EEE 452/552

Solutions to HW # 1

① (a) $(\hat{a}_y y' + \hat{a}_z z') \cdot \hat{a}_r = (\hat{a}_y y' + \hat{a}_z z') \cdot (\hat{a}_x \sin\theta \cos\phi + \hat{a}_y \sin\theta \sin\phi + \hat{a}_z \cos\theta)$

$= y' \sin\theta \sin\phi + z' \cos\theta$

(b) $(\hat{a}_x x' + \hat{a}_z z') \cdot \hat{a}_r = (\hat{a}_x x' + \hat{a}_z z') \cdot (\hat{a}_x \sin\theta \cos\phi + \hat{a}_y \sin\theta \sin\phi + \hat{a}_z \cos\theta)$

$= x' \sin\theta \cos\phi + z' \cos\theta$

(c) $(\hat{a}_x x' + \hat{a}_y y') \cdot \hat{a}_r = (\hat{a}_x x' + \hat{a}_y y') \cdot (\hat{a}_x \sin\theta \cos\phi + \hat{a}_y \sin\theta \sin\phi + \hat{a}_z \cos\theta)$

$= x' \sin\theta \cos\phi + y' \sin\theta \sin\phi$

(d) $\bar{\rho}' \cdot \hat{a}_r = (\rho' \hat{a}_\rho' \cdot \hat{a}_r) = \rho' (\cos\phi' \, \hat{a}_x + \sin\phi' \, \hat{a}_y) \cdot (\hat{a}_x \sin\theta \cos\phi + \hat{a}_y \sin\theta \sin\phi + \hat{a}_z \cos\theta)$

$= \rho' [\cos\phi' \sin\theta \cos\phi + \sin\phi' \sin\theta \sin\phi]$

$= \rho' \sin\theta \cos(\phi - \phi')$

② (a) $\sin[k(\ell/2 - |z|)] \quad -\ell/2 < z < \ell/2 = \begin{cases} \sin[k(\ell/2 - z)] & 0 \le z \le \ell/2 \\ \sin[k(\ell/2 + z)] & -\ell/2 \le z \le 0 \end{cases}$

$\Rightarrow \int_{-\ell/2}^{\ell/2} I_0 \sin[k(\ell/2 - |z|)] e^{jkz\cos\theta} dz$ becomes

$$I_0 \int_{-\ell/2}^{0} \sin[k(\ell/2 + z)] e^{jkz\cos\theta} dz + I_0 \int_{0}^{\ell/2} \sin[k(\ell/2 - z)] e^{jkz\cos\theta} dz$$

each integral above has the following form whose result is given below.

$$\int e^{\alpha x} \sin(\beta x + \gamma) \, dx = \frac{e^{\alpha x}}{\alpha^2 + \beta^2} [\alpha \sin(\beta x + \gamma) - \beta \cos(\beta x + \gamma)]$$

where $\gamma = \frac{k\ell}{2}$, $\alpha = jk\cos\theta$, $\beta = \pm k$

$$\Rightarrow \int_{-l/2}^{l/2} I_0 \sin\left[k\left(l/2 - |z|\right)\right] e^{jkz\cos\theta}\, dz$$

$$= I_0 \left\{ \frac{e^{jkz\cos\theta}}{k^2 - k^2\cos^2\theta}\left[jk\cos\theta \sin\left(kz + \frac{kl}{2}\right) - k\cos\left(kz + \frac{kl}{2}\right)\right]\Bigg[_{-l/2}^{0}\right.$$

$$\left. + \frac{e^{jkz\cos\theta}}{k^2 - k^2\cos^2\theta}\left[jk\cos\theta \sin\left(-kz + \frac{kl}{2}\right) + k\cos\left(-kz + \frac{kl}{2}\right)\right]\Bigg[_{0}^{l/2} \right\}$$

$$= I_0 \left\{ \frac{1}{k^2\sin^2\theta}\left[jk\cos\theta \sin\left(\frac{kl}{2}\right) - k\cos\left(\frac{kl}{2}\right)\right] - \frac{e^{-jk\frac{l}{2}\cos\theta}}{k^2\sin^2\theta}\left[\cancel{jk\cos\theta \sin\left(-\frac{kl}{2} + \frac{kl}{2}\right)}^{0}\right.\right.$$

$$\left. - k\overbrace{\cos\left(-\frac{kl}{2} + \frac{kl}{2}\right)}^{1}\right]$$

