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Solutions to HW # 4

① $\lim_{r\to 0} c \int_V e^{-jkr} r \sin\theta \, d\theta \, d\phi \, dr = c\,4\pi \int_0^{\alpha} e^{-jkr} r \, dr$ and then let $\alpha = 0$

$c\,4\pi \int_0^{\alpha} e^{-jkr} r \, dr$ let $u = -jkr \Rightarrow du = -jk \, dr$ and if $r \to 0$, $u \to 0$

$$c\,4\pi \int_0^{-jk\alpha} e^{u} \frac{u}{(-jk)} \frac{du}{(-jk)} = -\frac{c\,4\pi}{k^2}\left[u e^{u} - e^{u} \right]_0^{-jk\alpha} = -\frac{c\,4\pi}{k^2}\left[-jk\alpha e^{-jk\alpha} - e^{-jk\alpha} + 1 \right]$$

$$\lim_{\alpha\to 0} \left[-jk\alpha e^{-jk\alpha} - e^{-jk\alpha} + 1 \right] = 0$$

$\Rightarrow$ The integral is zero //

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② During the derivation of wave eqn. for $\bar{A}$ we found that

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \nabla \bar{\Phi} \quad \text{—} (*)$$

Also we defined the Lorentz's gauge as $\nabla \cdot \bar{A} = -\epsilon\mu \frac{\partial \Phi}{\partial t}$ — $(**)$

In phasor domain $(*)$ becomes $\bar{E} = -j\omega\bar{A} - \nabla\bar{\Phi}$ and $(**)$ becomes

$$\nabla \cdot \bar{A} = -j\omega\epsilon\mu\bar{\Phi} \Rightarrow \bar{\Phi} = \frac{1}{-j\omega\epsilon\mu} \nabla \cdot \bar{A} \quad \text{—} (***)$$

Using $(***)$ in $\bar{E} = -j\omega\bar{A} - \nabla\bar{\Phi}$ we find

$$\bar{E} = -j\omega\bar{A} - \nabla\left(\frac{1}{-j\omega\epsilon\mu} \nabla \cdot \bar{A} \right) \Rightarrow \boxed{\bar{E} = -j\omega\bar{A} - j\frac{1}{\omega\mu\epsilon} \nabla(\nabla \cdot \bar{A})}$$

//

(3) $\bar{J}\left(\bar{r}', t - \frac{|\bar{r}-\bar{r}'|}{v}\right) = Re\left\{\bar{J}(\bar{r}')\, e^{j\omega\left(t - \frac{1}{v}|\bar{r}-\bar{r}'|\right)}\right\}$

$$= Re\left\{\bar{J}(\bar{r}')\, e^{j\omega t}\, e^{-j\omega \frac{|\bar{r}-\bar{r}'|}{v}}\right\}$$

$$\Rightarrow \bar{A}(\bar{r},t) = Re\left\{\bar{A}(\bar{r})\, e^{j\omega t}\right\} = \frac{\mu}{4\pi}\int_{v'} \frac{Re\left\{\bar{J}(\bar{r}')\, e^{-j\frac{\omega}{v}|\bar{r}-\bar{r}'|}\, e^{j\omega t}\right\}}{|\bar{r}-\bar{r}'|}$$

$k = \frac{\omega}{v}$

$$= Re\left\{\frac{\mu}{4\pi}\underbrace{\int_{v'} \bar{J}(\bar{r}')\, \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}}\, e^{j\omega t}\, dv'\right\}$$

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\int_{v'} \bar{J}(\bar{r}')\, \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\, dv'$$

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(4) Consider $E_r \simeq -j\eta \frac{I_o l \cos\theta}{2\pi k r^3} e^{-jkr}$ $\qquad k = \omega\sqrt{\mu\epsilon}$

$\Rightarrow E_r \sim -j\sqrt{\frac{\mu}{\epsilon}}\, \frac{I_o l \cos\theta}{2\pi\, \omega\sqrt{\mu\epsilon}\; r^3}\, e^{-j\omega\sqrt{\mu\epsilon}\, r}$ $\qquad$ Also $i(t) = I_o \cos\omega t$ & $i(t) = \frac{dq}{dt}$

