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# Linearly time-varying systems and their fast implementation

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ABSTRACT

Linear time-invariant systems can be implemented in $O(N \log N)$ time, whereas the most general family of linear systems can be implemented as a vector-matrix product in $O(N^2)$ time. However, there are time-variant systems that can be implemented in $O(N \log N)$ time. In this paper, we introduce a particular family of such systems, which we refer to as the class of linearly time varying (LTV) systems. These systems interpolate between multiplicative systems and convolutive systems, and are characterized by their chirp-type eigenfunctions and their relationship to fractional Fourier domain filtering. We derive expressions for the linear transform kernel of LTV systems, and illustrate their use with examples. Recognizing LTV systems, or approximating linear systems with LTV systems when possible, can reduce the time of computation from $O(N^2)$ to $O(N \log N)$.



## 1. Introduction

In this paper we focus on a class of systems which we refer to as linearly time-varying systems (LTV systems). We underline that we are not dealing with the broader class of “linear time-variant” systems, which are linear systems which are not time-invariant. Rather, we focus on a subset of time-variant systems, whose time-variance is of a particular form. The term “linearly time-varying” is meant to refer to the particular kind of time-variance that is of interest to us.

We give two definitions of LTV systems. In the first, we characterize LTV systems with their chirp-type eigenfunctions. The eigenfunctions of linear time-invariant systems are harmonic functions, whose frequencies do not vary with time. In contrast, the frequencies of chirp-type functions vary linearly with time. Thus we refer to systems with chirp-type eigenfunctions as linearly time-varying systems. Chirp functions are known to interpolate between Dirac delta functions and pure harmonic functions [17, pp. 146-149]. Correspondingly, LTV systems interpolate between multiplicative systems and convolutive systems. (With these terms we refer to systems that correspond to multiplication or convolution by a function in the time domain.)

In our second definition, we define an LTV system to be a system which corresponds to multiplicative filtering in fractional Fourier domains. According to the second definition, an LTV system corresponds to taking the fractional Fourier transform of the input, multiplying the result with a filter function in the $a$th fractional Fourier domain, and returning back to the original domain by taking the inverse transform. We show that these two definitions are equivalent. As suggested by the previous paragraph, LTV systems have the same relationship with the fractional Fourier transform, as linear time-invariant systems have with the ordinary Fourier transform.

The fractional Fourier transform (FRT) has been widely studied in the context of signal processing and optics [1–23]. Its properties have been established [17] and it appears in standard handbooks of mathematical transforms [20]. The FRT is intimately related to time-frequency distributions that are widely used in signal analysis and processing, and most notably the Wigner distribution [3,10,21,43]. Notable among its properties are the product and convolution theorems [11,14,15,17] and eigenfunctions [18]. The fractional Fourier domain decomposition has been inspired by the singular value decomposition [16] and the sampling theorem has been generalized to FRT domains [19].

There are several important relationships between the FRT and optics. First, the FRT models the propagation of light in graded-index media [1,2,13]. Second, with appropriate scaling, it models the propagation of light not only in free space but also in systems consisting of arbitrary concatenations of lenses and sections of free space [13,17]. Third, it is possible to construct optical systems that perform analog fractional Fourier transforms with very high resolution and speed [5,6]. Fourth, it is intimately related to Gaussian-beam propagation in lasers and the Gouy phase [13].

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The fractional generalizations of many transforms and other concepts have been proposed [24–42]. Many applications of the FRT have been explored [43–51]. Specifically, the FRT has been shown to provide significant improvements in numerous signal reconstruction, signal recovery, filtering, and noise removal problems [52–70]. The concept of filtering in the ordinary Fourier domain has been extended to filtering in fractional Fourier domains, where the signal is fractional Fourier transformed, filtered, and then inverse transformed back to the time domain [3,39,52,56]. Closely related is the concept of fractional convolution [3,28,37]. It has been demonstrated that noise that cannot be separated either in the time or frequency domains can be separated in FRT domains [3,58]. Another example where the FRT has been found useful is the detection of linear FM signals [31,50]. Fractional correlation has been proposed as a generalization of ordinary correlation [24,26]. This has applications to signal detection and pattern recognition [54]. Multiplexing in FRT domains has been suggested as a strategy between time-domain and frequency-domain multiplexing [3]. The FRT has been applied to signal synthesis [9], as well as to signal recovery from multiple measurements of wavefields [62]. The concept of filtering in FRT domains has been generalized to filtering in multiple FRT domains [59]. These domains can be visited in sequential manner, which is referred to as repeated or multi-stage filtering [55,61]. They can also be visited at the same time and the results combined, which is known as parallel or multi-channel filtering [17,59]. Applications to filter design are considered in [66]. Reference [69] deals with applications to beamforming. Further applications are reviewed in [46].

Generally speaking, for any application where the ordinary Fourier transform plays a role, there is the potential for generalization or improvement with the fractional Fourier transform. The fractional Fourier transform has an order parameter $a$ which is equal to 1 for the ordinary Fourier transform. Thus the FRT includes the ordinary FT as a special case. This additional parameter provides greater generality and a degree of freedom over which to optimize. If the optimal value is $a = 1$, that means the FRT does not provide any improvement. However, if it is any other value, that means the FRT offers at least some improvement. The improvements are most often strongest when the system is not time-invariant and/or when the noise or other random characteristics are not stationary. This is because the ordinary FT works best with time-invariant systems; after all the basis functions of the FT are the eigenfunctions of time-invariant systems. For time-variant systems, the FRT often provides some improvement. As will be seen, the basis functions of the FRT are chirp functions, which are functions with linearly varying frequency. Thus one may expect the FRT to offer the greatest improvements when signals and systems exhibiting such behavior are involved. This is indeed frequently observed.

An analogy with harmonic analysis in physics may be illuminating. Oscillatory systems are widespread in nature but they do not always have quadratic potentials. However, if the potential function is expanded as a Taylor series around its equilibrium point, the first order term vanishes by definition and the leading term is the quadratic term. Thus, provided the oscillations are not large, most vibrating systems can be treated as harmonic oscillators as a first approximation. The situation with application of the FRT is similar. While all time-varying systems do not exhibit behavior in frequency that is linear, this can be taken as a first approximation so that many systems will benefit from the use of the fractional Fourier transform. This analogy directly carries over to the concept of linearly time-varying systems developed in this paper. While all linear time-variant systems are not linearly time-varying systems, many can be reasonably approximated by one, and thus their implementation can be accomplished in $O(N \log N)$ time instead of $O(N^2)$. Thus the motivation of this paper is to show that it is often possible to exactly or approximately model a system with a linearly time-varying system and thus implement it in $O(N \log N)$ time.

