H = 2(90^\circ - 45^\circ) = 90^\circ$

(c) $D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi U_{max}}{\int_0^\pi\int_0^\pi U(\theta,\phi)\sin\theta\, d\theta\, d\phi}$ where $U_{max} = \sin^2\theta\sin^2\phi = 1$

and $P_{rad} = \int_0^{\pi}\int_0^{\pi} \sin^2\phi \sin^3\theta \, d\theta \, d\phi = \underbrace{\int_0^{\pi} \sin^2\phi \, d\phi}_{\frac{\pi}{2}} \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} = \frac{4\pi}{6}$

$\Rightarrow D_0 = \dfrac{4\pi(1)}{4\pi/6} = 6 = 7.782 \text{ dB}$

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(3) In the design of an array, the maximum occurs at $\theta = \theta_0$ at the design frequency $f = f_0$ which has been used to determine the progressive phase between the elements. As the shifts from f_0, the maximum also shifts to some other angle θ array from θ_0.

At a frequency f_h, the maximum of the array factor will be 0.707 of the normalized maximum value of unity. The frequency f_h is referred to as the half-power frequency, and it is used to determine the frequency bandwidth over which the pattern varies over an amplitude of 3-dB from the maximum at f_0. To determine the frequency f_h and the 3-dB frequency bandwidth, the normalized array factor of (6-10c) is written using (6 - 21) as

$$\text{AF} = \frac{1}{N}\frac{\sin\left[\frac{N\pi\, d}{\lambda}(\cos\theta - \cos\theta_0)\right]}{\sin\left[\frac{\pi\, d}{\lambda}(\cos\theta - \cos\theta_0)\right]}$$

The frequency is obscured in the array factor. To be valid over a band of frequencies, the wavelengths λ and λ_0 and the frequencies f and f_0 should be shown explicitly. Using the relation $v = \lambda f$, the array factor can be written as

$$\text{AF} = \frac{1}{N}\frac{\sin\left[\frac{\pi N d}{v}(f\cos\theta - f_0\cos\theta_0)\right]}{\sin\left[\frac{\pi\, d}{v}(f\cos\theta - f_0\cos\theta_0)\right]}$$

which peaks at $\theta = \theta_0$ when $f = f_0$. At any other frequency, the array factor peaks when

$$f\cos\theta - f_0\cos\theta_0 = 0 \Rightarrow \cos\theta = \frac{f_0}{f}\cos\theta_0$$

The half-power of the array factor is obtained by letting $\theta = \theta_0$ and occurs

$$\frac{N\pi\, d}{v}(f_h\cos\theta_0 - f_0\cos\theta_0) = \frac{N\pi\, d}{v}\cos\theta_0(f_h - f_0) = 1.391$$

or

$$(f_h - f_0) = \frac{1.391v}{N\pi\, d\cos\theta} = \frac{0.886v}{Nd\cos\theta_0} = \frac{0.886v}{(L+d)\cos\theta}$$

Therefore the 3-dB frequency bandwidth is

$$\text{BW(3-dB)} = \frac{0.886v}{Nd\cos\theta_0} = \frac{0.886v}{(L+d)\cos\theta_0}$$

Therefore the bandwidth of an array depends not on the frequency operation but rather on the array length and scan angle. This is a fundamental constraint on wide-instantaneous bandwidth of arrays.

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(4). It is derived that the element pattern of a half-wave dipole is given by

$E_\theta \simeq j\eta I_0 \dfrac{e^{-jkr}}{2\pi r}\left[\dfrac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]$: element pattern

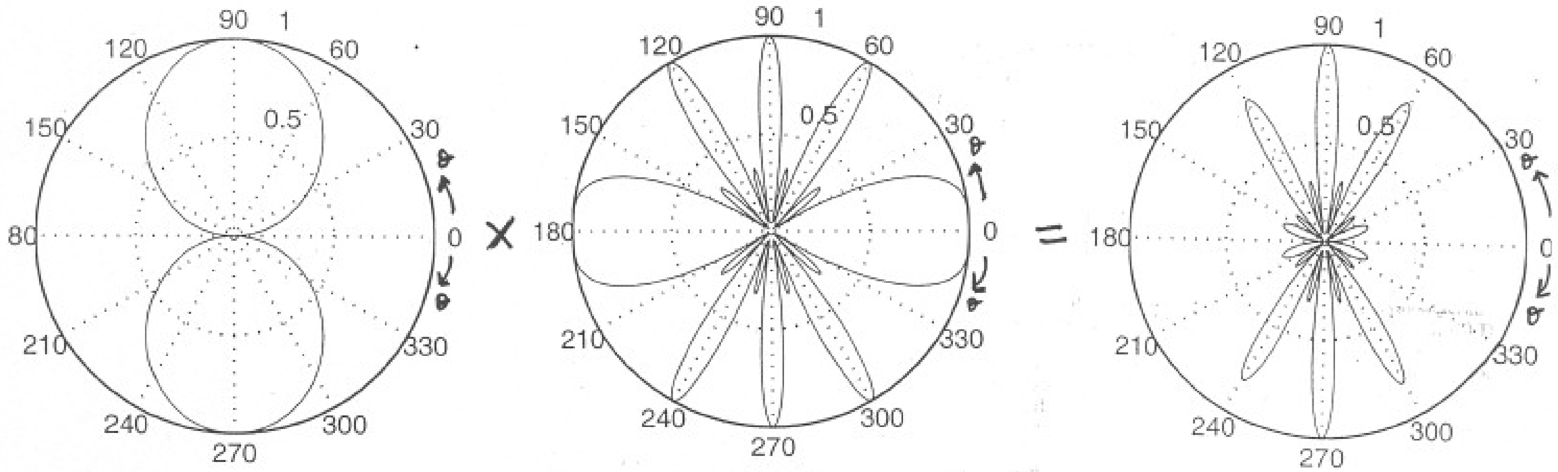
$\Rightarrow (E_e)_n = \cos(\pi/2\cos\theta)/\sin\theta$: normalized element pattern.

