EEE 452 / 552

Solutions to HW #8

1. $$\bar{A}(\bar{r}) = \frac{\mu}{4\pi} \int_C I_0 \hat{a}_\phi' \frac{e^{-jkR}}{R} dl'$$

$\hat{a}_\phi' = -\hat{a}_x \sin\phi' + \hat{a}_y \cos\phi'$ and $dl' = a\,d\phi'$

$$\Rightarrow \bar{A}(\bar{r}) = \frac{\mu}{4\pi} I_0 \int_0^{2\pi} (-\hat{a}_x \sin\phi' + \hat{a}_y \cos\phi') \frac{e^{-jkR}}{R} a\,d\phi'$$

$R \to r$ in the denominator & $R = r - \bar{r}' \cdot \hat{a}_r$ in the exponent

where $\hat{a}_r \cdot \bar{r}' = \hat{a}_r \cdot a\hat{a}_\rho' = a\left[\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta \sin\phi + \hat{a}_z \cos\theta\right] \cdot \left[\hat{a}_x \cos\phi' + \hat{a}_y \sin\phi'\right]$

$= a\left[\sin\theta \cos\phi \cos\phi' + \sin\theta \sin\phi \sin\phi'\right] = a \sin\theta \cos(\phi - \phi')$

$$\Rightarrow \bar{A}(\bar{r}) = \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r} \left[-\hat{a}_x \int_0^{2\pi} \sin\phi' e^{jka\sin\theta\cos(\phi-\phi')} d\phi' + \hat{a}_y \int_0^{2\pi} \cos\phi' e^{jka\sin\theta\cos(\phi-\phi')} d\phi'\right]$$

$\Rightarrow$ from A_x and A_y

$$A_r \simeq \mu I_0 \frac{e^{-jkr}}{4\pi r} a \int_0^{2\pi} [-\sin\theta\cos\phi\sin\phi' + \sin\theta\sin\phi\cos\phi'] e^{jka\sin\theta\cos(\phi-\phi')} d\phi'$$

$$A_\theta \simeq \mu I_0 \frac{e^{-jkr}}{4\pi r} a \int_0^{2\pi} [-\cos\theta\cos\phi\sin\phi' + \cos\theta\sin\phi\cos\phi'] e^{jka\sin\theta\cos(\phi-\phi')} d\phi'$$

$$A_\phi \simeq \mu I_0 \frac{e^{-jkr}}{4\pi r} a \int_0^{2\pi} [\sin\phi'\sin\phi + \cos\phi\cos\phi'] e^{jka\sin\theta\cos(\phi-\phi')} d\phi'$$

let $\phi = 0$ for simplicity. Results are expected to have ϕ-symmetry.

Therefore, a choice for ϕ will not change the final results.

$\Rightarrow \sin\phi = 0 \quad \cos\phi = 1$

$$\Rightarrow A_r \simeq -\mu I_0 \frac{e^{-jkr}}{4\pi r} a \sin\theta \int_0^{2\pi} \sin\phi' \, e^{jka\sin\theta\cos\phi'} d\phi'$$

$$A_\theta \simeq -\mu I_0 \frac{e^{-jkr}}{4\pi r} a\cos\theta \int_0^{2\pi} \sin\phi' \, e^{jka\sin\theta\cos\phi'} d\phi'$$

$$A_\phi \simeq \mu I_0 \frac{e^{-jkr}}{4\pi r} a \int_0^{2\pi} \cos\phi \, e^{jka\sin\theta\cos\phi'} d\phi'$$

consider

$$A_\phi \simeq \mu a I_0 \frac{e^{-jkr}}{4\pi r} \int_0^{2\pi} \cos\phi' \, e^{jka\sin\theta\cos\phi'} d\phi'$$

$$= \mu a I_0 \frac{e^{-jkr}}{4\pi r} \left[\underbrace{\int_0^{\pi} \cos\phi' \, e^{jka\sin\theta\cos\phi'} d\phi'}_{j\pi J_1(ka\sin\theta)} + \underbrace{\int_\pi^{2\pi} \cos\phi' \, e^{jka\sin\theta\cos\phi'} d\phi'}_{} \right]$$

