EEE 452/552

Solutions to HW #6

1 (a) $\bar{A}(\bar{r}) = \frac{\mu}{4\pi} \int_{V'} \bar{J}(\bar{r}') \frac{e^{-jkR}}{R} \, dv'$ when $|\bar{r}| = r >> |\bar{r}'| = r'$

$R = |\bar{R}| = |\bar{r} - \bar{r}'| \approx r - \hat{a}_r \cdot \bar{r}'$ in the exponent for phase variation

$\approx r$ in the denominator for amplitude variation.

$$\Rightarrow \bar{A}(\bar{r}) \simeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_{V'} \bar{J}(\bar{r}') e^{jk\hat{a}_r \cdot \bar{r}'} \, dv' = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \bar{f}(\theta,\phi)$$

$$\Rightarrow \bar{A}(\bar{r}) \simeq \frac{e^{-jkr}}{r} \bar{F}(\theta,\phi) = \frac{e^{-jkr}}{r} \left(\hat{a}_r F_r(\theta,\phi) + \hat{a}_\theta F_\theta(\theta,\phi) + \hat{a}_\phi F_\phi(\theta,\phi) \right)$$

all constants (i.e., $\frac{\mu}{4\pi}$) are included to $\bar{F}(\theta,\phi)$

(b) $\nabla \cdot \bar{A}(\bar{r}) = \nabla \cdot \left\{ \frac{e^{-jkr}}{r} \bar{F}(\theta,\phi) \right\}$ use the fact that $\bar{F}(\theta,\phi)$ is r independent

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{e^{-jkr}}{r} F_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{e^{-jkr}}{r} \sin\theta F_\theta \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \left(\frac{e^{-jkr}}{r} F_\phi \right)$$

neglect since it has $\frac{1}{r^2}$ dependence

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r e^{-jkr} F_r(\theta,\phi) \right) + \cancel{\frac{e^{-jkr}}{r^2} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta F_\theta(\theta,\phi) \right)}$$

neglect since it has $\frac{1}{r^2}$

$$+ \cancel{\frac{e^{-jkr}}{r^2} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \left(F_\phi(\theta,\phi) \right)}$$

$$\Rightarrow \nabla \cdot \bar{A}(\bar{r}) \simeq \frac{1}{r^2} F_r(\theta,\phi) \left[e^{-jkr} + r(-jk) e^{-jkr} \right]$$

neglect

$$\simeq \cancel{\frac{1}{r^2} F_r(\theta,\phi) e^{-jkr}} - jk F_r(\theta,\phi) \frac{e^{-jkr}}{r}$$

$$\Rightarrow \nabla \cdot \bar{A}(\bar{r}) \simeq -jk \frac{e^{-jkr}}{r} F_r(\theta,\phi)$$

$$\Rightarrow \nabla \left[\nabla \cdot \bar{A}(\bar{r}) \right] \simeq \nabla \left(-jk \frac{e^{-jkr}}{r} F_r(\theta,\phi) \right) = \hat{a}_r \frac{\partial}{\partial r} \left(-jk \frac{e^{-jkr}}{r} F_r(\theta,\phi) \right)$$

$$+\hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\left(-jk\frac{e^{-jkr}}{r}F_r(\theta,\phi)\right) + \hat{a}_\phi \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(-jk\frac{e^{-jkr}}{r}F_r(\theta,\phi)\right)$$

neglect $\frac{1}{r^2}$

$$\Rightarrow \nabla\left[\nabla\cdot\bar{A}(\bar{r})\right] \simeq \hat{a}_r F_r(\theta,\phi)(-jk)\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right) + \cancel{\hat{a}_\theta (-jk)\frac{e^{-jkr}}{r^2}\frac{\partial}{\partial\theta}F_r(\theta,\phi)}$$

neglect $(1/r^2)$

$$+\cancel{\hat{a}_\phi (-jk)\frac{e^{-jkr}}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}F_r(\theta,\phi)}$$

$$\Rightarrow \nabla\left[\nabla\cdot\bar{A}(\bar{r})\right] \simeq \hat{a}_r F_r(\theta,\phi)(-jk)\left[(-jk)\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right]$$ neglecting the second term

$$\nabla\cdot\left[\nabla\cdot\bar{A}(\bar{r})\right] \simeq \hat{a}_r(-k^2)F_r(\theta,\phi)\frac{e^{-jkr}}{r} \quad //$$

