EEE 452/552

Solutions to HW #5

1 (a) $$\bar{A}(x,y,t) = \hat{a}_x A_x = \hat{a}_x \frac{\mu_o}{4\pi} \int_V I_o \delta(z')\delta(y')\left[U(x'+l/2) - U(x'-l/2)\right] \frac{e^{-jkR}}{R}\, dv'$$

$$\simeq \hat{a}_x \frac{\mu_o}{4\pi} I_o l \frac{e^{-jkr}}{r}$$

$$\Rightarrow A_r = \mu_o I_o l \frac{e^{-jkr}}{4\pi r} \sin\theta\cos\phi$$

$$A_\theta = \mu_o I_o l \frac{e^{-jkr}}{4\pi r} \cos\theta\cos\phi$$

$$A_\phi = -\mu_o I_o l \frac{e^{-jkr}}{4\pi r} \sin\phi$$

(b) $$\bar{H} = \frac{1}{\mu_o}\nabla\times\bar{A} = \frac{1}{\mu}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\right\}$$

$$\Rightarrow H_r = \frac{1}{\mu r\sin\theta}\left[(-\mu_o I_o l)\frac{e^{-jkr}}{4\pi r}\sin\phi\cos\theta - \mu_o I_o l\frac{e^{-jkr}}{4\pi r}\cos\theta(-\sin\phi)\right] = 0$$

$$H_\theta = \frac{1}{\mu_o}\frac{1}{r}\left[\frac{1}{\sin\theta}\mu_o I_o l\frac{e^{-jkr}}{4\pi r}\sin\theta(-\sin\phi) - (-\mu_o I_o l)\sin\phi\frac{(-jk)}{4\pi}e^{-jkr}\right]$$

$$\Rightarrow H_\theta = \frac{1}{\cancel{\mu_o}}\frac{1}{r}\left[-\cancel{\mu_o} I_o l\sin\phi\frac{e^{-jkr}}{4\pi r} - \cancel{\mu_o} I_o l\sin\phi\frac{e^{-jkr}}{4\pi}jk\right]$$

$$\Rightarrow H_\theta = -I_o l\sin\phi\frac{e^{-jkr}}{4\pi r}\left(jk + \frac{1}{r}\right)$$

$$H_\phi = \frac{1}{\cancel{\mu_o}}\frac{1}{r}\left[\cancel{\mu_o} I_o l\cos\theta\cos\phi(-jk)\frac{e^{-jkr}}{4\pi} - \mu_o I_o l\frac{e^{-jkr}}{4\pi r}\cos\phi\cos\theta\right]$$

$$\Rightarrow H_\phi = -I_o l\cos\theta\cos\phi\frac{e^{-jkr}}{4\pi r}\left[jk + \frac{1}{r}\right]$$

(c) $\bar{E} = \frac{1}{j\omega\epsilon} \nabla \times \bar{H}$ and $H_r = 0$

$$E_r = \frac{1}{j\omega\epsilon}\left\{ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(H_\phi \sin\theta\right) - \frac{\partial}{\partial\phi} H_\theta \right)\right\}$$

$$= \frac{1}{j\omega\epsilon}\left\{ \frac{1}{r\sin\theta}\left[-I_o l \frac{e^{-jkr}}{4\pi r}\left[jk+\frac{1}{r}\right]\frac{\partial}{\partial\theta}\left(\cos\theta\cos\phi\sin\theta\right) - \frac{\partial}{\partial\phi}\left(-I_o l \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\sin\phi\right]\right\}$$

$$= \frac{1}{j\omega\epsilon}\left\{ \frac{1}{r\sin\theta}\left[-I_o l \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\cos\phi\cos 2\theta + I_o l \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\cos\phi\right]\right\}$$

$$\Rightarrow \boxed{E_r = \frac{I_o l \cos\phi}{j\omega\epsilon\sin\theta}\frac{e^{-jkr}}{4\pi r^2}\left[-\left(jk+\frac{1}{r}\right)\cos 2\theta + jk + \frac{1}{r}\right]}$$

$$E_\theta = \frac{1}{j\omega\epsilon}\left\{\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(0) - \frac{\partial}{\partial r}\left(r\left(-I_o l\cos\theta\cos\phi\frac{e^{-jkr}}{4\pi r}\right)\left(jk+\frac{1}{r}\right)\right)\right]\right\}$$