$\phi' = \phi'' + \pi$

$d\phi' = d\phi''$

$\cos(\phi''+\pi) = -\cos\phi''$

$$\underbrace{-\int_0^{\pi} \cos\phi'' \, e^{-jka\sin\theta\cos\phi''} d\phi''}_{j\pi J_1(-ka\sin\theta)}$$

$$\Rightarrow A_\phi \simeq \mu a I_0 \frac{e^{-jkr}}{4\pi r} \left[j\pi J_1(ka\sin\theta) - j\pi J_1(-ka\sin\theta) \right]$$

Note that $J_n(-z) = (-1)^n J_n(z) \Rightarrow J_1(-z) = -J_1(z)$

$$\Rightarrow \boxed{A_\phi \simeq \mu a I_0 \frac{e^{-jkr}}{4\pi r} 2j\pi J_1(ka\sin\theta)}$$

Also we have shown that $\int_0^{2\pi} \cos\phi' \, e^{jka\sin\theta\cos\phi'} d\phi' = j2\pi J_1(ka\sin\theta)$

Thus, for A_θ and A_ϕ we need to evaluate the followings:

using $j^n 2\pi J_n(x) = \int_0^{2\pi} [\cos(n\phi) + j\sin(n\phi)] e^{jx\cos\phi} d\phi$ for $n=1$, $x = ka\sin\theta$, $\phi = \phi'$

$$\Rightarrow \int_0^{2\pi} [\cos\phi' + j\sin\phi'] e^{jka\sin\theta\cos\phi'} d\phi' = j2\pi J_1(ka\sin\theta)$$

but for A_ϕ component we showed that $\int_0^{2\pi} \cos\phi' e^{jka\sin\theta\cos\phi'} d\phi' = j2\pi J_1(ka\sin\theta)$

$$\therefore \int_0^{2\pi} j\sin\phi' e^{jka\sin\theta\cos\phi'} d\phi' = 0$$

$$\Longrightarrow A_\theta = A_r = 0$$

$$\Rightarrow \bar{A} = \hat{a}_\phi A_\phi = \hat{a}_\phi j\mu a I_0 \frac{e^{-jkr}}{2r} J_1(ka\sin\theta)$$

$$\Rightarrow E_r \simeq E_\theta \simeq 0$$

$$E_\phi = -j\omega A_\phi = \mu a\omega I_0 \frac{e^{-jkr}}{2r} J_1(ka\sin\theta) = \eta k a I_0 \frac{e^{-jkr}}{2r} J_1(ka\sin\theta)$$

$$H_r = H_\phi = 0$$

$$H_\theta = -ak I_0 \frac{e^{-jkr}}{2r} J_1(ka\sin\theta)$$

(b) Small loop approximation $J_1(ka\sin\theta) \simeq \frac{1}{2} ka\sin\theta$

$$\Rightarrow E_r \simeq E_\theta \simeq 0$$

$$E_\phi \simeq \eta a^2k^2 I_0 \frac{e^{-jkr}}{4r} \sin\theta$$

$$H_r = H_\phi \simeq 0$$

$$H_\theta \simeq -a^2k^2 I_0 \frac{e^{-jkr}}{4r} \sin\theta$$

2 (a) $I = \dfrac{V_g}{Z_A + Z_g} = \dfrac{V_g}{125 + j125} = \dfrac{2}{125 + j125} = \dfrac{1-j}{125}$ (A)

P (supplied by the generator) $= P_s = \frac{1}{2} V_g I^* = \frac{1}{2} R_{total} |I|^2 \approx 8\times10^{-3}$ watts

P (dissipated in the generator) $= P_g = \frac{1}{2} R_g |I|^2 \approx 3.2\times10^{-3}$ watts

P (delivered to the antenna) $= P_L + P_r = \frac{1}{2}(R_r + R_L)|I|^2 \approx 4.8\times10^{-3}$ watts

P (radiated by the antenna) $= P_r = \frac{1}{2} R_r |I|^2 \approx 3.2\times10^{-3}$ watts

(b) $Z_g = Z_A^* \Rightarrow Z_g = 75 - j50\,\Omega$

$\Rightarrow I = \dfrac{V_g}{Z_A + Z_A^*} = \dfrac{2}{150} = 0.0133$ (A)

$P_s = \frac{1}{2} R_{total} |I|^2 \approx 13.3\times10^{-3}$ watts

$P_g = \frac{1}{2} R_g |I|^2 \approx 6.67\times10^{-3}$ watts.