The $a$th order FRT is a fractional generalization of the ordinary Fourier transform which interpolates between the identity operation and the ordinary Fourier transform operation, as $a$ increases from 0 to 1. The $a$th order FRT of a function $f(u)$ is commonly defined as [17]:

$$\begin{aligned} \mathcal{F}^a[f(u)] &= (\mathcal{F}^a f)(u_a) = f_a(u_a) = \int_{-\infty}^{\infty} K_a(u_a, u) f(u)\, du, \\ K_a(u_a, u) &= A_\alpha \exp(i\pi(\cot\alpha\, u_a^2 - 2\csc\alpha\, u_a u + \cot\alpha\, u^2)), \\ A_\alpha &= \sqrt{1 - i\cot\alpha}, \quad \alpha = \frac{a\pi}{2}, \end{aligned} \tag{1}$$

where $\mathcal{F}^a$ is the $a$th order FRT operator and $K_a(u_a, u)$ is the $a$th order FRT kernel. Here $u_a$ denotes the coordinate variable in the $a$th order fractional Fourier domain. All integrals in this paper are from $-\infty$ to $+\infty$. The above holds when $a \neq 2j$ for $j \in \mathbb{Z}$. If $a = 4j$, $K_a(u_a, u) = \delta(u_a - u)$, and the FRT is the identity. When $a = 4j + 2$, $K_a(u_a, u) = \delta(u_a + u)$ and the FRT becomes the parity operator.

The FRT can be computed in $O(N \log N)$ time, just as the Fourier transform [71,72]. These algorithms are based on careful analytical manipulation of the integrals prior to discretization. As we will more precisely see, LTV systems correspond to multiplicative filtering of their inputs in the $a$th fractional Fourier domain. Since multiplicative filtering involves an FRT followed by simple multiplication (which take $O(N)$ time), followed by another FRT, this implies that LTV systems can also be computed in $O(N \log N)$ time. We underline the fact that although LTV systems are not linear time-invariant, this allows us to implement them in $O(N \log N)$ time. Furthermore, even when a given system is not strictly LTV, we can use LTV systems to approximate non-LTV systems, with the purpose of reducing the computation times.

The base of logarithm is not important in complexity expressions but for concreteness it may be taken as 2.

The paper is organized as follows: In section 2, we introduce the two definitions of LTV systems, and demonstrate their equivalence. Next, in section 3, we derive the explicit form of the linear transform kernel for LTV systems, and examine its limits. In section 4, we illustrate the use of LTV systems to exactly or approximately model systems that we might encounter. We see that they can provide better performance than ordinary multiplicative or convolutive filtering, and lower cost than general linear filtering. In the last section, we conclude.

## 2. Definitions of LTV systems

In this section, we present two definitions of LTV systems. The first definition characterizes LTV systems with their eigenfunctions. The second definition specifies LTV systems as those systems which correspond to filtering in a fractional Fourier domain. Then the equivalence of these two definitions is established.

### 2.1. First definition

We define $a$th order LTV systems to have eigenfunctions given by $\text{chirp}_{-a,\zeta}(u)$, which are functions of $u$ indexed by $\zeta$. Chirp functions, which are functions whose instantaneous frequencies are linearly time varying, are here defined as:

$$\begin{aligned} \text{chirp}_{a,\zeta}(u) &= A_\alpha \exp\left(i\pi(\cot\alpha\, u^2 - 2\csc\alpha\, u\zeta + \cot\alpha\, \zeta^2)\right), \\ A_\alpha &= \sqrt{(1 - i\cot\alpha)}, \qquad \alpha = \frac{a\pi}{2}. \end{aligned} \tag{2}$$

The eigenvalues associated with these eigenfunctions can be chosen arbitrarily. Different eigenvalue choices result in different LTV systems.

Chirp functions are more commonly expressed in the form $\mathrm{chirp}(u) = \exp(i\pi(\chi u^2 + 2\xi u))$. The form given in (2) may be related to this by taking $\chi = \cot\alpha$ and $\xi = -\csc\alpha\,\zeta$, and multiplying by $A_\alpha \exp(i\pi\cot\alpha\zeta^2)$. This reparameterization produces more convenient harmonic and delta limits as $a \to 1$ and $a \to 0$. Furthermore, we recognize the parameterized family of chirps $\mathrm{chirp}_{a,\zeta}(u)$ as being essentially equal to the $a$th order FRT kernel $K_a(\zeta, u)$:

$$\mathrm{chirp}_{a,\zeta}(u) = K_a(\zeta, u). \tag{3}$$

We also note that, all chirp functions are fractional Fourier transforms of each other and satisfy the equality $\mathcal{F}^a[\mathrm{chirp}_{a',\zeta}(u)] = \mathrm{chirp}_{a+a',\zeta}(u_a)$ [17]. The Wigner distribution of this family of chirps is given by:

$$W_{\mathrm{chirp}_{a,\zeta}}(u,\mu) = \frac{1}{|\sin\alpha|}\delta\big(\mu - \cot\alpha\, u + \csc\alpha\,\zeta\big), \tag{4}$$

which is concentrated along the line $\mu = \cot\alpha\, u - \csc\alpha\,\zeta$ in the time-frequency plane. As the order $a$ varies, the axis crossings of the Wigner distribution changes. For more on the Wigner distribution, see [17].

For $a = 0$, we have $\mathrm{chirp}_{0,\zeta}(u) = \delta(u-\zeta)$ and the Wigner distribution $W_{\mathrm{chirp}_{0,\zeta}}(u,\mu) = \delta(u-\zeta)$ forms vertical lines in the time-frequency plane. Systems whose eigenfunctions are of the form $\delta(u-\zeta)$ are multiplicative systems; they simply multiply the input with some function of $u$ to produce the output. On the other hand, for $a = 1$, we have $\mathrm{chirp}_{-1,\zeta}(u) = \exp(i2\pi u\zeta)$ and the Wigner distribution $W_{\mathrm{chirp}_{-1,\zeta}}(u,\mu) = \delta(\mu-\zeta)$ forms horizontal lines in the time-frequency plane. In this case the general chirp family reduces to harmonics. Systems whose eigenfunctions are harmonics are convolutive systems (time-invariant systems). Therefore, according to definition 1, LTV systems reduce to multiplicative systems for $a = 0$ and convolutive systems for $a = 1$. As $a$ increases from 0 to 1, the chirp family employed in the definition evolves from delta functions to harmonic functions, and the LTV systems, as defined, evolve from multiplicative systems to convolutive systems.