$$= \frac{1}{j\omega\epsilon}\frac{1}{r}\left[I_o l\cos\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{4\pi}\left(jk+\frac{1}{r}\right)\right)\right]$$

$$= \frac{I_o l}{j\omega\epsilon}\frac{\cos\theta\cos\phi}{r}\left(jk(-jk)\frac{e^{-jkr}}{4\pi} + \frac{(-jk)e^{-jkr}}{4\pi}\frac{1}{r} + \frac{e^{-jkr}}{4\pi}\left(-\frac{1}{r^2}\right)\right)$$

$$= \frac{I_o l}{j\omega\epsilon}\frac{\cos\theta\cos\phi}{r}\left[k^2\frac{e^{-jkr}}{4\pi} - \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right]$$

$$\Rightarrow \boxed{E_\theta = \frac{I_o l\cos\theta\cos\phi}{j\omega\epsilon}\frac{e^{-jkr}}{4\pi r}\left[k^2 - \frac{jk}{r} - \frac{1}{r^2}\right]}$$

$$E_\phi = \frac{1}{j\omega\epsilon}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left(-I_o l \sin\phi \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right)\right) - \frac{\partial}{\partial\theta}(0)\right]$$

$$= -\frac{I_o l \sin\phi}{j\omega\epsilon r}\left(\frac{\partial}{\partial r}\frac{e^{-jkr}}{4\pi}\left(jk+\frac{1}{r}\right)\right)$$

$$= -\frac{I_o l \sin\phi}{j\omega\epsilon r}\frac{1}{4\pi}\left[k^2 e^{-jkr} - jk\frac{e^{-jkr}}{r} - \frac{1}{r^2}e^{-jkr}\right]$$

$$\Rightarrow \boxed{E_\phi = -\frac{I_o l \sin\phi}{j\omega\epsilon}\frac{e^{-jkr}}{4\pi r}\left[k^2 - \frac{jk}{r} - \frac{1}{r^2}\right]}$$

(d) If we write the dominant contributions for $kr \gg 1$ $\quad r \gg 1 \quad r \gg \lambda$

basically far-field

$$H_r = 0$$

$$H_\theta \simeq -jk I_o l \sin\phi \frac{e^{-jkr}}{4\pi r} \quad //$$

$$H_\phi \simeq -jk I_o l \cos\theta \cos\phi \frac{e^{-jkr}}{4\pi r} \quad //$$

$$E_r \simeq 0$$

$$E_\theta \simeq -j I_o l \frac{k^2}{\omega\epsilon} \cos\theta\cos\phi \frac{e^{-jkr}}{4\pi r} \quad //$$

$$E_\phi \simeq +j I_o l \frac{k^2}{\omega\epsilon} \sin\phi \frac{e^{-jkr}}{4\pi r} \quad //$$

note that $\frac{k}{\omega\epsilon} = \frac{\cancel{\omega}\sqrt{\mu\epsilon}}{\cancel{\omega}\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta$

2 (a) $\bar{A}(x,y,z) = \hat{a}_y A_y = \hat{a}_y \dfrac{\mu_0}{4\pi} \displaystyle\int_V I_0 \delta(z')\delta(x')\left[u(y'+\ell/2) - u(y'-\ell/2)\right] \dfrac{e^{-jkR}}{R}\, dv'$

$$\simeq \hat{a}_y \frac{\mu_0}{4\pi} I_0 \ell \frac{e^{-jkr}}{r}$$

$$\Rightarrow A_r = \mu_0 I_0 \ell \sin\theta \sin\phi \frac{e^{-jkr}}{4\pi r}$$

$$A_\theta = \mu_0 I_0 \ell \cos\theta \sin\phi \frac{e^{-jkr}}{4\pi r}$$

$$A_\phi = \mu_0 I_0 \ell \cos\phi \frac{e^{-jkr}}{4\pi r}$$

(b) $\bar{H} = \dfrac{1}{\mu_0} \nabla\times\bar{A}$

$$\Rightarrow H_r = \frac{1}{\mu_0}\left[\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\mu_0 I_0 \ell \cos\phi \frac{e^{-jkr}}{4\pi r}\sin\theta\right) - \frac{\partial}{\partial\phi}\left(\mu_0 I_0 \ell \cos\theta \sin\phi \frac{e^{-jkr}}{4\pi r}\right)\right)\right]$$

