EEE 452/552

Solutions to HW # 2

1) $\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$ — (1)

$\nabla\times\bar{H} = \bar{J}_i + \frac{\partial\bar{D}}{\partial t} + \sigma\bar{E}$ — (2)

$\nabla\cdot\bar{D} = \rho$ — (3)

$\nabla\cdot\bar{B} = 0$ — (4)

If $\nabla\cdot\bar{B} = 0 \Rightarrow \bar{B} = \nabla\times\bar{A}$ — (5)

using (5) in (1)

$\Rightarrow \nabla\times\bar{E} = -\frac{\partial}{\partial t}(\nabla\times\bar{A})$ that can be written as

$\Rightarrow \nabla\times\bar{E} = -\nabla\times\frac{\partial\bar{A}}{\partial t} \Rightarrow \nabla\times\left(\bar{E}+\frac{\partial\bar{A}}{\partial t}\right) = 0$

$\Rightarrow \bar{E}+\frac{\partial\bar{A}}{\partial t} = -\nabla\Phi \Rightarrow \boxed{\bar{E} = -\frac{\partial\bar{A}}{\partial t} - \nabla\Phi}$ — (6)

Multiply (2) by μ

$\Rightarrow \nabla\times\bar{B} = \mu\bar{J}_i + \mu\epsilon\frac{\partial\bar{E}}{\partial t} + \mu\sigma\bar{E}$ — (7)

Substitute (5) and (6) into (7)

$\Rightarrow \nabla\times\nabla\times\bar{A} = \mu\bar{J}_i + \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial\bar{A}}{\partial t} - \nabla\Phi\right) + \mu\sigma\left(-\frac{\partial\bar{A}}{\partial t} - \nabla\Phi\right)$

$\Rightarrow \nabla\times\nabla\times\bar{A} = \mu\bar{J}_i - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} - \mu\epsilon\frac{\partial}{\partial t}\nabla\Phi - \mu\sigma\frac{\partial\bar{A}}{\partial t} - \mu\sigma\nabla\Phi$ — (8)

$\Rightarrow \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} = \mu\bar{J}_i - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} - \mu\sigma\frac{\partial\bar{A}}{\partial t} - \nabla\left(\mu\epsilon\frac{\partial\Phi}{\partial t} + \mu\sigma\Phi\right)$ — (9)

Rearranging (9)

$\nabla^2\bar{A} - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} - \mu\sigma\frac{\partial\bar{A}}{\partial t} = -\mu\bar{J}_i + \nabla\left[\nabla\cdot\bar{A} + \mu\epsilon\frac{\partial\Phi}{\partial t} + \mu\sigma\Phi\right]$ — (10)

In (10) let $\boxed{\nabla\cdot\bar{A} = -\mu\epsilon\frac{\partial\Phi}{\partial t} - \mu\sigma\Phi}$ ← This should be your gauge! — (11)

$\Rightarrow \boxed{\nabla^2\bar{A} - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} - \mu\sigma\frac{\partial\bar{A}}{\partial t} = -\mu\bar{J}_i}$ wave eqn. for $\bar{A}$ in time domain.

— (12)

The phasor forms of (11) and (12)

$\nabla\cdot\bar{A} = -j\omega\mu\epsilon\Phi - \mu\sigma\Phi \Rightarrow \nabla\cdot\bar{A} = \mu\Phi(-j\omega\epsilon - \sigma)$ // — (13)

$\nabla^2\bar{A} + k^2\bar{A} - j\omega\mu\sigma\bar{A} = -\mu\bar{J}_i$ — (14)

From (3) $\nabla\cdot\bar{D} = \rho \Rightarrow \nabla\cdot\bar{E} = \rho/\epsilon$ and using (6)

$\nabla\cdot\left(-\nabla\Phi - \frac{\partial\bar{A}}{\partial t}\right) = \rho/\epsilon \Rightarrow \nabla^2\Phi + \frac{\partial}{\partial t}(\nabla\cdot\bar{A}) = -\rho/\epsilon$ — (15)

Substituting (11) into (15)

$$\nabla^2 \Phi + \frac{\partial}{\partial t}\left(-\mu\epsilon \frac{\partial \Phi}{\partial t} - \mu\sigma \Phi\right) = -\rho/\epsilon$$

$$\Rightarrow \boxed{\nabla^2\Phi - \mu\epsilon\frac{\partial^2\Phi}{\partial t^2} - \mu\sigma\frac{\partial\Phi}{\partial t} = -\rho/\epsilon} \quad \text{—— (16)}$$

wave eqn. for Φ in time-domain

$$\boxed{\nabla^2\Phi + k^2\Phi - j\omega\mu\sigma\Phi = -\rho/\epsilon} \quad \leftarrow \text{phasor form of (16)}$$

—— (17)