### 2.2. Second definition

Here we define LTV systems of order $a$ as multiplicative filtering in the $a$th order fractional Fourier domain. In other words, we take the $a$th order FRT of the input $f(u)$, multiply the result with some filter function $H(u_a)$, and then take the inverse FRT of the product to obtain the output $g(u)$. Thus, mathematically, the output $g(u)$ for an input $f(u)$ is given by:

$$g(u) = \mathcal{L}_a\{f(u)\} = \mathcal{F}^{-a}\Big[H(u_a)\,(\mathcal{F}^a f)(u_a)\Big](u), \tag{5}$$

where $\mathcal{L}_a\{\cdot\}$ denotes the $a$th order LTV system, and $\mathcal{F}^a$ denotes the $a$th order FRT operator. The filter function in the $a$th domain $H(u_a)$ can be chosen arbitrarily; choice of different filter functions results in different LTV systems.

When $a = 0$, the FRT operators in (5) boil down to identity operators, and the system becomes a multiplicative system, $g(u) = f(u)H(u)$. When $a = 1$, (5) becomes Fourier domain filtering and the system becomes a convolutive system, $g(u) = f(u) * h(u)$, where $h(u)$ denotes the inverse Fourier Transform of $H(u)$. As $a$ varies from 0 to 1, the LTV systems evolve from multiplicative systems to convolutive systems. Therefore, the second definition conforms with the first definition, in terms of the behavior of the systems with respect to $a$.

Since LTV systems are systems which correspond to multiplicative filtering in the $a$th fractional Fourier domain, one is also led to inquire the systems which correspond to convolutive filtering in the $a$th domain. It can trivially be shown that such systems are also members of the family of LTV systems defined in this section, but with order $(a+1)$, rather than $a$. The reason is that convolution in the $a$th domain is equivalent to multiplication in the $(a+1)$th domain.

The second definition of LTV systems enables representing these systems as matrix-vector products in the discrete-time domain. For that purpose, we construct the diagonal $\mathbf{\Lambda_H}$ matrix which contains the samples of $H(u)$ along its diagonal. Given the input vector $\mathbf{f}$ containing the input samples, the output vector $\mathbf{g}$ can be calculated as

$$\mathbf{g} = \mathbf{F}^{-a}\mathbf{\Lambda_H}\mathbf{F}^a\mathbf{f} \tag{6}$$

where $\mathbf{F}^a$ is the $a$th order discrete fractional Fourier transform matrix [73].

The FRT can be computed in $O(N\log N)$ time, employing the algorithm discussed in [71,72]. This algorithm, in the light of (6), also implies a fast implementation technique for the LTV systems defined in definition 2. This is what gives significance to LTV systems. Although they are not time-invariant, they can be calculated fast in $O(N\log N)$ time.

### 2.3. Equivalence of the two definitions

In this section, we establish the equivalence of the two definitions given. First, we prove that systems with eigenfunctions of the form $\mathrm{chirp}_{-a,\zeta}(u)$ indeed filter their inputs in the $a$th fractional Fourier domain. Conversely, we prove that systems which correspond to $a$th order fractional Fourier domain filtering, have eigenfunctions of the form $\mathrm{chirp}_{-a,\zeta}(u)$.

To prove the first step, we express $f(u)$ in terms of the following inverse FRT relationship, given the $a$th order FRT $f_a(u_a)$ of $f(u)$:

$$\begin{aligned} f(u) &= \int_{-\infty}^{\infty} f_a(u_a)K_{-a}(u,u_a)\,du_a \\ &= \int_{-\infty}^{\infty} f_a(u_a)K_{-a}(u_a,u)\,du_a \\ &= \int_{-\infty}^{\infty} f_a(u_a)\mathrm{chirp}_{-a,u_a}(u)\,du_a, \end{aligned} \tag{7}$$

where we used symmetry of the FRT kernel $K_a(u,u_a) = K_a(u_a,u)$ in the second equality. The rightmost expression expresses $f(u)$ as a linear superposition of the $\mathrm{chirp}_{-a,\zeta}(u)$ functions with weights $f_a(\zeta)$, if we replace the dummy variable $u_a$ with $\zeta$. However, the functions $\mathrm{chirp}_{-a,\zeta}(u)$ are the eigenfunctions that characterize an $a$th order LTV system in the first definition. Therefore, the rightmost form in equation (7) expresses $f(u)$ in terms of the eigenfunctions of the $a$th order LTV system according to the first definition:

$$f(u) = \int_{-\infty}^{\infty} f_a(\zeta)\mathrm{chirp}_{-a,\zeta}(u)\,d\zeta. \tag{8}$$

Hence, when $f(u)$ is fed into the $a$th order LTV system defined in definition 1, the output can be trivially obtained by multiplying the eigenfunctions with the corresponding eigenvalues. We will denote the eigenvalues by $H(\zeta)$, where $\zeta$ is the index. Then the output of the system becomes

$$\mathcal{L}_a\{f(u)\} = \int_{-\infty}^{\infty} f_a(\zeta)H(\zeta)\text{chirp}_{-a,\zeta}(u)\,d\zeta$$
$$= \int_{-\infty}^{\infty} f_a(\zeta)H(\zeta)\text{chirp}_{-a,u}(\zeta)\,d\zeta \tag{9}$$
$$= \int_{-\infty}^{\infty} f_a(u_a)H(u_a)\text{chirp}_{-a,u}(u_a)\,du_a,$$

where we first used $\text{chirp}_{-a,u}(\zeta) = \text{chirp}_{-a,\zeta}(u)$ which followed from the symmetry of the kernel, and switched back to the dummy variable $u_a$. Equation (9) amounts to taking the $a$th order FRT of the input $f(u)$ to obtain $f_a(u_a)$, then multiplying $f_a(u_a)$ with $H(u_a)$ in the $a$th order domain, and then taking the inverse fractional Fourier transform of the result, since $\text{chirp}_{-a,u}(u_a) = K_{-a}(u,u_a)$ is merely the inverse FRT kernel. This sequence of operations precisely corresponds to our second definition of LTV systems with the indexed eigenvalues $H(\zeta)$ appearing in the first definition corresponding to the filter function in the second definition. Thus we have shown that any system covered by definition 1 is also a member of the systems defined by definition 2.

