Solutions to HW # 9

1 (a)

$$\bar{I}_e(z') = \begin{cases} I_o\left(1+\frac{2z'}{l}\right) & -\frac{l}{2} < z' < 0 \\ I_o\left(1-\frac{2z'}{l}\right) & 0 < z' < l/2 \end{cases}$$

$$\bar{A}(\bar{r}) \simeq \hat{a}_z \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} I_e(z')\, e^{jk\hat{a}_r\cdot\bar{r}'}dz' = \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2}\left(1-2\frac{|z'|}{l}\right)e^{jkz'\cos\theta}dz'$$

$$\Rightarrow \bar{A}(\bar{r}) \simeq \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left[l\,\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} - 2\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}dz'\right]$$

$$\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}dz' = \int_{0}^{l/2}\frac{z'}{l}e^{jkz'\cos\theta}dz' - \int_{-l/2}^{0}\frac{z'}{l}e^{jkz'\cos\theta}dz'$$

$$= 2\int_0^{l/2}\frac{z'}{l}\cos\left[kz'\cos\theta\right]dz' = \frac{l}{2}\left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} + \frac{\cos\left(\frac{kl}{2}\cos\theta\right)-1}{\left(\frac{kl}{2}\cos\theta\right)^2}\right]$$

$$\Rightarrow \bar{A}(\bar{r}) \simeq \hat{a}_z\frac{\mu l}{4\pi}\frac{e^{-jkr}}{r}\left[\frac{1-\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2}\right] \Rightarrow A_\theta = -\sin\theta A_z$$

In the far-zone $E_r \simeq 0$, $E_\theta = -j\omega A_\theta = j\omega\mu\frac{l}{4\pi}\frac{e^{-jkr}}{r}\sin\theta\left\{\frac{1-\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}$

$E_\phi \simeq 0$, $H_r = 0$, $H_\theta \simeq 0$

$H_\phi = E_\theta/\eta$

(b) from part (a) $E_\theta \simeq j\eta k\frac{e^{-jkr}}{4\pi r}\sin\theta\left[\int_{-l/2}^{l/2} I_o\cos^2\left(\frac{\pi}{l}z'\right)e^{jkz'\cos\theta}dz'\right]$

use $\int \cos^2 bz\, e^{az}dz = \frac{e^{az}}{2a} + \frac{e^{az}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bz + b\sin 2bz\right)$

where $a = jk\cos\theta$, $b = \pi/l$

$$\Rightarrow E_\theta \simeq j\eta k \frac{e^{-jkr}}{4\pi r} \sin\theta\, I_0 \left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \right]$$

$$H_\phi = E_\theta/\eta$$

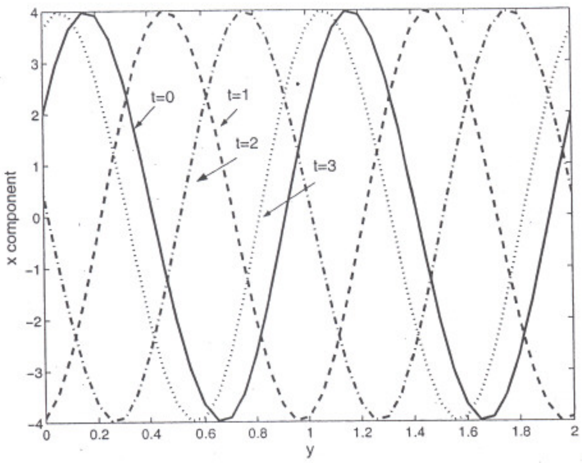
2

$$\bar{E}(\bar{r},t) = \mathrm{Re}\{\bar{E}(\bar{r})\exp(jwt)\} = \mathrm{Re}\left\{\left[\hat{x}(2+2\sqrt{3}j) + \hat{z}4e^{j5\pi/6}\right]\exp(jwt - jky)\right\}$$

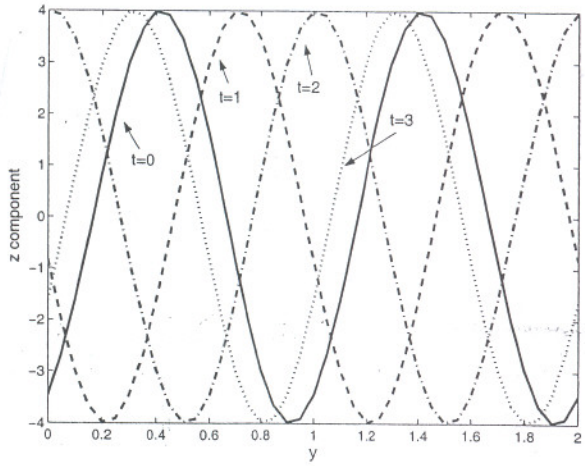
$$= \hat{x}\left[2\cos(wt-ky) - 2\sqrt{3}\sin(wt-ky)\right] + \hat{z}4\cos\left(wt - ky + \frac{5\pi}{4}\right)$$

$$= \hat{x}4\cos(wt - ky + \frac{2\pi}{6}) + \hat{z}4\cos\left(wt - ky + \frac{5\pi}{6}\right)$$

(a)