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② (a) $$\frac{\partial^2}{\partial z^2}\left[\cos(t - z/v) + \cos(t + z/v)\right] = -\frac{1}{v^2}\left[\cos(t - z/v) + \cos(t + z/v)\right] = -\frac{1}{v^2}\psi(z,t)$$

$$\frac{\partial^2}{\partial t^2}\left[\cos(t - z/v) + \cos(t + z/v)\right] = -\left[\cos(t - z/v) + \cos(t + z/v)\right] = -\psi(z,t)$$

$$\Rightarrow \text{since } \frac{1}{v^2} = \mu\epsilon \Rightarrow -\mu\epsilon\,\psi(z,t) - \mu\epsilon(-\psi(z,t)) = 0 = 0$$

(b) $$\frac{\partial^2}{\partial z^2}\left[C_0^+\cos(\omega t - kz) + C_0^-\cos(\omega t + kz)\right] = -k^2\left[C_0^+\cos(\omega t - kz) + C_0^-\cos(\omega t + kz)\right] = -k^2\psi(z,t)$$

$$\frac{\partial^2}{\partial t^2}\left[C_0^+\cos(\omega t - kz) + C_0^-\cos(\omega t + kz)\right] = -\omega^2\left[C_0^+\cos(\omega t - kz) + C_0^-\cos(\omega t + kz)\right] = -\omega^2\psi(z,t)$$

$$\Rightarrow -k^2\psi(z,t) - \mu\epsilon\left(-\omega^2\psi(z,t)\right) = -k^2\psi(z,t) + \underbrace{\omega^2\mu\epsilon}_{k^2}\psi(z,t) = 0 = 0$$

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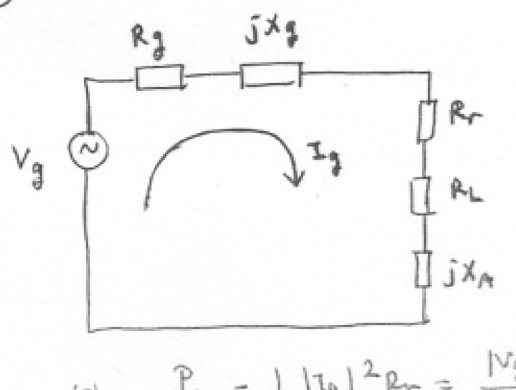
③



$$I_g = \frac{V_g}{(R_g + R_r + R_L) + j(X_g + X_A)}$$

$$\Rightarrow |I_g| = \frac{|V_g|}{\left[(R_g + R_r + R_L)^2 + (X_g + X_A)^2\right]^{1/2}}$$

(a) $$P_{R_r} = \frac{1}{2}|I_g|^2 R_r = \frac{|V_g|^2}{2}\,\frac{R_r}{(R_g + R_r + R_L)^2 + (X_g + X_A)^2}$$

(b) $$P_{R_L} = \frac{1}{2}|I_g|^2 R_L = \frac{|V_g|^2}{2}\,\frac{R_L}{(R_g + R_r + R_L)^2 + (X_g + X_A)^2}$$

(c) $$P_{R_g} = \frac{1}{2}|I_g|^2 R_g = \frac{|V_g|^2}{2}\,\frac{R_g}{(R_g + R_r + R_L)^2 + (X_g + X_A)^2}$$

(d) when $X_A = -X_g$ and $R_g = (R_r + R_L)$

$$\Rightarrow \quad P_{R_r} = \frac{|V_g|^2}{8} \frac{R_r}{(R_r + R_L)^2} \qquad \left(\text{or} \quad \frac{|V_g|^2}{8} \frac{R_r}{R_g^2} \right)$$

$$\Rightarrow \quad P_{R_L} = \frac{|V_g|^2}{8} \frac{R_L}{(R_r + R_L)^2} \qquad \left(\text{or} \quad \frac{|V_g|^2}{8} \frac{R_L}{R_g^2} \right)$$

$$\Rightarrow \quad P_{R_g} = \frac{|V_g|^2}{8} \frac{R_g}{(R_r + R_L)^2} = \frac{|V_g|^2}{8} \frac{1}{R_g}$$

(e) power supplied by the generator : $P_s = \frac{1}{2} V_g I_g^* = \frac{1}{2} V_g \frac{V_g^*}{2 R_g} = \frac{|V_g|^2}{4 R_g}$

$$= \frac{|V_g|^2}{4 (R_L + R_r)}$$

Notice that

$$P_{R_r} + P_{R_L} + P_{R_g} = \frac{|V_g|^2}{8} \left(\frac{R_r + R_L}{(R_r + R_L)^2} + \frac{1}{R_g} \right) = \frac{|V_g|^2}{8} \left(\frac{1}{R_g} + \frac{1}{R_g} \right)$$

$$= \frac{|V_g|^2}{4 R_g} = P_s .$$