$\Rightarrow I_o = j\omega q$

$\Rightarrow E_r \sim -j\sqrt{\frac{\mu}{\epsilon}}\, \frac{j\cancel{\omega} q\, l \cos\theta}{2\pi \cancel{\omega}\sqrt{\mu\epsilon}}\, \frac{e^{-j\omega\sqrt{\mu\epsilon}\, r}}{r^3}$ $\qquad$ let $\bar{p}_e = q l$

$\Rightarrow E_r \sim \frac{p_e\, 2\cos\theta}{4\pi\epsilon r^3}\, e^{-j\omega\sqrt{\mu\epsilon}\, r}$ $\qquad$ but $\omega \to 0$ $\quad e^{-j\omega\sqrt{\mu\epsilon}\, r} \approx 1$

$\Rightarrow E_r \sim \frac{p_e\, 2\cos\theta}{4\pi\epsilon r^3}$ $\qquad$ Similarly $E_\theta \simeq -j\eta \frac{I_o l \sin\theta}{4\pi k r^3} e^{-jkr}$ becomes

$$E_\theta \sim \frac{p_e}{4\pi\epsilon r^3}\sin\theta$$

$\Rightarrow \bar{E} = \frac{p_e}{4\pi\epsilon r^3}\left(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta\right)$ //

① Write the units of the following electromagnetic quantities!
(10 points)

(a) Magnetic field intensity : $\bar{H}$ (A/m)

(b) Electric field intensity : $\bar{E}$ (V/m)

(c) Surface current density : $\bar{J}_s$ (A/m)

(d) Volume current density : $\bar{J}$ (A/m²)

(e) Surface charge density : ρ_s (C/m²)

(f) Volume charge density : ρ (C/m³)

(g) The displacement current vector (density) : $\frac{\partial \bar{D}}{\partial t}$ (A/m²)

(h) Power density : w watts/m³

(i) The term $\sigma\bar{E}$ (A/m²)

(j) Magnetic flux density : $\bar{B}$ (Tesla or Wb/m²)

② If the <u>transmitting pattern</u> of an antenna is given, what can you say the <u>receiving</u> and <u>scattering patterns</u> of the same antenna?
(10 points)

The transmitting and receiving patterns are the SAME! $P^r = P^t$

Scattering pattern is different from them! $P^s \neq P^r$

$P^s \neq P^t$

③ Express $3\cos(wt) - 4\sin(wt)$ in the form of
(20 points)

(a) $A_1 \cos(wt + \theta_1)$

(b) $A_2 \sin(wt + \theta_2)$

You need to find A_1 and A_2 exactly and express θ_1 and θ_2 as

$\theta_1 = \tan^{-1}\left(\frac{\alpha}{\beta}\right)$, where α & β are just some numbers to indicate how you should give your answer.

(a) Use $\cos(wt)$ as the reference. In phasor domain $3\cos wt \longleftrightarrow 3$

$-4\sin wt \longleftrightarrow -4e^{-j\pi/2}$
$= j4$

$$\Rightarrow 3 + j4 = 5e^{j\tan^{-1}(4/3)}$$

$$\Rightarrow A_1 = 5 \quad \& \quad \theta_1 = \tan^{-1}(4/3)$$

$$\Rightarrow 3\cos(wt) - 4\sin(wt) = 5\cos[wt + \tan^{-1}(4/3)] \quad //$$

(b) Use $\sin(wt)$ as the reference. In phasor domain $3\cos wt \longleftrightarrow 3j$

$-4\sin wt \longleftrightarrow -4$

$$\Rightarrow 3j - 4 = 5e^{j\tan^{-1}(3/-4)}$$

$$\Rightarrow A_2 = 5 \quad \& \quad \theta_2 = \tan^{-1}\left(\frac{3}{-4}\right)$$

$$\Rightarrow 3\cos(wt) - 4\sin(wt) = 5\sin\left[wt + \tan^{-1}\left(\frac{3}{-4}\right)\right] \quad //$$

(4) Perform the following dot + cross products!
(20 points)

(a) $\hat{a}_y \cdot \hat{a}_\theta = \hat{a}_y \cdot \left(\hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta \right)$

$= \cos\theta\sin\phi$ //

(b) $\hat{a}_x \cdot \hat{a}_\rho = \hat{a}_x \cdot \left(\hat{a}_x \cos\phi + \hat{a}_y \sin\phi \right) = \cos\phi$ //