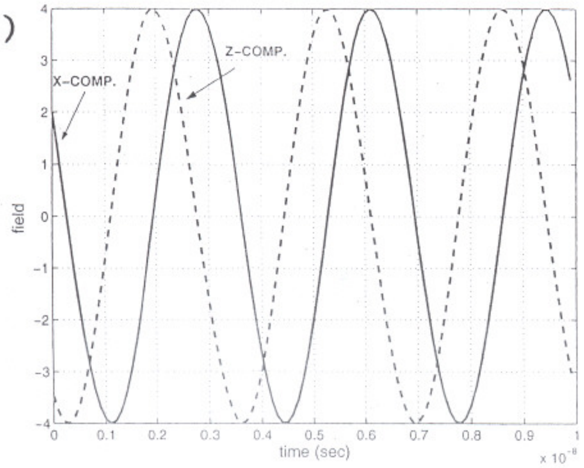
(b)



Wave propagates in the y-direction as the time goes on.

(c) Distance between two similar points on two curves gives the distance that the wave travels in 1 ns.

(d)



when $E_x = max$, $E_z = 0$ (goes from positive to negative); and when $E_z = max$, $E_x = 0$ (goes from negative to positive). In addition $|E_z|_{max} = |E_x|_{max}$

$\Rightarrow$ RHCP

(e) $|E_x| = |E_z|$; $\Delta\phi = \phi_z - \phi_x = \pi/2 \longrightarrow$ RHCP

(f) $\bar{E}(\bar{r}) = \left[\hat{a}_x(2+2\sqrt{3}j) - \hat{a}_z 4 e^{j5\pi/6}\right] e^{jky}$ is still RHCP

3 (a) on the y-z plane $\bar{E}(\bar{r}) = -\hat{a}_\phi \cos(\theta) f(r,\theta,\phi)$ $\phi = \pi/2$

linearly polarized antenna for the minimum PL is $\hat{a}_a = \hat{a}_x$

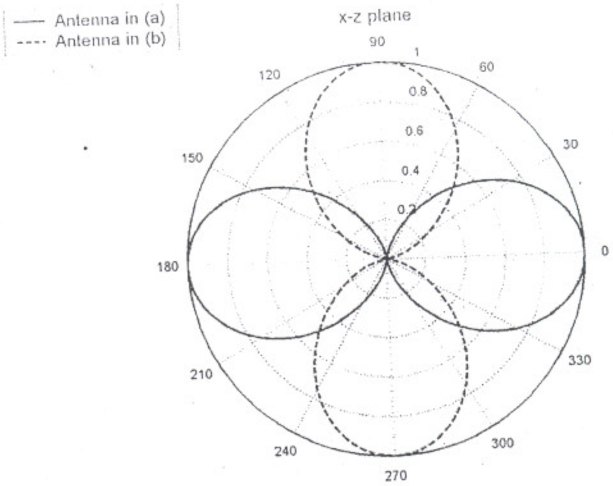
(b) on the x-y plane $\bar{E}(\bar{r}) = \hat{a}_\theta \cos(\phi) f(r,\theta,\phi)$ $\theta = \pi/2$

linearly polarized antenna for the minimum PL is $\hat{a}_a = \hat{a}_z$

(c) x-z plane $\bar{E}(\bar{r}) = [\hat{a}_\theta \sin\theta] f(r,\theta,\phi)$ $\phi = 0°$

$\hat{a}_a = \hat{a}_x \Rightarrow PLF = |\hat{a}_x \cdot \hat{a}_\theta|^2 = \cos^2\theta$

$\hat{a}_a = \hat{a}_z \Rightarrow PLF = |\hat{a}_z \cdot \hat{a}_\theta|^2 = \sin^2\theta$



(d) wave traveling in the $-x$ direction $\hat{a}_w = \frac{1}{\sqrt{2}}\hat{a}_y \mp \frac{1}{\sqrt{2}}\hat{a}_z$; $\hat{a}_a = -\hat{a}_z$

$\Rightarrow PLF = 3\,dB$

wave traveling in the $-y$ direction $\hat{a}_w = -\frac{1}{\sqrt{2}}\hat{a}_x \pm \frac{1}{\sqrt{2}}\hat{a}_z$; $\hat{a}_a = -\hat{a}_z$

$\Rightarrow PLF = 3\,dB$

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4 (a) $P_{rad} = U_0 \int_0^{2\pi} d\phi \int_0^{\theta_0} \cos^3\theta \sin\theta \, d\theta = \frac{2\pi U_0}{4}(1-\cos^4\theta_0) \Rightarrow D_0 = \frac{16\pi}{2\pi(1-\cos^4\theta_0)} = \frac{4\pi}{\lambda^2} A_E = 4\pi$

$\Rightarrow 1-\cos^4\theta_0 = 2/\pi \Rightarrow \theta_0 = 39.067°$ OR $\theta_0 = 140.93°$

not possible. It makes radiation intensity negative.

(b) $P_{rad} = U_0 \int_0^{2\pi} d\phi \int_0^{\theta_0 = 3°} \cos^3\theta \sin\theta \, d\theta = \frac{2\pi U_0}{4}(1-\cos^4 3°)$

$\Rightarrow A_e = \frac{2\lambda^2}{\pi(1-\cos^4 3°)} = 29.093\, m^2$