$$= \frac{1}{\mu_0}\frac{\mu_0 I_0 \ell}{r\sin\theta}\frac{e^{-jkr}}{4\pi r}\left[\cos\phi\cos\theta - \cos\theta\cos\phi\right] = 0$$

$$\Rightarrow \boxed{H_r = 0}$$

$$H_\theta = \frac{1}{\mu_0}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\mu_0 I_0 \ell \sin\theta \sin\phi \frac{e^{-jkr}}{4\pi r}\right) - \frac{\partial}{\partial r}\left(r \mu_0 I_0 \ell \cos\phi \frac{e^{-jkr}}{4\pi r}\right)\right]$$

$$= \frac{1}{\cancel{\mu_0}}\frac{1}{r}\cancel{\mu_0} I_0 \ell \left[\frac{e^{-jkr}}{4\pi r}\frac{\cancel{\sin\theta}\cos\phi}{\cancel{\sin\theta}} - \cos\phi(-jk)\frac{e^{-jkr}}{4\pi}\right]$$

$$= \frac{I_0 \ell}{r}\cos\phi \frac{e^{-jkr}}{4\pi}\left[jk + \frac{1}{r}\right]$$

$$\Rightarrow \boxed{H_\theta = I_0 \ell \cos\phi \frac{e^{-jkr}}{4\pi r}\left(jk + \frac{1}{r}\right)}$$

$$H_\phi = \frac{1}{\mu_0}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\mu_0 I_0 l\cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\right) - \frac{\partial}{\partial\theta}\left(\mu_0 I_0 l\sin\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\right)\right]$$

$$\Rightarrow H_\phi = \frac{1}{\cancel{\mu_0}}\frac{1}{r}\cancel{\mu_0} I_0 l\left[\cos\theta\sin\phi\,(-jk)\,\frac{e^{-jkr}}{4\pi} - \cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\right]$$

$$\Rightarrow \boxed{H_\phi = -I_0 l\cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)}$$

(c) $\bar{E} = \frac{1}{j\omega\epsilon}\nabla\times\bar{H}$ with $H_r = 0$

$$E_r = \frac{1}{j\omega\epsilon}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(-I_0 l\cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\sin\theta\right) - \frac{\partial}{\partial\phi}\left(I_0 l\cos\phi\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right)\right]$$

$$= \frac{1}{j\omega\epsilon}\frac{1}{r\sin\theta}\left[-I_0 l\sin\phi\cos 2\theta\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right) + \sin\phi\, I_0 l\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right]$$

$$\Rightarrow \boxed{E_r = \frac{I_0 l\sin\phi}{j\omega\epsilon\sin\theta}\,\frac{e^{-jkr}}{4\pi r^2}\left[-\left(jk+\frac{1}{r}\right)\cos 2\theta + jk + \frac{1}{r}\right]}$$

$$E_\theta = \frac{1}{j\omega\epsilon}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(0) - \frac{\partial}{\partial r}\left(r(-I_0 l)\cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right)\right]$$

$$= \frac{I_0 l}{j\omega\epsilon r}\cos\theta\sin\phi\,\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{4\pi}\left(jk+\frac{1}{r}\right)\right)$$

$$= \frac{I_0 l}{j\omega\epsilon r}\cos\theta\sin\phi\left[k^2\frac{e^{-jkr}}{4\pi} - jk\,\frac{e^{-jkr}}{4\pi r} - \frac{e^{-jkr}}{4\pi r^2}\right]$$

$$\Rightarrow \boxed{E_\theta = \frac{I_0 l}{j\omega\epsilon}\cos\theta\sin\phi\,\frac{e^{-jkr}}{4\pi r}\left[k^2 - \frac{jk}{r} - \frac{1}{r^2}\right]}$$

$$E_\phi = \frac{1}{j\omega\epsilon}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, I_0 l \cos\phi \frac{e^{-jkr}}{4\pi r}\left(jk+\frac{1}{r}\right)\right) - \frac{\partial}{\partial\theta}(0)\right]$$

$$= \frac{1}{j\omega\epsilon}\frac{1}{r} I_0 l \cos\phi \frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{4\pi}\left(jk+\frac{1}{r}\right)\right)$$

$$= \frac{I_0 l}{j\omega\epsilon r}\cos\phi\left(k^2\frac{e^{-jkr}}{4\pi} - jk\frac{e^{-jkr}}{4\pi r} - \frac{e^{-jkr}}{4\pi r^2}\right)$$

$$\boxed{E_\phi = \frac{I_0 l \cos\phi}{j\omega\epsilon}\frac{e^{-jkr}}{4\pi r}\left(k^2 - \frac{jk}{r} - \frac{1}{r^2}\right)}$$

