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Solutions to HW # 3

(1) $$I = \frac{1}{T}\int_0^T \bar{E}(\bar{r}) \times \bar{H}(\bar{r})\, e^{j2\omega t}\, dt = \frac{1}{T}\, \bar{E}(\bar{r}) \times \bar{H}(\bar{r}) \int_0^T e^{j2\omega t}\, dt \qquad \omega = 2\pi f = \frac{2\pi}{T}$$

$\Rightarrow$ let's focus on the integral $$\int_0^T e^{j2\,2\pi/T\, t}\, dt = \left.\frac{e^{j4\pi t/T}}{j4\pi/T}\right|_0^T$$

$$= \frac{e^{j4\pi} - e^{j0}}{j4\pi/T} = 0 \;//$$

(2) let $$\bar{E}(\bar{r}) = \bar{E}_o e^{-j\bar{k}\cdot\bar{r}} = \bar{E}_o e^{-jk\hat{a}_k\cdot\bar{r}} \qquad \text{where } \bar{E}_o \perp \hat{a}_k$$

$$\Rightarrow \bar{H}(\bar{r}) = \frac{\hat{a}_k \times \bar{E}_o}{\eta} e^{-jk\hat{a}_k\cdot\bar{r}} \qquad \Rightarrow \bar{H}(\bar{r})^* = \frac{\hat{a}_k \times \bar{E}_o^*}{\eta} e^{jk\hat{a}_k\cdot\bar{r}}$$

$$\Rightarrow (\bar{E}\times\bar{H}^*) = \frac{1}{2}\underbrace{\frac{\bar{E}_o \times (\hat{a}_k \times \bar{E}_o^*)}{\eta} = \frac{1}{2}\frac{(\bar{E}_o\cdot\bar{E}_o^*)\,\hat{a}_k - \overbrace{(\bar{E}_o\cdot\hat{a}_k)}^{\to 0}\,\bar{E}_o^*}{\eta}}_{\text{vector identity}}$$

$$\Rightarrow P_{ave} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \frac{1}{2}\mathrm{Re}\left\{\frac{\overbrace{(\bar{E}_o\cdot\bar{E}_o^*)}^{=|\bar{E}_o|^2}}{\eta}\hat{a}_k\right\} = \frac{1}{2}\frac{|\bar{E}_o|^2}{\eta}\hat{a}_k \;//$$

(3) (a) $$\bar{P}(r,\theta,t) = \bar{E}(r,\theta,t)\times\bar{H}(r,\theta,t) = \mathrm{Re}\{\bar{E}(r,\theta)e^{j\omega t}\} \times \mathrm{Re}\{\bar{H}(r,\theta)e^{j\omega t}\}$$

$$= (\hat{a}_\theta \times \hat{a}_\phi)\, 30\pi \left(\frac{I\,dl}{\lambda r}\right)^2 \sin^2\theta\, \sin^2(\omega t - kr)$$

$$= \hat{a}_r\, 15\pi\left(\frac{I\,dl}{\lambda r}\right)^2 \sin^2\theta \left\{1 - \cos[2(\omega t - kr)]\right\}$$

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(b) $$\bar{P}_{ave}(r,\theta) = \hat{a}_r\, 15\pi\left(\frac{I\,dl}{\lambda r}\right)^2 \sin^2\theta$$

(c) $\bar{P}_{ave}(r,\theta) = \oint_S \bar{P}_{ave}(r,\theta)\cdot\hat{a}_r\, ds = \int_0^{2\pi}\int_0^{\pi} 15\pi \left(\frac{I\,dl}{\lambda r^2}\right) \sin^2\theta \; r^2 \sin\theta \, d\theta \, d\phi$

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$$= 40\pi^2 \left(\frac{dl}{\lambda}\right)^2 I^2 \quad \text{(watts)}$$

(4) $\bar{E} = \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi \Rightarrow \bar{H} = \frac{1}{\eta}\hat{a}_k \times \bar{E} = \frac{1}{\eta}\hat{a}_r \times \bar{E} = \frac{1}{\eta}\left(\hat{a}_\phi E_\theta - \hat{a}_\theta E_\phi\right)$

(Radiated fields are assumed to be plane waves. It does not violate the generality,

$$\Rightarrow \bar{P}_{ave} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \frac{1}{2}\mathrm{Re}\left\{\left(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi\right)\times\frac{1}{\eta}\left(\hat{a}_\phi E_\theta^* - \hat{a}_\theta E_\phi^*\right)\right\}$$

$$= \frac{1}{2\eta}\left(|E_\theta|^2 + |E_\phi|^2\right)\hat{a}_r \quad //$$

(5) $\bar{J} = \hat{a}_z \frac{I}{\pi b^2} \Rightarrow \bar{E} = \frac{\bar{J}}{\sigma} = \hat{a}_z \frac{I}{\sigma\pi b^2}$

From Ampere's law $\bar{H} = \hat{a}_\phi \frac{I}{2\pi b}$ on the surface of the conducting wire!

$\therefore$ The poynting vector at the surface of the wire is

$$\bar{P} = \bar{E}\times\bar{H} = \left(\hat{a}_z\times\hat{a}_\phi\right)\frac{I^2}{2\sigma\pi^2 b^3} = -\hat{a}_\rho \frac{I^2}{2\sigma\pi^2 b^3}$$

(directed everywhere into the wire surface.)

Integrate $\bar{P}$ over the wall of the wire-segment

$$-\oint_S \bar{P}\cdot d\bar{s} = -\oint_S \bar{P}\cdot\hat{a}_\rho\, ds = \left(\frac{I}{2\sigma\pi^2 b^3}\right)2\pi b l = I^2\left(\frac{l}{\sigma\pi b^2}\right) \doteq I^2 R$$

a the formula for the resistance of a straight wire, $R = \frac{l}{\sigma S}$ is used.

As a result, the negative surface integral of the Poynting vector $= IR^2$