(c) $\hat{a}_\phi \cdot \hat{a}_z = 0$ //

(d) $\hat{a}_r \cdot \hat{a}_\rho = \left(\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta \right) \cdot \left(\hat{a}_x \cos\phi + \hat{a}_y \sin\phi \right)$

$= \sin\theta\cos^2\phi + \sin\theta\sin^2\phi = \sin\theta\left[\cos^2\phi + \sin^2\phi\right] = \sin\theta$ //

(e) $\hat{a}_\theta \cdot \hat{a}_\rho = \left(\hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta \right) \cdot \left(\hat{a}_x \cos\phi + \hat{a}_y \sin\phi \right)$

$= \cos\theta\cos^2\phi + \cos\theta\sin^2\phi = \cos\theta\left[\cos^2\phi + \sin^2\phi\right] = \cos\theta$ //

(f) $\hat{a}_z \cdot \hat{a}_r = \hat{a}_z \cdot \left(\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta \right) = \cos\theta$ //

(g) $\hat{a}_x \cdot \hat{a}_\phi = \hat{a}_x \cdot \left(-\sin\phi\, \hat{a}_x + \cos\phi\, \hat{a}_y \right)$

$= -\sin\phi$ //

(h) $\hat{a}_x \times \hat{a}_r = \hat{a}_x \times \left(\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta \right)$

$= \hat{a}_z \sin\theta\sin\phi - \hat{a}_y \cos\theta$ //

(i) $\hat{a}_\theta \times \hat{a}_z = \left(\hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta \right) \times \hat{a}_z$

$= -\hat{a}_y \cos\theta\cos\phi + \hat{a}_x \cos\theta\sin\phi$ //

(j) $\hat{a}_\rho \times \hat{a}_r = \left(\hat{a}_x \cos\phi + \hat{a}_y \sin\phi \right) \times \left(\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta \right)$

$= \hat{a}_z \sin\theta\sin\phi\cos\phi - \hat{a}_y \cos\phi\cos\theta - \hat{a}_z \sin\phi\sin\theta\cos\phi + \hat{a}_x \cos\theta\sin\phi$

$= \hat{a}_x \cos\theta\sin\phi - \hat{a}_y \cos\theta\cos\phi + \hat{a}_z [0] = \cos\theta\,[\underbrace{\hat{a}_x \sin\phi - \hat{a}_y \cos\phi}_{-\hat{a}_\phi}]$

Note that $\hat{a}_x, \hat{a}_y, \hat{a}_z$ unit vectors in cartesian coordinate system
$\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ " " " cylindrical " "
$\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ " " " spherical " "

(6) Assume you are given the following expression

(27 points)

$$f(\Psi) = \frac{1}{N}\left|\frac{\sin(N\Psi/2)}{\sin \Psi/2}\right| \quad \text{where } N \text{ is a constant}$$

but Ψ is a variable given by $\Psi = \beta d \cos\phi + \xi$ with

β is the propagation constant

d is a distance parameter

ϕ is the standard ϕ-angle in spherical coordinates

ξ is an angle quantity

Answer the following questions:

(a) What is the maximum value of $f(\Psi)$ and for which Ψ value does it occur?

(7 points)

$f(\Psi) = \cancel{\frac{1}{N}}$ 1 when $\Psi \to 0$

(b) For the Ψ value that you find in part (a) give an expression for $\cos\phi$ in terms of ξ, β, d.

(7 points)

$$\beta d \cos\phi + \xi = 0 \implies \cos\phi = \frac{-\xi}{\beta d}$$

(7 points) (c) Find an expression (i.e., value) for ψ such that $f(\psi)$ becomes zero!

$$\frac{N\psi}{2} = \pm k\pi \quad \text{where} \quad k = 1, 2, 3$$

$$\Rightarrow \psi = \pm 2k\pi / N$$

(5) Perform the following integrals:

(20 points)

(a) $$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi$$

(b) $$\int_0^{\pi}\int_0^{\pi/2} \sin\theta \, d\theta \, d\phi = \cancel{2\pi} \; \pi$$

(c) $$\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, d\theta \, d\phi = \pi^2$$

(d) $$\int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi = \frac{8\pi}{3}$$