(d) in the far-zone

$H_r = 0$

$H_\theta \simeq jk I_0 l \cos\phi \dfrac{e^{-jkr}}{4\pi r}$ //

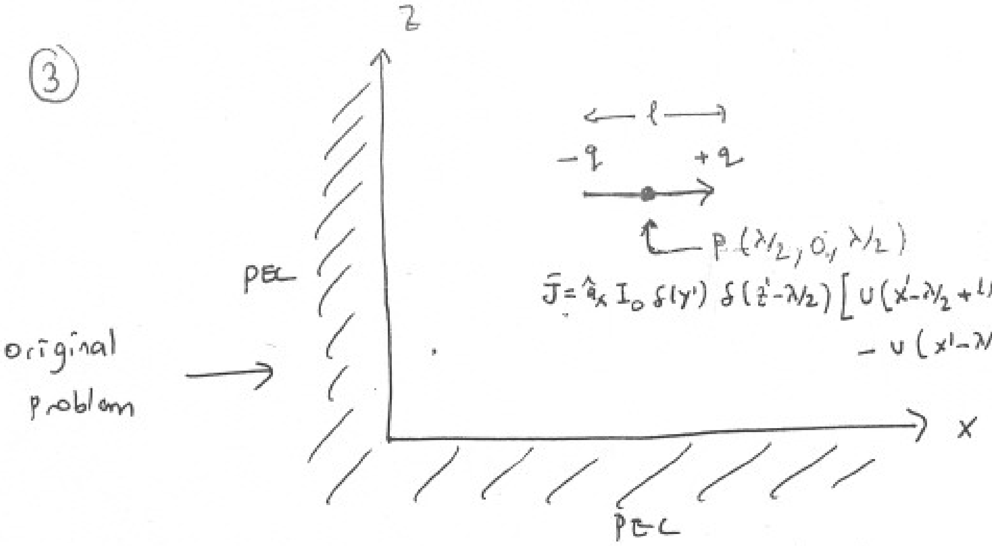
$H_\phi \simeq -jk I_0 l \cos\theta \sin\phi \dfrac{e^{-jkr}}{4\pi r}$ //

$E_r \simeq 0$

$E_\theta \simeq -j I_0 l \dfrac{k^2}{\omega\epsilon} \cos\theta \sin\phi \dfrac{e^{-jkr}}{4\pi r}$ //

$E_\phi \simeq -j I_0 l \dfrac{k^2}{\omega\epsilon} \cos\phi \dfrac{e^{-jkr}}{4\pi r}$ //

(3)