$P_L + P_r = \frac{1}{2}(R_r + R_L)|I|^2 \approx 6.67\times10^{-3}$ watts

$P_r = \frac{1}{2} R_r |I|^2 \approx 4.44\times10^{-3}$ watts

As expected, power delivered to the antenna is larger for the conjugate matched case. Actually, as you know, it is the max. power that can be delivered

(c) part (a) $P_r \approx 3.2\times10^{-3}$ watts $\Rightarrow 3.2\times10^{-3} = B_1 \pi/2$

$\Rightarrow B_1 \approx 2.038\times10^{-3}$ watts/unit solid angle

part (b) $P_r \approx 4.44\times10^{-3}$ watts $\Rightarrow 4.44\times10^{-3} = B_1 \pi/2 \Rightarrow B_1 \approx 2.826\times10^{-3}$ watts/unit solid angle

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3 (a) Experiment 1

since real

$I_{sc} = \dfrac{V_{in}}{Z_A^a + Z_T^a}$ and $P_{in} = \frac{1}{2} I_{in}^* V_{in} = \frac{1}{2} |I_{sc}|^2 (Z_A^a + Z_T^a) = 10$ watts.

$P_{out} = \frac{1}{2} \dfrac{|V_{oc}|^2}{(Z_A^b + Z_T^b)^2} Z_T^b = 1$ watts

real

If we define Z_{ab} as the mutual coupling between antenna A and antenna B

$\Rightarrow V_{oc} = I_{sc} Z_{ab}$

$$\Rightarrow \quad P_{out} = \frac{1}{2} \frac{|I_{sc}|^2 \, |Z_{ab}|^2}{(Z_A{}^b + Z_T{}^b)^2} Z_T{}^b \qquad \text{but} \quad |I_{sc}|^2 = \frac{P_{in}\, 2}{(Z_A{}^a + Z_T{}^a)}$$

$$\Rightarrow P_{out} = \frac{1}{\cancel{2}} \frac{P_{in} \times \cancel{2} \times |Z_{ab}|^2}{(Z_A{}^a + Z_T{}^a)(Z_A{}^b + Z_T{}^b)^2} Z_T{}^b$$

$$\Rightarrow \frac{P_{out}}{P_{in}} \frac{(Z_A{}^a + Z_T{}^a)(Z_A{}^b + Z_T{}^b)^2}{Z_T{}^b} = |Z_{ab}|^2 = \frac{1 \times 100^6}{25 \times 10} = \frac{10^5}{25} \,\Omega^2$$

Experiment 2

$$I^b = \frac{V_{in}}{Z_T{}^b + Z_A{}^b} \Rightarrow P_{in} = \frac{1}{2} |I^b|^2 (Z_T{}^b + Z_A{}^b) \Rightarrow |I^b|^2 = \frac{2 P_{in}}{(Z_T{}^b + Z_A{}^b)}$$

$$P_{out} = \frac{1}{2} \frac{|V_{oc}|^2}{(Z_A{}^a + Z_T{}^a)^2} Z_T{}^a \qquad \text{but} \quad V_{oc} = I^b Z_{ba} \quad \& \quad Z_{ab} = Z_{ba}$$

$$\Rightarrow P_{out} = \frac{1}{2} \frac{|I^b|^2 \, |Z_{ab}|^2}{(Z_A{}^a + Z_T{}^a)^2} Z_T{}^a = \frac{1}{\cancel{2}} \frac{2 P_{in} \, |Z_{ab}|^2}{(Z_A{}^a + Z_T{}^a)^2 (Z_T{}^b + Z_A{}^b)} Z_T{}^a$$

$$\Rightarrow P_{out} = \frac{1\cancel{0}}{\cancel{100^2 \times 100}} \times \frac{\cancel{10^5}}{25} \times 50 = \underline{\underline{2 \text{ watts}}}$$