The proof of the converse is established by considering the eigenfunction equation of an $a$th order LTV system as defined by definition 2, in the form of a multiplicative filtering operation in the $a$th order FRT domain. Let $f(u)$ be an eigenfunction and $\lambda$ be the corresponding eigenvalue of this LTV system. Here, $\Lambda_H$ is a multiplicative system operator which multiplies its input with $H(u_a)$:

$$\big(\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a f\big)(u) = \lambda f(u), \tag{10}$$
$$\big(\Lambda_H\mathcal{F}^a f\big)(u_a) = \lambda(\mathcal{F}^a f)(u_a), \tag{11}$$
$$\big(\Lambda_H f_a\big)(u_a) = \lambda f_a(u_a). \tag{12}$$

This is an eigenvalue equation in terms of $(\mathcal{F}^a f)(u_a) = f_a(u_a)$. Since $\Lambda_H$ is a multiplicative system operator, its eigenfunctions are $\delta(u_a - \zeta)$ for arbitrary $\zeta$. Therefore $(\mathcal{F}^a f)(u_a) = f_a(u_a) = \delta(u_a - \zeta) = \text{chirp}_{0,\zeta}(u_a)$. Performing an inverse FRT, we obtain

$$f(u) = \mathcal{F}^{-a}[\text{chirp}_{0,\zeta}(u_a)] = \text{chirp}_{-a,\zeta}(u), \tag{13}$$

which implies that the systems defined by definition 2 have the eigenfunctions $\text{chirp}_{-a,\zeta}(u)$, which are precisely the eigenfunctions that were used to define LTV systems in definition 1. Thus, the family of systems in definition 1 contain the systems defined in definition 2, and the equivalence of the definitions is established.

Definition 2 forms the basis for fast implementation of LTV systems. The FRT has a fast $O(N\log N)$ algorithm [71,72]. These algorithms are obtained by carefully manipulating the linear transform integral before discretization. Special attention is required to ensure the chirp-like features inside the integral are properly sampled, without requiring a sample rate that is above that dictated by the time-bandwidth product of the signals to be transformed [72]. Ordinary multiplication of two functions takes $O(N)$ time. Since this definition involves an FRT followed by a multiplication followed by an FRT, we deduce that LTV systems can be digitally implemented in $O(N\log N)$ time. If we are to implement LTV systems optically, this time we can use the fact that the FRT can be realized with an optical system whose space-bandwidth product is $O(N)$, to conclude that LTV systems can be realized with an optical system whose space-bandwidth product is $O(N)$ [17,59]. On the other hand, Definition 1 reveals the eigenfunctions of LTV systems to be chirps, which are functions of linearly changing frequency. This suggests that linear changes in frequency might be a feature of LTV systems, as we will later further discuss.

## 3. Kernel of LTV systems

In this section we present the explicit linear transform kernel of LTV systems. Assuming the input signal $f(u)$ is a well-behaving function, the output signal $g(u)$ is given by the following relationship:

$$g(u) = \mathcal{F}^{-a}\big[H(u_a)\,(\mathcal{F}^a f)(u_a)\big](u) \tag{14}$$
$$= \int_{-\infty}^{\infty} K_{-a}(u,u_a)H(u_a)\int_{-\infty}^{\infty} K_a(u_a,u')f(u')\,du'\,du_a, \tag{15}$$
$$= \int_{-\infty}^{\infty} f(u')\int_{-\infty}^{\infty} K_{-a}(u,u_a)H(u_a)K_a(u_a,u')\,du_a\,du', \tag{16}$$
$$= \int_{-\infty}^{\infty} L_a(u,u')f(u')du', \tag{17}$$

where we identify $L_a(u,u')$ as the kernel of the $a$th order LTV system:

$$L_a(u,u') = \int_{-\infty}^{\infty} K_{-a}(u,u_a)H(u_a)K_a(u_a,u')\,du_a, \tag{18}$$
$$= \frac{1}{|\sin\alpha|}\exp[i\pi\cot\alpha(u'^2 - u^2)]\,h[\csc\alpha(u-u')], \tag{19}$$

where $h(u)$ is the inverse Fourier transform of $H(u)$. Since $H(u)$ is an arbitrary function, so is $h(\csc\alpha\, u)$. Thus, the kernel is essentially formed by a chirp, multiplied with an arbitrary function of $u-u'$. If we expand the exponent as a difference of squares, we can also say that the kernel is an arbitrary function of $u-u'$, multiplied with an exponent of $u+u'$. The exponent of $u+u'$ causes the deviation from LTI systems.

The input-output relationship of LTV systems for an input $f(u)$ and output $g(u)$ can be expressed as follows, by using (19):

$$g(u) = \frac{1}{|\sin\alpha|}\exp(-i\pi\cot\alpha u^2) \times \Big[h\big(\csc\alpha\, u\big) * \big(f(u)\exp(i\pi\cot\alpha u^2)\big)\Big]. \tag{20}$$

From this equation it can be observed that an LTV system multiplies its input with a chirp, then convolves this with some function, and finally multiplies the result with the complex conjugate of the initial chirp. In other words, LTV systems are equivalent to linear time-invariant systems with a specific form of pre and post chirp multiplication.

### 3.1. The limit $a \to 1$

When $a \to 1$, the kernel of the LTV system given in (19) approaches to a convolutive kernel:

$$\lim_{a\to 1} L_a(u,u') = h(u-u'). \tag{21}$$

This kernel form is equivalent to convolution with the inverse Fourier transform of the filter function. This is expected since when $a = 1$, the system in (5) first takes the ordinary Fourier transform of the input, then multiplies this with $H(u)$ and then takes the inverse Fourier transform. Since multiplication in the frequency domain corresponds to convolution in the time domain, we obtain the convolutive kernel in (21).

We now present an expression for the kernel which is valid when $a$ is close to, but not necessarily equal to 1, by employing a Taylor series expansion of the kernel in (19):

$$L_a(u,u') \approx h(u-u') + i\frac{\pi^2}{2}(a-1)\,h(u-u')\,(u^2-u'^2). \tag{22}$$

From the equation above, it can be observed that when $a$ is close to 1, the convolutive kernel is perturbed by a term which gets more significant as the order moves away from $a=1$. This perturbation term destroys time-invariance. Note that when $h(u)$ is real, the perturbation is purely imaginary; in other words, the departure of LTV systems from time-invariance is of an imaginary nature, at least to first order.

The corresponding input-output relationship for (22) is:

$$y(u) = h(u) * f(u) + i\frac{\pi^2}{2}(a-1)\left[u^2\big(h(u)*f(u)\big) - h(u)*\big(u^2 f(u)\big)\right]. \tag{23}$$