$$+ \frac{e^{jk\,l/2\cos\theta}}{k^2\sin^2\theta}\left[jk\cos\theta\, \cancel{\sin\left(-\frac{kl}{2} + \frac{kl}{2}\right)}^{0} + k\overbrace{\cos\left(-\frac{kl}{2} + \frac{kl}{2}\right)}^{1}\right] - \frac{1}{k^2\sin^2\theta}\left[jk\cos\theta\sin\left(\frac{kl}{2}\right)\right.$$

$$\left.\left. + k\cos\left(\frac{kl}{2}\right)\right]\right\}$$

$$= I_0 \left\{ \frac{1}{k^2\sin^2\theta}\left(\cancel{jk\cos\theta\sin\left(\frac{kl}{2}\right)} - k\cos\left(\frac{kl}{2}\right) + e^{-jk\,l/2\cos\theta}\,k + e^{jk\,l/2\cos\theta}\,k\right.\right.$$

$$\left.\left. - \cancel{jk\cos\theta\sin\left(\frac{kl}{2}\right)} - k\cos\left(\frac{kl}{2}\right)\right)\right.$$

$$= I_0 \frac{1}{k\sin^2\theta}\left[2\cos\left(\frac{kl}{2}\cos\theta\right) - 2\cos\left(\frac{kl}{2}\right)\right] = \frac{2}{k}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin^2\theta}\right]$$

if $l = \lambda/2 \Rightarrow \frac{kl}{2} = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$ & $\cos(\pi/2) = 0$

$$\Rightarrow \int_{-l/2}^{l/2} I_0 \sin\left[k\left(l/2 - |z|\right)\right] e^{jkz\cos\theta}\, dz = \frac{2}{k}\,\frac{\cos\left(\pi/2\cos\theta\right)}{\sin^2\theta} \quad //$$

(b) $$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\,dx'\,dy'$$

Notice that x' & y' are separable!

$$= \int_{-b/2}^{b/2} e^{jky'\sin\theta\sin\phi}\,dy' \int_{-a/2}^{a/2} e^{jkx'\sin\theta\cos\phi}\,dx'$$

and each integral is in the form $$\int_{-c/2}^{c/2} e^{j\alpha z}\,dz = c\left[\frac{\sin\left(\frac{\alpha}{2}c\right)}{\frac{\alpha}{2}c}\right]$$

Thus; $$\int_{-b/2}^{b/2} e^{jk\sin\theta\sin\phi\, y'}\,dy' = b\left[\frac{\sin\left[\frac{kb}{2}\sin\theta\sin\phi\right]}{\frac{kb}{2}\sin\theta\sin\phi}\right]$$

$$\int_{-a/2}^{a/2} e^{jkx'\sin\theta\cos\phi}\,dx' = a\left[\frac{\sin\left[\frac{ka}{2}\sin\theta\cos\phi\right]}{\frac{ka}{2}\sin\theta\cos\phi}\right]$$

$$\Rightarrow \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\,dx'\,dy' = ab\left[\left(\frac{\sin X}{X}\right)\left(\frac{\sin Y}{Y}\right)\right] //$$

with $X = \frac{ka}{2}\sin\theta\cos\phi$

$Y = \frac{kb}{2}\sin\theta\sin\phi$

if $\phi = \pi/2 \Rightarrow \cos\phi = 0,\ \sin\phi = 1 \Rightarrow X = 0 \Rightarrow \frac{\sin X}{X} = 1$

result of the integral becomes $ab\,\dfrac{\sin\left(\frac{kb}{2}\sin\theta\right)}{\frac{kb}{2}\sin\theta}$ //

if $\phi = 0 \Rightarrow \cos\phi = 1,\ \sin\phi = 0 \Rightarrow Y = 0 \Rightarrow \frac{\sin Y}{Y} = 1$

result of the integral becomes $ab\,\dfrac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta}$ //

(c) $\int_0^{2\pi} e^{jk\rho' \sin\theta \cos(\phi-\phi')} d\phi'$

We'll use the integral identity $J_n(x) = \frac{j^{-n}}{2\pi} \int_0^{2\pi} e^{jx\cos\phi} e^{jn\phi} d\phi$.