(b) $$P_{out} = 1 = \frac{P_{in} \, |Z_{ab}|^2}{(Z_A{}^a + Z_T{}^a)^2 (Z_T{}^b + Z_A{}^b)} Z_T{}^a = \frac{10 \times 10^5/25}{100^2 \, (Z_T{}^b + 75)} \times 50$$

$$\Rightarrow (Z_T{}^b + 75) = \frac{100 \times 50}{25} = 200 \Rightarrow \underline{\underline{Z_T{}^b = 125\,\Omega}}$$

(c) $$P_{out} = 1 = \frac{P_{in} \, |Z_{ab}|^2}{(Z_A{}^a + Z_T{}^a)^2 (Z_T{}^b + Z_A{}^b)} Z_T{}^a = \frac{10 \times 10^5/25}{(50 + Z_T{}^a)^2 \, 100} Z_T{}^a$$

$$(50 + Z_T{}^a)^2 = 400 Z_T{}^a \Rightarrow (Z_T{}^a)^2 - 300 Z_T{}^a + 2500 = 0 \Rightarrow Z_T{}^a = 8.58\,\Omega \text{ or } 291.42\,\Omega$$

4 $\quad \nabla \times \bar{E}_1 = -j\omega\mu \bar{H}_1 - \bar{M}_1 \qquad \nabla \times \bar{E}_2 = -j\omega\mu \bar{H}_2 - \bar{M}_2$

$\quad \nabla \times \bar{H}_1 = j\omega\epsilon \bar{E}_1 + \bar{J}_1 \qquad \nabla \times \bar{H}_2 = j\omega\epsilon \bar{E}_2 + \bar{J}_2$

$$\Rightarrow \nabla\cdot\left(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1\right) = \bar{H}_2\cdot\underbrace{\nabla\times\bar{E}_1}_{\substack{-j\omega\mu\bar{H}_1\\ -\bar{M}_1}} - \bar{E}_1\cdot\underbrace{\nabla\times\bar{H}_2}_{\substack{j\omega\epsilon\bar{E}_2\\ +\bar{J}_2}} - \bar{H}_1\cdot\underbrace{\nabla\times\bar{E}_2}_{\substack{-j\omega\mu\bar{H}_2\\ -\bar{M}_2}} + \bar{E}_2\cdot\underbrace{\nabla\times\bar{H}_1}_{j\omega\epsilon\bar{E}_1+\bar{J}_1}$$

$$\Rightarrow \nabla\cdot\left(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1\right) = \bar{H}_2\cdot\left[-j\omega\mu\bar{H}_1 - \bar{M}_1\right] - \bar{E}_1\cdot\left[j\omega\epsilon\bar{E}_2 + \bar{J}_2\right] - \bar{H}_1\cdot\left[-j\omega\mu\bar{H}_2 - \bar{M}_2\right]$$

$$+ \bar{E}_2\cdot\left[j\omega\epsilon\bar{E}_1 + \bar{J}_1\right]$$

$$= \cancel{-j\omega\mu\bar{H}_1\cdot\bar{H}_2} - \bar{M}_1\cdot\bar{H}_1 \cancel{-j\omega\epsilon\bar{E}_1\cdot\bar{E}_2} - \bar{E}_1\cdot\bar{J}_2$$

$$\cancel{+j\omega\mu\bar{H}_1\cdot\bar{H}_2} + \bar{H}_1\cdot\bar{M}_2 \cancel{+j\omega\epsilon\bar{E}_1\cdot\bar{E}_2} + \bar{E}_2\cdot\bar{J}_1$$

$$= \bar{J}_1\cdot\bar{E}_2 + \bar{M}_2\cdot\bar{H}_1 - \bar{J}_2\cdot\bar{E}_1 - \bar{M}_1\cdot\bar{H}_1$$

integrate over the volume V

$$\int_V \nabla\cdot\left(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1\right) dv = \int_V \left(\bar{J}_1\cdot\bar{E}_2 + \bar{M}_2\cdot\bar{H}_1 - \bar{J}_2\cdot\bar{E}_1 - \bar{M}_1\cdot\bar{H}_1\right) dv$$

apply divergence theorem

$$\oint_S \left(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1\right)\cdot\hat{a}_n\, ds =$$

S: surface that completely encloses V