To find the AF_n, replace the dipoles by isotropic point sources

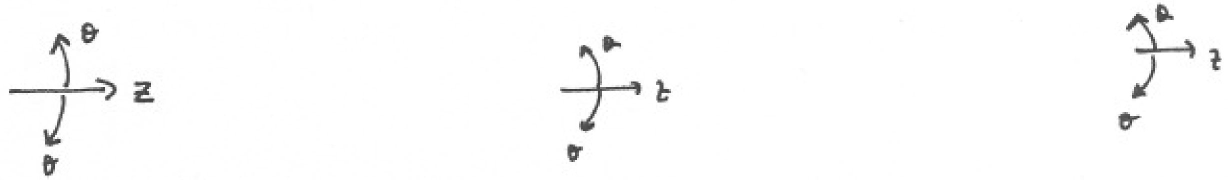
$$AF_n = \frac{1}{3}\,\frac{\sin\left[\frac{3}{2}kd\cos\theta\right]}{\sin\left(\frac{kd}{2}\cos\theta\right)} \qquad d = 2\lambda$$

$$\Rightarrow AF_n = \frac{1}{3}\,\frac{\sin(6\pi\cos\theta)}{\sin(2\pi\cos\theta)}$$

$$\Rightarrow E_{total_n}(\theta,\phi) = E_e(\theta,\phi)_n \times AF_n = \frac{1}{3}\,\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\frac{\sin(6\pi\cos\theta)}{\sin(2\pi\cos\theta)}$$



Element Pattern　　　Array Factor　　　Complete Array Pattern



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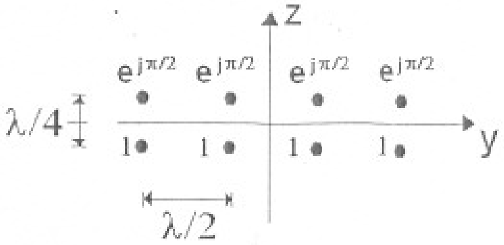
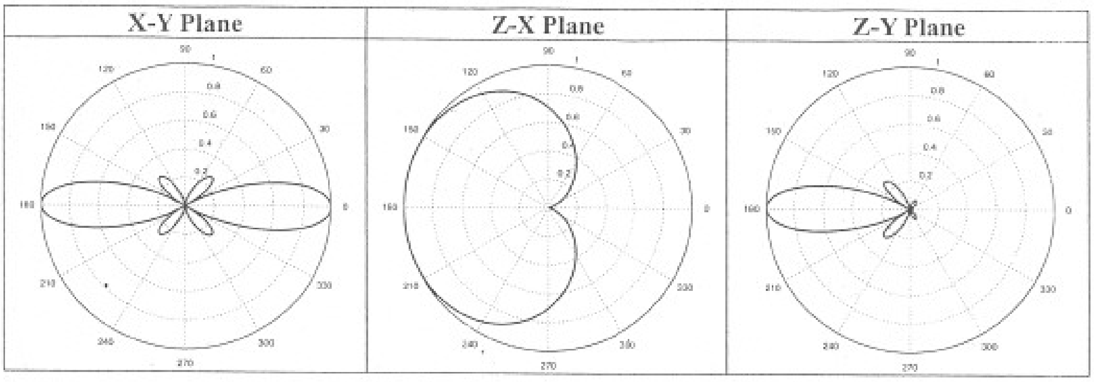


a,b)

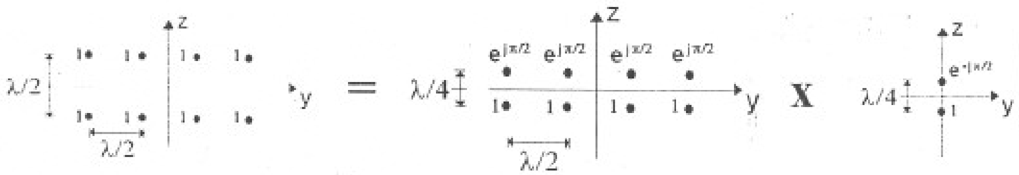
$$AF = AF_z^{(2)} AF_y^{(4)} = AF_y^{(4)} AF_z^{(2)}$$

$$= \frac{1}{4}\frac{\sin(2\pi\sin\theta\sin\phi)}{\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)}\frac{1}{2}\frac{\sin\left(\frac{\pi}{2}\cos\theta+\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4}\cos\theta+\frac{\pi}{4}\right)}$$

$$= \frac{1}{4}\cos\left(\frac{\pi}{4}\cos\theta+\frac{\pi}{4}\right)\frac{\sin(2\pi\sin\theta\sin\phi)}{\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)}$$



c)



$$AF = AF_z^{(2)}\frac{1}{8}\frac{\sin\left(\frac{\pi}{2}\cos\theta+\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4}\cos\theta+\frac{\pi}{4}\right)}\frac{\sin(2\pi\sin\theta\sin\phi)}{\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)}$$

$$= \frac{1}{8}\frac{\sin\left(\frac{\pi}{2}\cos\theta-\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4}\cos\theta-\frac{\pi}{4}\right)}\frac{\sin\left(\frac{\pi}{2}\cos\theta+\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4}\cos\theta+\frac{\pi}{4}\right)}\frac{\sin(2\pi\sin\theta\sin\phi)}{\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)}$$

$$= \frac{1}{4}\cos\left(\frac{\pi}{2}\cos\theta\right)\frac{\sin(2\pi\sin\theta\sin\phi)}{\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)}$$