(c) $$\bar{E}(\bar{r}) = -j\omega\bar{A} - j\frac{\nabla(\nabla\cdot\bar{A})}{\omega\mu\epsilon} \cong -j\omega\left(A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi\right) - j\frac{(-k^2)F_r(\theta,\phi)e^{-jkr}}{\omega\mu\epsilon r}$$

$$\Rightarrow \bar{E}(\bar{r}) \cong -j\omega\left[\frac{e^{-jkr}}{r}\left(\hat{a}_r F_r(\theta,\phi) + \hat{a}_\theta F_\theta(\theta,\phi) + \hat{a}_\phi F_\phi(\theta,\phi)\right)\right] + \frac{jk^2}{\omega\mu\epsilon}\frac{e^{-jkr}}{r}F_r(\theta,\phi)\hat{a}_r$$

$$\cong -j\omega\left[\hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi\right] + \left(-j\omega\frac{e^{-jkr}}{r}F_r + \frac{j\omega^2\mu\epsilon}{\omega\mu\epsilon}\frac{e^{-jkr}}{r}F_r\right)\hat{a}_r$$

cancel each other

$$\Rightarrow \bar{E}(\bar{r}) \simeq -j\omega\left\{\hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi\right\} \quad //$$

2 $R = |\bar{r} - \bar{r}'| = \sqrt{r^2 - 2\bar{r}.\bar{r}' + r'^2} = r\left[1 - \frac{2r'\cos\psi}{r} + \frac{r'^2}{r^2}\right]^{1/2}$

$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots$ if $|x| \ll 1$

$\Rightarrow R \simeq r\left[1 + \frac{1}{2}\left(-\frac{2r'\cos\psi}{r} + \frac{r'^2}{r^2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{2r'\cos\psi}{r} + \frac{r'^2}{r^2}\right)^2 + \cdots\right]$

$\simeq r\left[1 - \frac{r'\cos\psi}{r} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{2}\frac{r'^2\cos^2\psi}{r^2} + \text{higher order terms in } \frac{1}{r}\right]$

$\simeq r\left[1 - \frac{r'\cos\psi}{r} + \frac{1}{2}\frac{r'^2}{r^2}\underbrace{\left[1 - \cos^2\psi\right]}_{\sin^2\psi} + \text{higher order terms}\right]$

$\Rightarrow R \simeq r - r'\cos\psi + \frac{1}{r}\left(\frac{r'^2}{2}\sin^2\psi\right) + \text{higher order terms}]$

If $r - r'\cos\psi$ is used for the phase variation $\Rightarrow$ error term $= \frac{1}{r}\frac{r'^2}{2}\sin^2\psi$

Max. error occurs at $\psi = \frac{\pi}{2}$ $(\sin^2\psi = 1) \Rightarrow [\text{error term}]_{max} = r'^2/2r$

$\frac{kr'^2}{2r} \leq \frac{\pi}{8}$ and $k = \frac{2\pi}{\lambda}$ and $-\frac{D}{2} \leq r' \leq \frac{D}{2}$

↑
phase error

$\Rightarrow r'^2 \leq \frac{D^2}{4}$

$\frac{2\pi}{\lambda}\frac{D^2}{4}\frac{1}{2r} \leq \frac{\pi}{8}$

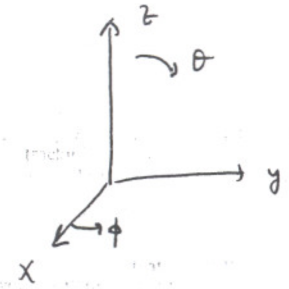
$\Rightarrow \boxed{\frac{2D^2}{\lambda} \leq r}$

3 For an x-directed Hertzian dipole the far zone components are:

$$E_r \simeq 0$$

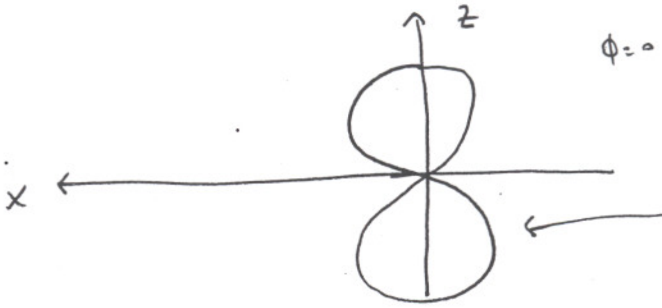
$$E_\theta \cong -j I_o l \, \eta k \cos\theta \cos\phi \, \frac{e^{-jkr}}{4\pi r}$$

$$E_\phi \simeq j I_o l \, \eta k \sin\phi \, \frac{e^{-jkr}}{4\pi r}$$



(a) if $\phi = 0 \Rightarrow$ $\sin\phi = 0 \Rightarrow E_\phi = 0$

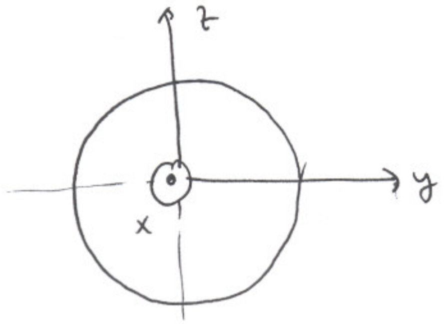
(x-z plane) $\cos\phi = 1 \Rightarrow E_\theta \simeq -j I_o l \, \eta k \cos\theta \, \frac{e^{-jkr}}{4\pi r}$