### 3.2. The limit $a \to 0$

For the case of $a \to 0$, the $\csc\alpha$ and $\cot\alpha$ terms in (19) diverge. Setting $a = b-1$ and using the index additivity of FRT, we rewrite the operator $\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a$ as $\mathcal{F}^1(\mathcal{F}^{-b}\Lambda_H\mathcal{F}^b)\mathcal{F}^{-1}$. The $a \to 0$ limit corresponds to $b \to 1$. Using (21) with $b$ in place of $a$, we see that the kernel corresponding to the operator $\mathcal{F}^{-b}\Lambda_H\mathcal{F}^b$ is simply $h(u-u')$, which amounts to convolution of the input with $h(u)$. Therefore the operator $\mathcal{F}^1(\mathcal{F}^{-b}\Lambda_H\mathcal{F}^b)\mathcal{F}^{-1}$ corresponds to multiplication with $H(u)$, so that we can write

$$\lim_{a\to 0} L_a(u,u') = H(u')\delta(u-u'). \tag{24}$$

When $a$ is perturbed around 0, it is not possible to directly use a Taylor series expansion around 0 since the kernel is not differentiable with respect to $a$ at this point. Again using $\mathcal{F}^1(\mathcal{F}^{-b}\Lambda_H\mathcal{F}^b)\mathcal{F}^{-1}$, we can write a Taylor expansion for the kernel of the operator $\mathcal{F}^{-b}\Lambda_H\mathcal{F}^b$ around $b=1$ as in (22), but with $b$ in place of $a$:

$$L_b(u,u') \approx h(u-u') + i\frac{\pi^2}{2}(b-1)\,h(u-u')\,(u^2-u'^2). \tag{25}$$

Now, to obtain the expansion of $L_a(u,u')$ around $a=0$, we perform ordinary Fourier transformation from the right and inverse Fourier transformation from the left:

$$L_a(u,u') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1(u,u''')L_b(u''',u'')K_{-1}(u'',u')\,du''\,du''', \tag{26}$$

which, using standard Fourier transform properties, evaluates to

$$L_a(u,u') \approx H(u')\delta(u-u') + i\frac{a}{8}\Big(H(u)-H(u')\Big)\delta''(u-u'). \tag{27}$$

The input-output relationship corresponding to (27) is as follows:

$$g(u) = H(u)f(u) - i\frac{a}{8}\left(f(u)\frac{d^2H(u)}{du^2} + 2\frac{df(u)}{du}\frac{dH(u)}{du}\right). \tag{28}$$

If $H(u)$ is real-valued, the first multiplication term is real and the perturbation is purely imaginary, as it was in the $a \to 1$ case.

## 4. Examples and applications of LTV systems

### 4.1. Elementary systems

First we consider the case $h(u) = |\sin\alpha|\,\delta(\sin\alpha\, u)$, which leads to the identity operator with kernel $L_a(u,u') = \delta(u-u')$. This is expected since in this case $h(u)$ reduces to a constant filter function $H(u)$, and the FRTs in (5) cancel each other.

Next we consider $h(u) = |\sin\alpha|\,\delta(\sin\alpha\, u - \epsilon)$, which leads to

$$L_a(u,u') = \delta(u-u'-\epsilon)\exp(-i\pi\cot\alpha\,(\epsilon^2 + 2u'\epsilon)), \tag{29}$$

which means that this kernel will cause a coordinate shift and phase shift. We observe that a shift of the filter function leads to both a coordinate and phase shift at the output.

Now we consider $h(u) = |\sin\alpha|\,\mathrm{rect}(\sin\alpha\, u)$. The kernel now becomes,

$$L_a(u,u') = \exp(i\pi\cot\alpha\,(u'^2-u^2))\,\mathrm{rect}(u-u'). \tag{30}$$

Convolution with the rectangle causes a blur and the chirp term causes time varying phase changes.

We now turn our attention to the relationship between LTV systems and chirplet transforms, which are defined as follows [74]:

$$C_f(u,\mu,\Delta_u,s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sqrt{\pi}\,\Delta_u}}\exp(-i2\pi(s(u'-u)^2 + \mu(u'-u))) \times \exp(-\frac{1}{2}(\frac{u'}{\Delta_u})^2)f(u')\,du', \tag{31}$$

where $u$ and $\mu$ are time and frequency centers, $\Delta_u$ is the duration and $s$ is the chirprate of the chirplet. We now show how to express this transform in terms of LTVs. If we specify $f(u') \times \exp(-\frac{1}{2}(\frac{u'}{\Delta_u})^2)\exp(-i\pi\cot\alpha\, u'^2)$ as the input of an LTV system with filter function $h(u) = |\sin\alpha|\,\frac{1}{\sqrt{\sqrt{\pi}\Delta_u}}\exp(-i2\pi(s(\sin\alpha\, u)^2 - \mu(\sin\alpha\, u)))$, the output of the LTV system will be $g(u) = C_f(u,\mu,\Delta_u,s)\exp(-i\pi\cot\alpha\, u^2)$. In other words, chirplet transforms can be interpreted as LTV systems with pre and post chirp-multiplication.

In the next section, we discuss a class of systems which are exactly LTV systems.

### 4.2. Linearly sliding frequency filters

In this section, we focus on a specific type of system and prove that its members are exactly LTV systems. This particular system can be referred to as a *linearly sliding frequency filter*. This system is essentially a bandpass frequency filter whose center frequency linearly shifts with time. Such systems are encountered in real life. For example, if we rotate an analog radio tuner knob with constant angular velocity, we will have such a system.

We mathematically represent the linearly sliding filter with the following kernel:

$$p(u,u') = \exp(-i2\pi cu(u-u'))\,w(u-u'), \tag{32}$$

where $w(u)$ is the filter window and $c$ is a parameter. The exponential term provides the linear shift.

To understand the nature of this kernel, we first consider a linear time-invariant version of it for which $p(u,u') = p'(u-u')$. To obtain this we will assume the term $cu$ appearing in $p(u,u')$ to be a constant $\xi$, instead of linearly varying with $u$. In this case, the impulse response of the time-invariant kernel and the associated frequency response become

$$p'(u) = \exp(-i2\pi\xi u)\, w(u) \tag{33}$$

$$\Longrightarrow P'(\mu) = W(\mu + \xi). \tag{34}$$

Thus $P'(\mu)$ is nothing but a $\xi$ shifted version of $W(\mu)$. Now, if we recall that we had used $\xi$ to replace $cu$ and substitute this back, we obtain $W(\mu + cu)$, a filter whose center shifts linearly with time $u$. Obviously, as strictly defined, linear time-invariant systems cannot have impulse responses or frequency responses that depend on time. Thus the argument above is clearly not a legitimate derivation and is merely meant to be suggestive. It can be thought to be approximately true if $c$ is small and the change in center frequency is slow compared to other changes. In this case, it does become meaningful to speak of a frequency filter with linearly shifting center frequency.

Now we prove that the kernel given in (32) is an LTV system kernel. We emphasize that while the interpretation of this kernel as a frequency filter with linearly shifting center frequency was approximate and suggestive, the demonstration that this kernel is an LTV system is exact.

