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Relationships between two definitions of the discrete Wigner distribution and the continuous Wigner distribution

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ABSTRACT

We present a very simple relationship between two widely used discrete-time discrete-frequency Wigner distributions. The first one is obtained through sampling and the second one is obtained from the representation theory of the finite Heisenberg group. This relation shows that the values of one can simply be obtained by permuting the values of the other along the frequency axis, which in turn implies a relationship of the second definition to the samples of the continuous Wigner distribution, and the first definition to group representation theory. In the process, we derive a simplified form for the second definition which is completely analogous to the continuous Wigner distribution, and develop a set of relationships relating this definition to a discrete ambiguity function and auxiliary functions.

1. Introduction

The Wigner distribution (WD) of a function $f(u)$, defined by

$$W_f(u,v) = \int_{-\infty}^{\infty} f(u+\bar{u}/2) f^*(u-\bar{u}/2) e^{-j2\pi\bar{u}v} d\bar{u}, \tag{1}$$

is one of the most studied time-frequency representations [1–5] and has found many applications in signal processing and other areas [6–9]. It is closely related to the fractional Fourier transform [10–12] and linear canonical transforms [13–15]. The discrete Wigner distribution is likewise of great importance [16–21] and is similarly related to the discrete fractional Fourier transform [22] and discrete linear canonical transforms [23,24].

The definition of the Discrete Fourier Transform (DFT) is standardized and widely accepted (save for the factor of $1/N$ versus $1/\sqrt{N}$ which determines whether it is unitary or not). Apart from being highly elegant in its construction, the Wigner distribution satisfies a surprisingly large number of analytical properties and has found wide application. It has a special place among all time-frequency representations and is indispensable in mathematics, physics, and signal analysis and processing. The establishment and standardization of the discrete Wigner distribution would contribute substantially to the consolidation of signal theory more broadly. Unfortunately, despite many definitions having been proposed, none have emerged as the main standard definition, nor has it been possible to establish all the promising connections with fractional Fourier transforms and linear canonical transforms that are well established in the continuous case. The purpose of this paper is to bring us closer to the establishment of a widely accepted definition by demonstrating what we believe are previously unnoticed connections among two discrete definitions and the continuous distribution.

A widely accepted definition for the discrete-time discrete-frequency Wigner distribution of a discrete-time signal $f(n)$ with time extent M, here referred to as WD$^{\mathrm{s}}$, is given by [1,18],

$$W_f^{\mathrm{s}}(n,k) = \sum_{\bar{n}=0}^{M-1} f(n+\bar{n}) f^*(n-\bar{n}) e^{-j2\pi k\bar{n}/M}, \tag{2}$$

where the shifts in the above definition are linear. If the signal is zero padded properly to N and periodically replicated so that the linear shifts can be replaced by circular shifts without overlap, then this definition takes the following form:

$$W_f^{\mathrm{s}}[n,k] = \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}] f^*[n-\bar{n}] e^{-j2\pi k\bar{n}/N}, \tag{3}$$

which is periodic in both n and k with period N. As can be seen, this definition is not strictly analogous to the continuous definition (1) in the sense that the continuous Fourier transform (FT) is analogous to the discrete Fourier transform (DFT), because of the absence of the factors $1/2$.

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This definition suffers from aliasing and thus provides an approximation of the continuous WD over only half of the spectrum [1,18].

A distinct definition for the discrete WD, here referred to as WD$^{\mathrm{m}}$, is given by

$$W_f^{\mathrm{m}}[n,k] = \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]\, e^{-j2\pi k\bar{n}/N}, \tag{4}$$

where $\bar{n}2^{-1}$ is evaluated modulo N. This definition originates from mathematical studies on the discrete Weyl correspondence [25,26] and has also been extensively studied in the physics literature [27–31] and the signal processing literature [32,16,17,33]. However, it should be noted that this definition of the discrete WD does not appear in these papers in the form of Equation (4), but rather in a different form which will be given below as Equation (5). In this paper we show how to obtain Equation (4) from Equation (5). The form of Equation (4) is advantageous in that it allows direct comparison with the continuous definition (1) without referring to any group theoretical concepts. This definition exhibits a high degree of structural analogy to the continuous WD; not only is Equation (4) fully analogous to Equation (1), but this definition of the WD satisfies many of the operational properties of the continuous WD [17,30,31]. However, it has been generally noted that it does not provide an approximation to the samples of the continuous WD, and therefore is apparently not suited to serve as a discrete WD for most signal processing purposes. We also note that the superscript “s” is chosen because WD$^{\mathrm{s}}$ is related to sampling, and the superscript “m” is chosen because WD$^{\mathrm{m}}$ arises from purer mathematical considerations.

In this paper, we show that for the case of odd N there is a very simple relationship between these two definitions of the discrete WD, and relate WD$^{\mathrm{m}}$ to the samples of the continuous WD [21]. Although both of the definitions discussed here have been widely studied and used in the literature, to the best of our knowledge, the relationship between them was not observed. In the process, we develop a set of relationships relating this definition to a discrete ambiguity function and auxiliary functions. We also connect WD$^{\mathrm{s}}$ to group representation theory.