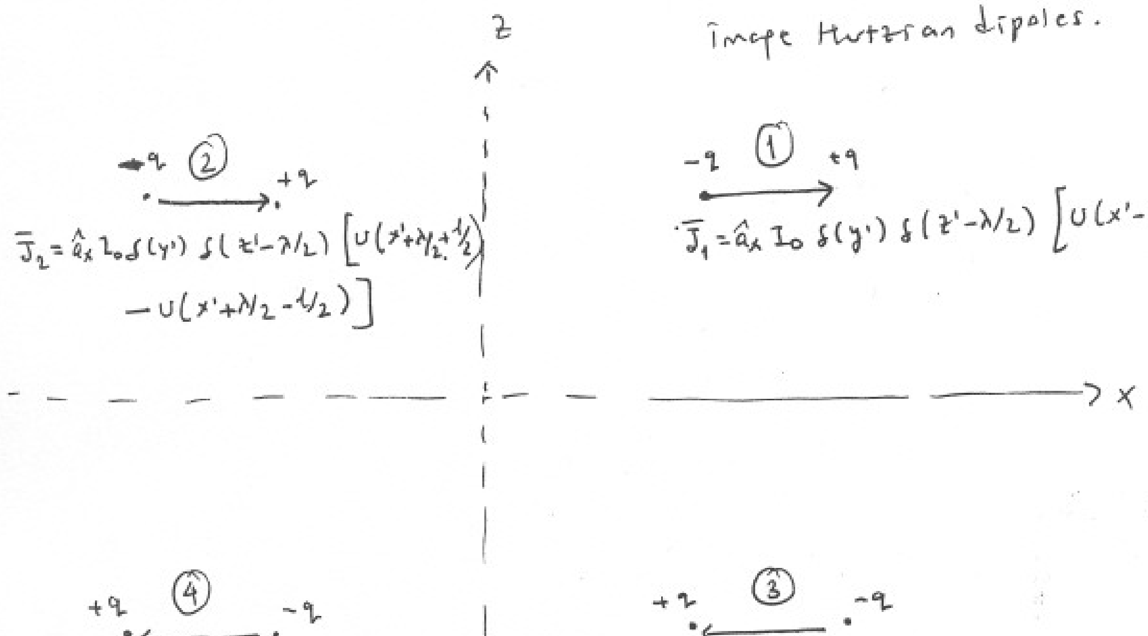
$\bar{J} = \hat{a}_x I_o \delta(y') \delta(z'-\lambda/2) \left[U(x'-\lambda/2 + l/2) - U(x'-\lambda/2 - l/2) \right]$

Each dipole (Hertzian) is equivalent to two charges ($+q$ & $-q$) separated by l

$l << \lambda$

$l << r$

Use Image Theory to get rid of PEC walls then we have one real Hertzian dipole and 3 image Hertzian dipoles.



$\bar{J}_2 = \hat{a}_x I_o \delta(y') \delta(z'-\lambda/2) \left[U(x'+\lambda/2 + l/2) - U(x'+\lambda/2 - l/2) \right]$

$\bar{J}_1 = \hat{a}_x I_o \delta(y') \delta(z'-\lambda/2) \left[U(x'-\lambda/2 + l/2) - U(x'-\lambda/2 - l/2) \right]$

$\bar{J}_4 = -\hat{a}_x I_o \delta(y') \delta(z'+\lambda/2) \cdot \left[U(x'+\lambda/2 + l/2) - U(x'+\lambda/2 - l/2) \right]$

$\bar{J}_3 = -\hat{a}_x I_o \delta(y') \delta(z'+\lambda/2) \left[U(x'-\lambda/2 + l/2) - U(x'-\lambda/2 - l/2) \right]$

$$\Rightarrow \bar{A}(x, y, z) = \frac{\mu_o}{4\pi} \int_{V_1} \bar{J}_1(x', y', z') \frac{e^{-jkR}}{R} dv' + \frac{\mu_o}{4\pi} \int_{V_2} \bar{J}_2(x', y', z') \frac{e^{-jkR}}{R} dv'$$

$$+ \frac{\mu_o}{4\pi} \int_{V_3} \bar{J}_3(x', y', z') \frac{e^{-jkR}}{R} dv' + \frac{\mu_o}{4\pi} \int_{V_4} \bar{J}_4(x', y', z') \frac{e^{-jkR}}{R} dv'$$

$$\Rightarrow \bar{A}(x, y, z) = \hat{a}_x \overbrace{I_0 l \frac{\mu}{4\pi}\left[\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} - \frac{e^{-jkr_3}}{r_3} - \frac{e^{-jkr_4}}{r_4} \right]}^{A_x}$$

where $r_1 = \sqrt{(x-\lambda/2)^2 + y^2 + (z-\lambda/2)^2}$

$$r_2 = \sqrt{(x+\lambda/2)^2 + y^2 + (z-\lambda/2)^2}$$

$$r_3 = \sqrt{(x-\lambda/2)^2 + y^2 + (z+\lambda/2)^2}$$

$$r_4 = \sqrt{(x+\lambda/2)^2 + y^2 + (z+\lambda/2)^2}$$

$$\Rightarrow \quad A_r = A_x \sin\theta \cos\phi$$

$$A_\theta = A_x \cos\theta \cos\phi$$

$$A_\phi = -A_x \sin\phi$$