We will establish the proof by showing that the kernel $p(u, u')$ given in (32) can be made equal to the kernel of an LTV system $L_a(u, u')$ given in (19), both in magnitude and phase. We first equate the magnitudes of the two kernels:

$$|L_a(u, u')| = |p(u, u')| \tag{35}$$

$$\Longrightarrow \frac{1}{|\sin\alpha|}\, |h(\csc\alpha(u - u'))| = |w(u - u')| \tag{36}$$

$$\Longrightarrow |h(u)| = \big|\sin\alpha\, w(\sin\alpha\, u)\big|. \tag{37}$$

The final equation tells us what the inverse Fourier transform of LTV system's filter function $h(u)$ must be chosen in terms of the window function $w(u)$. Next, we equate the phases of the two kernels:

$$\angle(L_a(u, u')) = \angle(p(u, u')) \tag{38}$$

$$\Longrightarrow \angle h\big(\csc\alpha(u - u')\big) = \angle w(u - u') - \pi(u - u')\big((2c - \cot\alpha)u - \cot\alpha\, u'\big). \tag{39}$$

Notice that the left hand side is a function of $u - u'$ only. Therefore, equality requires that the right hand side also be so. This is possible if $c = \cot\alpha$. Using this,

$$\angle h\big(\csc\alpha(u - u')\big) = \angle w(u - u') - \pi\cot\alpha(u - u')^2 \tag{40}$$

$$\Longrightarrow \angle h(u) = \angle w\big(\sin\alpha\, u\big) - \pi\sin\alpha\cos\alpha\, u^2. \tag{41}$$

Thus, we conclude that, by choosing $c = \cot\alpha$ and

$$h(u) = |\sin\alpha|\, w(\sin\alpha\, u)\exp(-i\pi\sin\alpha\cos\alpha u^2), \tag{42}$$

we can express the given linearly sliding frequency filter kernel $p(u, u')$ in the form of an LTV kernel.

The order $a$ of the equivalent LTV system is given by $a = (2/\pi)\,\mathrm{arccot}(c)$ and its defining $h(u)$ is given by (42). The multiplicative filter to be applied in the $a$th order FRT domain is $H(u)$, which is the Fourier transform of $h(u)$.

For larger values of $c$, the rate at which the center frequency slides increases, and the equivalent LTV system has a lower FRT order $a$. For smaller values of $c$, the center frequency slides more slowly, and the LTV system has a higher order $a$. The limiting case of $c = 0$ corresponds to no sliding, and the system becomes a time-invariant (convolution) system. The FRT order associated with the equivalent LTV system is $a = 1$. On the other hand, as $c \to \infty$, the sliding rate increases without bound, and the system becomes a purely multiplicative system. The associated FRT order is $a = 0$.

Linear time-invariant systems can be expressed either as a convolution, or as a multiplicative filtering operation in the frequency domain. Since discrete Fourier transforms can be computed in $O(N\log N)$ time, it is possible to simulate time-invariant systems in $O(N\log N)$ time. In general, non-time-invariant systems cannot be computed in $O(N\log N)$ time and must be computed in $O(N^2)$ time. However, if we are able to show that a non-time-invariant system can be expressed or approximated as an LTV system, then they can also be simulated in $O(N\log N)$ time.

While the eigenfunctions of time-invariant systems are harmonic functions of constant frequency, the eigenfunctions of LTV systems are chirp functions, which have linearly changing frequency. This suggests that LTV systems might be useful when the system is not time-invariant or stationary, but has features that are linearly changing with time. However, it matters what feature is changing linearly. Above we saw that, if the center frequency of a band-pass filter is changing linearly, that indeed leads to an LTV system. On the other hand, if it was the width of the band-pass filter that was changing linearly, we would not arrive at an exact LTV system.

### *4.3. Approximating linear systems with LTV systems*

LTV systems can be useful when we have some general linear system $\mathcal{L}$ that we wish to implement. Ordinarily, discrete implementation of a general linear system takes the form of a matrix-vector product which requires $O(N^2)$ time. LTV systems may in some cases allow fast $O(N\log N)$ time computation of general linear systems. To this end, we try to approximate $\mathcal{L}$ with an LTV system of the form $\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a$, which we know can be computed in $O(N\log N)$ time. In other words, we choose $a$ and the diagonal operator $\Lambda_H$ so as to make $\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a$ as close as possible to $\mathcal{L}$, which amounts to approximately modeling $\mathcal{L}$ as an LTV system. To select $a$ and $\Lambda_H$ appropriately, we try to minimize the error measure $||\mathcal{L} - \mathcal{F}^{-a}\Lambda_H\mathcal{F}^a||$ over $\Lambda_H$ and the order $a$. Since the FRT is a unitary operation, this minimization is the same as minimizing $||\mathcal{F}^a\mathcal{L}\mathcal{F}^{-a} - \Lambda_H||$. Therefore, the optimal filter function turns out to be $\Lambda_H = \mathrm{diag}(\mathcal{F}^a\mathcal{L}\mathcal{F}^{-a})$ for a specific $a$. For each $a$, the error norm can be calculated after finding the corresponding optimal $\Lambda_H$. Following this, the LTV order $a$ producing the minimum error norm can be identified and employed for the approximation.

If the kernel of $\mathcal{L}$ is purely real, it can be approximated by the real part of an LTV system: $\mathrm{Re}\{\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a\}$. We will refer to systems of the form $\mathrm{Re}\{\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a\}$ as *real LTV systems*. If the filter function $\Lambda_H$ is real, real LTVs exhibit the following useful property:

$$\mathrm{Re}\{\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a\} = \mathrm{Re}\{\mathcal{F}^{a}\Lambda_H\mathcal{F}^{-a}\} \tag{43}$$

$$= \frac{1}{2}\big(\mathcal{F}^{-a}\Lambda_H\mathcal{F}^a + \mathcal{F}^{a}\Lambda_H\mathcal{F}^{-a}\big), \tag{44}$$

which implies that the real LTV can be expressed as half of the sum of an $a$th order and an $-a$th order LTVs (if the filter function is real). This equality holds because the complex conjugate of an $a$th order LTV system with a real filter function is the $-a$th order LTV system with the same filter function. The last statement can easily be obtained from (19).