$\cos(\phi-\phi') = \cos(\phi'-\phi)$ cosine function is always even!

let $\alpha = \phi' - \phi \Rightarrow d\alpha = d\phi'$

$$\Rightarrow \int_0^{2\pi} e^{jk\rho' \sin\theta \cos(\phi-\phi')} d\phi' = \int_{-\phi}^{2\pi-\phi} e^{jk\rho' \sin\theta \cos\alpha} d\alpha$$

let $X = k\rho' \sin\theta$. Then, we need to show that

$$\int_{-\phi}^{2\pi-\phi} e^{jX\cos\alpha} d\alpha = \int_0^{2\pi} e^{jX\cos\alpha} d\alpha$$

. Remember $\cos\alpha$ is periodic with 2π

$$\Rightarrow \int_{-\phi}^{0} e^{jX\cos\alpha} d\alpha + \underbrace{\int_0^{2\pi-\phi} e^{jX\cos\alpha} d\alpha}_{\text{same}} \stackrel{?}{=} \underbrace{\int_0^{2\pi-\phi} e^{jX\cos\alpha} d\alpha}_{\text{same}} + \int_{2\pi-\phi}^{2\pi} e^{jX\cos\alpha} d\alpha$$

$$\Rightarrow \int_{-\phi}^{0} e^{jX\cos\alpha} d\alpha \stackrel{?}{=} \int_{2\pi-\phi}^{2\pi} e^{jX\cos\alpha} d\alpha$$

let $\beta = 2\pi - \alpha \Rightarrow d\alpha = -d\beta \Rightarrow \int_{2\pi-\phi}^{2\pi} e^{jX\cos\alpha} d\alpha = \int_{\phi}^{0} e^{jX\cos(2\pi-\beta)} (-d\beta)$

$$= \int_{\phi}^{0} e^{jX\cos\beta} (-d\beta)$$

let $\beta' = -\beta \Rightarrow d\beta' = -d\beta$

$$\Rightarrow \int_{-\phi}^{0} e^{jX\cos(-\beta')} d\beta' = \int_{-\phi}^{0} e^{jX\cos\beta'} d\beta'$$

. Since β', β, α are dummy variables

same

$$\int_{-\phi}^{2\pi-\phi} e^{jx\cos\alpha}\,d\alpha = \int_{0}^{2\pi} e^{jx\cos\alpha}\,d\alpha \Rightarrow J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\cos\alpha}\,d\alpha$$

$$\Rightarrow \int_0^{2\pi} e^{jk\rho'\sin\theta\cos(\phi-\phi')} = 2\pi J_0(k\rho'\sin\theta)$$

(3) $\nabla\times\bar{H} = \epsilon\frac{\partial\bar{E}}{\partial t} + \bar{J}$; $\nabla\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t}$; $\nabla\cdot\bar{E} = \rho/\epsilon$; $\nabla\cdot\bar{H} = 0$

$$\nabla\times(\nabla\times\bar{H}) = \epsilon\frac{\partial}{\partial t}\nabla\times\bar{E} + \nabla\times\bar{J} = \epsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) + \nabla\times\bar{J}$$

$$\nabla(\cancelto{0}{\nabla\cdot\bar{H}}) - \nabla^2\bar{H} = -\epsilon\mu\frac{\partial^2\bar{H}}{\partial t^2} + \nabla\times\bar{J} \Rightarrow \boxed{\nabla^2\bar{H} - \epsilon\mu\frac{\partial^2\bar{H}}{\partial t^2} = -\nabla\times\bar{J}}$$ solution to part (c)

if $\bar{J} = 0 \Rightarrow \boxed{\nabla^2\bar{H} - \epsilon\mu\frac{\partial^2\bar{H}}{\partial t^2} = 0}$ solution to part (a)

$$\nabla\times(\nabla\times\bar{E}) = -\mu\frac{\partial}{\partial t}\nabla\times\bar{H} = -\mu\frac{\partial}{\partial t}\left(\epsilon\frac{\partial\bar{E}}{\partial t} + \bar{J}\right)$$

$$\Rightarrow \nabla(\underbrace{\nabla\cdot\bar{E}}_{\rho/\epsilon}) - \nabla^2\bar{E} = -\epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} - \mu\frac{\partial\bar{J}}{\partial t} \Rightarrow \nabla(\rho/\epsilon) - \nabla^2\bar{E} = -\epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} - \mu\frac{\partial\bar{J}}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2\bar{E} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} = \mu\frac{\partial\bar{J}}{\partial t} + \nabla(\rho/\epsilon)}$$ solution to part (b)