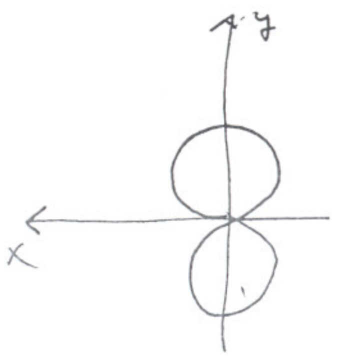


same!

(b) if $\phi = 90^\circ$ $\cos\phi = 0 \Rightarrow E_\theta = 0$

(y-z plane) $\sin\phi = 1 \Rightarrow E_\phi \simeq j I_o l \, \eta k \, \frac{e^{-jkr}}{4\pi r}$





(c) if $\theta = 90^\circ$ $\Rightarrow$ $\cos\theta = 0 \Rightarrow E_\theta = 0$

(x-y plane) $E_\phi \simeq j I_o l \, \eta k \sin\phi \, \frac{e^{-jkr}}{4\pi r}$

4 $$\bar{A}(x,y,z) = \frac{\mu}{4\pi}\hat{a}_z \int_{-l/2}^{0} I_o \sin\left[k\left(\frac{l}{2}+z'\right)\right] \frac{e^{-jkR}}{R}\, dz' + \frac{\mu}{4\pi}\hat{a}_z \int_{0}^{l/2} I_o \sin\left[k\left(\frac{l}{2}-z'\right)\right] \frac{e^{-jkR}}{R}\, dz'$$

$$R = |\bar{r}-\bar{r}'| = \begin{cases} r - \bar{r}'\cdot\hat{a}_r = r - z'\cos\theta & \text{for the phase variation} \\ r & \text{for the amplitude variation} \end{cases}$$

$$\Rightarrow \bar{A}(x,y,z) = \hat{a}_z A_z = \hat{a}_z \frac{\mu}{4\pi}\frac{e^{-jkr}}{r} I_o \left[\int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2}+z'\right)\right] e^{jkz'\cos\theta}\, dz' + \int_{0}^{l/2} \sin\left[k\left(\frac{l}{2}-z'\right)\right] e^{jkz'\cos\theta}\, dz' \right]$$

$$\Rightarrow A_\theta = -\sin\theta A_z$$

$$A_\phi = 0$$

$\Rightarrow \bar{E} \simeq -j\omega\bar{A}$ (ignore r-component)

$$\Rightarrow E_\theta \simeq j\omega \sin\theta \frac{\mu}{4\pi}\frac{e^{-jkr}}{r} I_o \left[\int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2}+z'\right)\right] e^{jkz'\cos\theta}\, dz' + \int_{0}^{l/2} \sin\left[k\left(\frac{l}{2}-z'\right)\right] e^{jkz'\cos\theta}\, dz \right]$$

$$\Rightarrow E_\theta \simeq j\eta k \frac{e^{-jkr}}{4\pi r} \sin\theta \left[\quad \right]$$

Note that $\eta k = \sqrt{\frac{\mu}{\epsilon}}\, \omega\sqrt{\mu\epsilon} = \omega\mu$

Using the integral identity

$$\int e^{\alpha x}\sin(\beta x+\gamma)\, dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta x+\gamma) - \beta\cos(\beta x+\gamma)\right]$$

where $\alpha = jk\cos\theta$, $\beta = \pm k$, $\gamma = l/2$ and using the result of HW#1 Q#2 (a)

$$E_\theta \simeq j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]$$

for the plots, refer to Fig 4.8 on pp. 176 of your textbook.

5 Bonus question:

$$\bar{H}=\frac{1}{\mu}\nabla\times\bar{A}\;;\quad \bar{A}=\frac{\mu}{4\pi}\int_{V'}\bar{J}(\bar{r}')\frac{e^{-jkR}}{R}\,dv'\quad;\quad \bar{E}=\frac{1}{j\omega\epsilon}\nabla\times\bar{H}$$

$$\Rightarrow \bar{H}(\bar{r})=\frac{1}{\mu}\nabla\times\left[\frac{\mu}{4\pi}\int_{V'}\bar{J}(\bar{r}')\frac{e^{-jkR}}{R}\,dv'\right]=\frac{1}{4\pi}\nabla\times\int_{V'}\bar{J}(\bar{r}')\frac{e^{-jkR}}{R}\,dv'$$

$$\Rightarrow \bar{H}(\bar{r})=\frac{1}{4\pi}\int_{V'}\nabla\times\left[\bar{J}(\bar{r}')\frac{e^{-jkR}}{R}\,dv'\right]$$