LTV systems are also useful in restoring, recovering, or reconstructing signals that have been distorted by systems that are not time-invariant or stationary. Let us call the distorting system $\mathcal{L}_d$. Our recovery operator $\mathcal{L}_r$ will be an LTV system of the form $\mathcal{F}^{-a}\Lambda_{H_r}\mathcal{F}^a$. Ideally, we would want this operator to reverse the effect of $\mathcal{L}_d$; that is $\mathcal{F}^{-a}\Lambda_{H_r}\mathcal{F}^a\mathcal{L}_d = \mathcal{I}$. Thus we want $\mathcal{L}_d = \mathcal{F}^{-a}(\Lambda_{H_r})^{-1}\mathcal{F}^a$. If there exists a diagonal operator $\Lambda_{H_r}$ satisfying this equation, perfect recovery is possible. This would mean

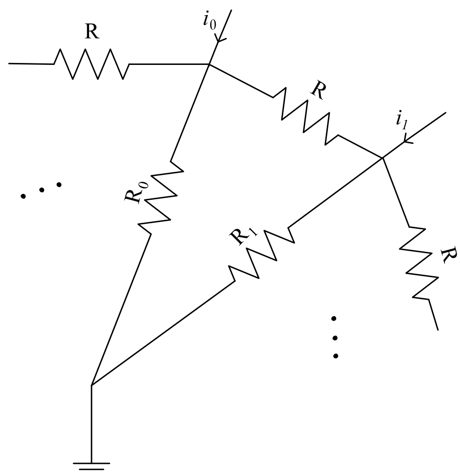


**Fig. 1.** A sector of the example circuit. Resistors between outer nodes are constant and have value $R$. The resistor between the outer node $k$ and the ground at the center is $R_k$. The values of $R_k$ are linearly changing.

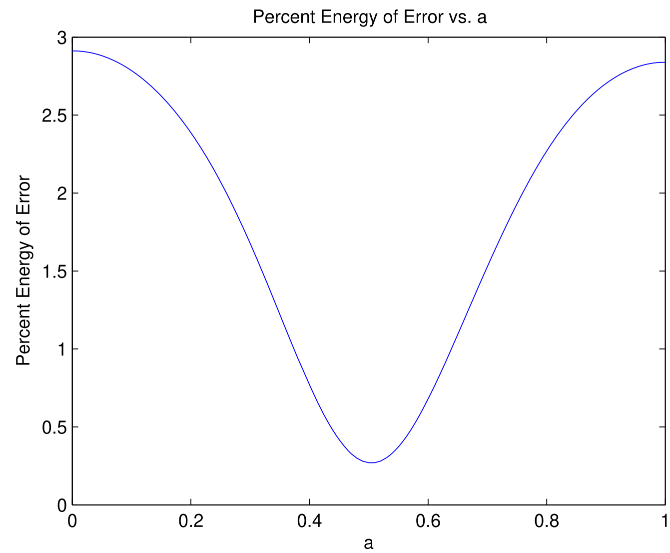


**Fig. 2.** The error energy as a function of order $a$. The minimum error is attained at $a = 0.50$.

that $\mathcal{L}_d$ itself was an LTV system. Thus we observe that if a distortion can be modeled as an LTV system, then the original signal can be completely recovered by another LTV system. However, if $\mathcal{L}_d$ cannot be expressed in the form $\mathcal{L}_d = \mathcal{F}^{-a}(\Lambda_{H_r})^{-1}\mathcal{F}^a$, then we might instead aim to approximate it with an LTV system. After finding the best approximation $\mathcal{F}^{-a}(\Lambda_{H_r})^{-1}\mathcal{F}^a$ for $\mathcal{L}_d$, we can achieve fast recovery by choosing the recovery filter associated with the LTV approximation of the original system. Alternatively, in some situations, the general linear optimal recovery operator $\mathcal{L}_r$ may have been obtained through one of several means and available to us. In this case, rather than implementing this in $O(N^2)$ time, we might choose to employ the LTV approximation procedure and implement the resulting system in $O(N \log N)$ time.

To provide an example, we consider 8 nodes on a circularly shaped circuit. We assume that each node is connected to its left and right neighbors with resistive elements of the same value. We further assume that each node is connected to the ground at the center of the circle, with resistive elements whose values change linearly from node to node. A section of the designated circuit is given in Fig. 1.

In this configuration, the input vector $\mathbf{i}$ consists of the currents $i_k$ that flow into the nodes and the output vector $\mathbf{v}$ consists of the voltages $v_k$ of each node relative to the ground at the center. The input output relationship of this system is trivial and can be computed as $\mathbf{v} = \mathbf{S}\mathbf{i}$, where $\mathbf{S}$ is the matrix formed by the Kirchhoff's laws. However, finding $\mathbf{v}$ from $\mathbf{i}$ by direct matrix multiplication requires $O(N^2)$ operations.

In our example, we choose the constant resistances between the outer nodes to be $R = 10\,\mathrm{k}\Omega$. The values of the resistances between the outer nodes and the ground node are $R_k = 2000$, 1750, 1500, 1250, 1000, 1250, 1500, 1750 $\Omega$ for $k \in \{1, \ldots, 8\}$, so that the values of these resistances are decreasing linearly from $2000\,\Omega$ to $1000\,\Omega$, and then increasing back from $1000\,\Omega$ to $2000\,\Omega$ as we go around the circle. Given this configuration, we approximate the $\mathbf{S}$ matrix with a real LTV system of the form $\mathrm{Re}\{\mathbf{F}^{-a}\boldsymbol{\Lambda}_{\mathbf{H}}\mathbf{F}^a\}$, where $\mathbf{F}^a$ is the $a$th order discrete FRT matrix [73] and $\boldsymbol{\Lambda}_{\mathbf{H}}$ is a diagonal matrix.

In this approximation, both the order $a$ and the diagonal matrix $\boldsymbol{\Lambda}_{\mathbf{H}}$ are free parameters to be optimized over with the goal of minimizing the Frobenius norm of the difference. This leads us to the following optimization problem for a given value of $a$:

$$\min \left\| \mathbf{S} - \mathrm{Re}\{\mathbf{F}^{-a}\boldsymbol{\Lambda}_{\mathbf{H}}\mathbf{F}^a\} \right\|, \tag{45}$$

subject to $\boldsymbol{\Lambda}_{\mathbf{H}}$ being a diagonal matrix. This problem is solved using YALMIP [75], which is a MATLAB toolbox for solving optimization problems. Once we find the optimal $\boldsymbol{\Lambda}_{\mathbf{H}}$ for each $a$, we calculate the resulting error energy and express it as a percentage of the energy of $\mathbf{S}$. The resulting error energy as a function of the order $a$ is given in Fig. 2.

The optimal order is $a = 0.50$. The error energy at $a = 0.50$ is about 0.27%, which amounts to a more than 10-fold improvement compared to approximating the same system in the ordinary time ($a = 0$) or Fourier ($a = 1$) domains, where the error energies are 2.91 and 2.84, respectively. The error versus $a$ function is not necessarily symmetric, although in this case it is nearly so. For different resistance values and different circuit structures, the optimal order could be other values between 0 and 1.

Having found the LTV system that best approximates the $\mathbf{S}$ matrix, we compare the performances of the LTV system and the original system for arbitrary inputs. The resulting approximate voltage outputs for these inputs are given in Fig. 3, along with the exact voltage outputs. The normalized error energy between the approximate and exact voltage outputs is also indicated at the top of each figure. In Fig. 3.a, the same current value of 10 mA is made to flow into each node located on the circle. In Fig. 3.b, uniformly distributed random current values between 5 mA and 15 mA flow into the nodes. Lastly, in Fig. 3.c, the values of currents flowing into these nodes increase linearly from 10 mA at node 1 to 20 mA at node 8.