(4) (a) If $\bar{H}_e = j\omega\epsilon\nabla\times\bar{\Pi}_e$ — (1)

$\nabla\times\bar{E}_e = -j\omega\mu\bar{H}_e \Rightarrow \nabla\times\bar{E}_e = -j\omega\mu(j\omega\epsilon\nabla\times\bar{\Pi}_e) = \omega^2\mu\epsilon\nabla\times\bar{\Pi}_e$

$$\Rightarrow \nabla\times(\bar{E}_e - \omega^2\mu\epsilon\bar{\Pi}_e) = \nabla\times(\bar{E}_e - k^2\bar{\Pi}_e) = 0 \Rightarrow \bar{E}_e - k^2\bar{\Pi}_e = -\nabla\Phi_e$$

$$\Rightarrow \bar{E}_e = -\nabla\Phi_e + k^2\bar{\Pi}_e \quad \text{—— (2)}$$

By taking curl of (1)

$$\nabla\times\bar{H}_e = j\omega\epsilon\nabla\times\nabla\times\bar{\Pi}_e = j\omega\epsilon\left[\nabla(\nabla\cdot\bar{\Pi}_e) - \nabla^2\bar{\Pi}_e\right] \quad \text{—— (3)}$$

from Maxwell's eqns. $\nabla \times \bar{H}_e = \bar{J} + j\omega\epsilon \bar{E}_e$

$$\Rightarrow \bar{J} + j\omega\epsilon \bar{E}_e = j\omega\epsilon \left[\nabla(\nabla \cdot \bar{\Pi}_e) - \nabla^2 \bar{\Pi}_e \right] \quad \text{——} \quad (4)$$

substitute (2) into (4)

$$\Rightarrow \bar{J} + j\omega\epsilon\left(-\nabla\Phi_e + k^2 \bar{\Pi}_e\right) = j\omega\epsilon\, \nabla(\nabla \cdot \bar{\Pi}_e) - j\omega\epsilon\, \nabla^2 \bar{\Pi}_e$$

$$\Rightarrow \nabla^2 \bar{\Pi}_e + k^2 \bar{\Pi}_e = \frac{-\bar{J}}{j\omega\epsilon} + \nabla(\nabla \cdot \bar{\Pi}_e) + \nabla\Phi_e$$

$$= \frac{-\bar{J}}{j\omega\epsilon} + \nabla\left(\nabla \cdot \bar{\Pi}_e + \Phi_e\right)$$

let $\Phi_e = -\nabla \cdot \bar{\Pi}_e$

$$\Rightarrow \nabla^2 \bar{\Pi}_e + k^2 \bar{\Pi}_e = j\frac{\bar{J}}{\omega\epsilon} \quad //$$

(b) In (2) $\bar{E}_e = -\nabla\Phi_e + k^2 \bar{\Pi}_e$ but $\Phi_e = -\nabla \cdot \bar{\Pi}_e \Rightarrow \nabla\Phi_e = -\nabla(\nabla \cdot \bar{\Pi}_e)$

$$\Rightarrow \bar{E}_e = k^2 \bar{\Pi}_e + \nabla(\nabla \cdot \bar{\Pi}_e)$$

(c) Now compare $\nabla^2 \bar{\Pi}_e + k^2 \bar{\Pi}_e = j\frac{\bar{J}}{\omega\epsilon}$ with $\nabla^2 \bar{A} + k^2 \bar{A} = -\mu \bar{J}$

let $\bar{\Pi}_e = -j\frac{1}{\omega\mu\epsilon}\bar{A}$

$$\Rightarrow \nabla^2\left(-j\frac{1}{\omega\mu\epsilon}\bar{A}\right) + k^2\left(-j\frac{1}{\omega\mu\epsilon}\bar{A}\right) = j\frac{\bar{J}}{\omega\epsilon}$$

$$\Rightarrow -j\frac{1}{\omega\mu\epsilon}\nabla^2 \bar{A} + \left(-j\frac{1}{\omega\mu\epsilon}\right)k^2 \bar{A} = j\frac{\bar{J}}{\omega\epsilon}$$

$$\Rightarrow \nabla^2 \bar{A} + k^2 \bar{A} = \frac{\omega\mu\epsilon}{(-j)}\, j\frac{\bar{J}}{\omega\epsilon} = -\mu \bar{J} \quad \checkmark$$