Due 26/03/2008

EEE 452/552

Homework # 6

(1) Vector potential due to a current distribution in free-space is given by

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\int_{V'} \bar{J}(\bar{r}')\, \frac{e^{-jkR}}{R}\, dv' \qquad \text{with} \quad R = |\bar{r}-\bar{r}'|, \text{ distance}$$

between the source and observation points.

(a) Show that the expression for the vector potential is in the form of

$$\bar{A}(\bar{r}) \approx \frac{e^{-jkr}}{r}\left[\hat{a}_r F_r(\theta,\phi) + \hat{a}_\theta F_\theta(\theta,\phi) + \hat{a}_\phi F_\phi(\theta,\phi)\right],$$

where $F_r(\theta,\phi)$, $F_\theta(\theta,\phi)$ and $F_\phi(\theta,\phi)$ are the r, θ and ϕ components of the integral given by

$$\bar{F}(\theta,\phi) = \frac{\mu}{4\pi}\int_{V'} \bar{J}(\bar{r}')\, e^{jk\hat{a}_r\cdot\bar{r}'}\, dv'$$

when the observation point is in the far-field.

(b) The electric field can be written in terms of the vector potential as

$$\bar{E}(\bar{r}) = -j\omega\bar{A}(\bar{r}) - j\,\frac{\nabla(\nabla\cdot\bar{A}(\bar{r}))}{\omega\mu\epsilon}.$$

Show that $\nabla(\nabla\cdot\bar{A}(\bar{r})) \approx \left(-\frac{k^2}{r} F_r(\theta,\phi)\, e^{-jkr}\,\hat{a}_r\right)$ by using the far-field approximation of the vector potential and keeping only $\frac{1}{r}$ terms.

(c) Find the far-field expression for the electric field in terms of the vector potentio

(2) Show that the approximation $R = r - r'\cos\psi$ for the phase variation in the far-field approximation (ψ is the angle between $\bar{r}$ and $\bar{r}'$) yields a maximum phase error of $\pi/8$ ($=22.5°$) provided the observations are made at distances $r \gtrsim 2D^2/\lambda$ where D is the largest dimension of the radiator or scatterer.

(3) A horizontal infinitesimal dipole (i.e., a Hertzian dipole) of constant current I_0 is placed symmetrically about the origin and directed along the x-axis (basically, the same geometry as question #1 of HW #5). Consider the far-zone and answer the followings:

(a) For $\phi = 0°$ plane determine which $\bar{E}$-field components exist (E_θ, E_ϕ) and sketch the amplitude pattern of the existent component(s).

(b) Repeat part (a) for $\phi = 90°$ plane.

(c) Repeat part (a) for $\theta = 90°$ plane.

(4) Find the far-field electric field expression of a very thin dipole (ideally zero-diameter) whose current distribution is approximated as

$$I(x'=0, y'=0, z') = \begin{cases} \hat{a}_z I_0 \sin\left[k\left(\frac{l}{2} - z'\right)\right] ; & 0 \le z' \le l/2 \\ \hat{a}_z I_0 \sin\left[k\left(\frac{l}{2} + z'\right)\right] , & -l/2 \le z' \le 0 \end{cases}$$

Also plot the current distributions along the length of this antenna for $l = \lambda/4$, $l = \lambda/2$, $l = \lambda$, $l = 3\lambda/2$, $l = 2\lambda$.

Note that all plots should be on the same figure and plotted using a different line-style.

(5) (Bonus) In class, we have seen that

$$\bar{A} = \frac{\mu}{4\pi} \int_{V'} \bar{J}(\bar{r}') \frac{e^{-jkR}}{R} dv' , \quad \bar{H} = \frac{1}{\mu} \nabla \times \bar{A} , \quad \bar{E} = \frac{\nabla \times \bar{H}}{j\omega\epsilon}$$

where $\bar{A} = \bar{A}(\bar{r})$, $\bar{E} = \bar{E}(\bar{r})$ and $\bar{H} = \bar{H}(\bar{r})$. Obtain an expression for $\bar{E}(\bar{r})$ and $\bar{H}(\bar{r})$ in terms of $\bar{J}(\bar{r}')$. You may need to use the following vector identities if necessary

$\nabla \times (\alpha \bar{B}) = \alpha \nabla \times \bar{B} - \bar{B} \times \nabla \alpha$ ($\bar{B}$ is a vector quantity, α is scalar)

$\nabla \times (\bar{B} \times \bar{C}) = \bar{B} \nabla \cdot \bar{C} - \bar{C} \nabla \cdot \bar{B} + (\bar{C} \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)\bar{C}$

Hint: $\bar{H}(\bar{r}) = -\frac{1}{4\pi} \int \left[\bar{J}(\bar{r}') \times \nabla\left(\frac{e^{-jkR}}{R}\right)\right] dv'$