In all of these examples, the LTV system serves as a very good approximation of the original system represented by $\mathbf{S}$. Thus without losing much in terms of accuracy, we are able to compute the output in $O(N \log N)$ time instead of the usual $O(N^2)$.

## 5. Conclusions

In this paper we consider a class of systems we refer to as linearly time-varying (LTV) systems, and define them in two different ways: The first definition focuses on the eigenfunctions of LTV systems, stating that an $a$th order LTV system has chirp-type eigenfunctions given by $\mathrm{chirp}_{-a,\zeta}(u)$. Our second definition states that $a$th order LTV systems correspond to multiplicative filtering in the $a$th order fractional Fourier domain. We demonstrate the equivalence of the two definitions. We also show that LTV systems interpolate between multiplicative systems and convolutive systems, as $a$ changes between 0 and 1.

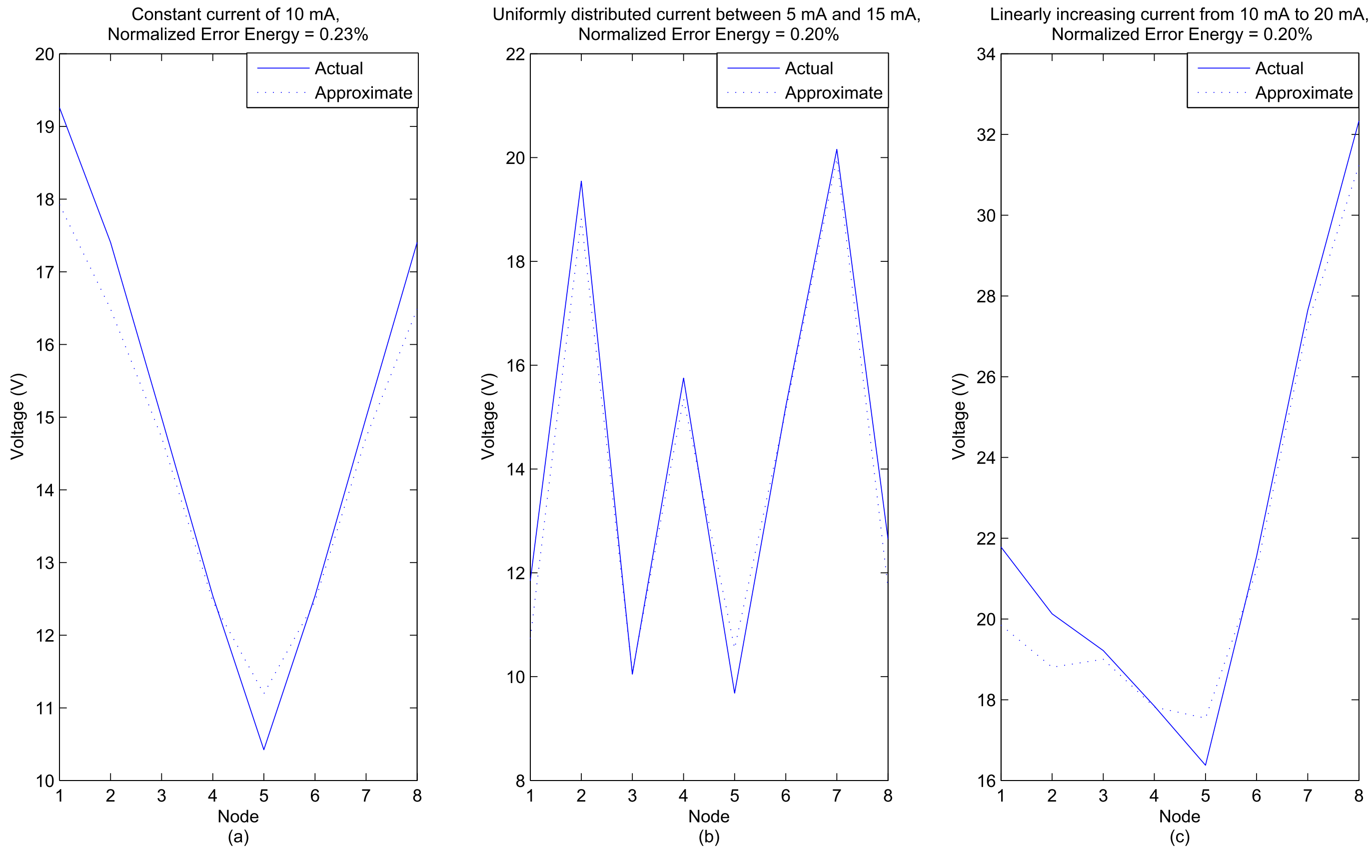


**Fig. 3.** Actual and approximate voltage distributions for three different current distributions.

In addition to some elementary systems, we discussed a linearly sliding window filter, and noted that it corresponds to an LTV.

The second definition of LTV systems shows how to compute them with a fast $O(N \log N)$ algorithm, despite the fact that they are not time-invariant systems. Being able to implement LTV systems with a $O(N \log N)$ algorithm can be beneficial in a number of ways. If we can recognize a system to be an LTV system, we can compute it in $O(N \log N)$. When this is not the case, we can still try to approximate the linear system at hand with an LTV system. If an acceptable approximation is found, we can compute the approximation to this system in $O(N \log N)$ time, leading to considerable computational savings.

In the process of approximating a given linear system as an LTV system, we should try to choose both the fractional order $a$ and the filter function optimally in order to make the approximation as close as possible. In general, use of the optimal order $a$ will give better results than use of $a = 0$ or $a = 1$, which correspond to the ordinary time and frequency domains respectively. So, we are able to improve performance with respect to time and frequency-domain filtering, without giving up the $O(N \log N)$ fast computation advantage we are accustomed to with time and frequency-domain filtering. We provided a numerical example which illustrates these points, showing more than a 10-fold improvement in error.

Other properties and applications of LTV systems remain to be investigated. The concept of real LTVs also seems to deserve further study.

It may also be of interest to contrast the present work with [76, 77], where a completely different approach is taken to deal with the problem of correcting *weakly* time-varying systems.

## CRediT authorship contribution statement

Utkan Candogan: Methodology, Software, Validation, Formal analysis, Investigation, Writing.
Ozan Candogan: Methodology, Software, Validation, Formal analysis, Investigation.
Haldun M. Ozaktas: Conceptualization, Methodology, Formal analysis, Writing, Supervision, Funding acquisition.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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