Use $\nabla\times(\alpha\bar{B})=\alpha\nabla\times\bar{B}-\bar{B}\times\nabla\alpha$ with $\alpha=\frac{e^{-jkR}}{R}$ and $\bar{B}=\bar{J}(\bar{r}')$

$$\Rightarrow \nabla\times\left[\bar{J}(\bar{r}')\frac{e^{-jkR}}{R}\right]=\cancelto{0}{\frac{e^{-jkR}}{R}\nabla\times\bar{J}(\bar{r}')}-\bar{J}(\bar{r}')\times\nabla\left(\frac{e^{-jkR}}{R}\right)$$

∇ is acting on field points $(\bar{r})$, $\bar{J}(\bar{r}')$ is a function of source points $(\bar{r}')$. Thus, the first term is zero

$$\Rightarrow \boxed{\bar{H}(\bar{r})=-\frac{1}{4\pi}\int_{V'}\left[\bar{J}(\bar{r}')\times\nabla\left(\frac{e^{-jkR}}{R}\right)\right]dv'}$$

$$\bar{E}=\frac{1}{j\omega\epsilon}\nabla\times\bar{H}=\frac{-1}{j\omega\epsilon}\nabla\times\left\{\frac{1}{4\pi}\int_{V'}\left[\bar{J}(\bar{r}')\times\nabla\left(\frac{e^{-jkR}}{R}\right)\right]dv'\right\}$$

$$\Rightarrow \bar{E} = \frac{-1}{j\omega\epsilon 4\pi} \int_{V'} \nabla \times \left[\bar{J}(\bar{r}') \times \nabla \left(\frac{e^{-jkR}}{R} \right) \right] dv'$$

Use $\nabla \times (\bar{B} \times \bar{C}) = B\, \nabla \cdot \bar{C} - C\, \nabla \cdot \bar{B} + (\bar{C} \cdot \nabla)\, \bar{B} - (\bar{B} \cdot \nabla)\, \bar{C}$

$$\Rightarrow \nabla \times \left[\bar{J}(\bar{r}') \times \nabla \left(\frac{e^{-jkR}}{R} \right) \right] = \bar{J}(\bar{r}')\, \nabla \cdot \nabla \left(\frac{e^{-jkR}}{R} \right) - \cancelto{0}{\nabla \left(\frac{e^{-jkR}}{R} \right) \nabla \cdot \bar{J}(\bar{r}')}$$

$$+ \cancelto{0}{\left(\nabla \left(\frac{e^{-jkR}}{R} \right) \cdot \nabla \right) \bar{J}(\bar{r}')} - \left(\bar{J}(\bar{r}') \cdot \nabla \right) \nabla \left(\frac{e^{-jkR}}{R} \right)$$

The second and third terms vanish since derivatives are w.r.t. unprimed coordinates but they are acting on $\bar{J}(\bar{r}')$, which depends only on primed coordinates.

$$\Rightarrow \nabla \times \left[\bar{J}(\bar{r}') \times \nabla \left(\frac{e^{-jkR}}{R} \right) \right] = \bar{J}(\bar{r}')\, \nabla \cdot \nabla \left(\frac{e^{-jkR}}{R} \right) - \left(\bar{J}(\bar{r}') \cdot \nabla \right) \nabla \left(\frac{e^{-jkR}}{R} \right)$$

$$\Rightarrow \bar{E}(\bar{r}) = \frac{j}{4\pi\omega\epsilon} \int_{V'} \left\{ \bar{J}(\bar{r}')\, \nabla^2 \left(\frac{e^{-jkR}}{R} \right) - \left(\bar{J}(\bar{r}') \cdot \nabla \right) \nabla \left(\frac{e^{-jkR}}{R} \right) \right\} dv'$$

Notice that $\nabla^2 \left(\frac{e^{-jkR}}{R} \right) = \frac{1}{R^2} \frac{\partial}{\partial R} \left[R^2 \frac{\partial}{\partial R} \left(\frac{e^{-jkR}}{R} \right) \right] = -k^2 \frac{e^{-jkR}}{R}$

$$\Rightarrow \bar{E}(\bar{r}) = \frac{-j}{4\pi\omega\epsilon} \int_{V'} \left\{ \left[k^2 \bar{J}(\bar{r}') + \left(\bar{J}(\bar{r}') \cdot \nabla \right) \nabla \right] \frac{e^{-jkR}}{R} \right\} dv'$$