2. Some relationships for WD$^{\mathrm{m}}$ and auxiliary functions

The usual way of obtaining WD$^{\mathrm{m}}$ is from group representation theory, where this discrete WD is defined as (for odd N) [16]:

$$W_f^{\mathrm{m}}[n,k] = \frac{1}{N} \sum_{\bar{n},\bar{k}=0}^{N-1} \langle \rho[\bar{n},\bar{k}]f, f\rangle\, e^{-j2\pi(n\bar{k}+k\bar{n})/N}, \tag{5}$$

where

$$\rho[\bar{n},\bar{k}]f[n] = e^{j2\pi(2^{-1}\bar{n}\bar{k})/N} e^{j2\pi\bar{k}n/N} f[n+\bar{n}], \tag{6}$$

and $2^{-1} = (N+1)/2$ in modulo N. (2^{-1}, when multiplied by 2, should give unity. Indeed $2\times(N+1)/2 = N+1$, which in modulo N is unity [17].) The inner product in the above expression is defined as

$$\langle \rho[\bar{n},\bar{k}]f, f\rangle = \sum_{n=0}^{N-1} e^{j2\pi(2^{-1}\bar{n}\bar{k})/N} e^{j2\pi\bar{k}n/N} f[n+\bar{n}]f^*[n]. \tag{7}$$

This inner product essentially corresponds to the ambiguity function ($A_f[\bar{n},\bar{k}] = \langle \rho[\bar{n},-\bar{k}]f, f\rangle$). We now show that Equation (5) can be reduced to the form (4) which is more analogous to the continuous WD (1). First note that the above inner product can be further simplified by a change of variables $n \to n-\bar{n}2^{-1}$ as follows:

$$\begin{aligned}\langle \rho[\bar{n},\bar{k}]f, f\rangle &= \sum_{n=0}^{N-1} e^{j2\pi(2^{-1}\bar{n}\bar{k})/N} e^{j2\pi\bar{k}(n-\bar{n}2^{-1})/N} \\ &\quad\times f[n-\bar{n}2^{-1}+\bar{n}]f^*[n-\bar{n}2^{-1}] \\ &= \sum_{n=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{j2\pi\bar{k}n/N}.\end{aligned} \tag{8}$$

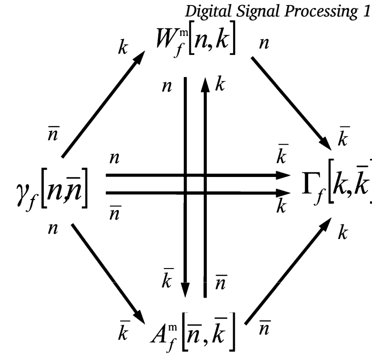

Fig. 1. The graphical representation of the relationships between W_f^{m}, A_f^{m} and the auxiliary functions. The arrows indicate DFTs.

In passing from the first line of the above equation to the second we used

$$n - 2^{-1}\bar{n} + \bar{n} = n + 2^{-1}\bar{n}. \tag{9}$$

To see why this is true, using $2^{-1} = (N+1)/2$ in modulo N, the left hand side of this equation can be shown to equal, in modulo N:

$$\begin{aligned}&= n + (-2^{-1}+1)\bar{n} \\ &= n + (-((N+1)/2)+1)\bar{n} \\ &= n + (-((N+1)/2)+1+N)\bar{n} \\ &= n + ((N+1)/2)\bar{n} \\ &= n + 2^{-1}\bar{n}\end{aligned} \tag{10}$$

which is the right side of Equation (9).

Now, let us define the two discrete auxiliary functions γ_f and Γ_f as follows:

$$\gamma_f[n,\bar{n}] = f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}], \tag{11}$$

$$\Gamma_f[k,\bar{k}] = F[k+\bar{k}2^{-1}]F^*[k-\bar{k}2^{-1}] \tag{12}$$

where F denotes the DFT. Note that our auxiliary functions are defined on rectangular grids, as opposed to those of [33] which proposes discrete auxiliary functions on hexagonal sampling grids. By combining Equations (8) and (5) a simple expression for WD$^{\mathrm{m}}$ can be obtained as:

$$\begin{aligned}W_f^{\mathrm{m}}[n,k] &= \frac{1}{N} \sum_{\bar{n},\bar{k},n'=0}^{N-1} \gamma_f[n',\bar{n}]e^{j2\pi\bar{k}n'/N} e^{-j2\pi(n\bar{k}+k\bar{n})/N} \\ &= \sum_{\bar{n},n'=0}^{N-1} \gamma_f[n',\bar{n}]e^{-j2\pi k\bar{n}/N}\,\delta[n-n'] \\ &= \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{-j2\pi k\bar{n}/N},\end{aligned} \tag{13}$$

which is the same as (4). Equations (8), (13) and a corresponding pair of equations for Γ_f which can be similarly derived, can be summarized in graphical form (Fig. 1) which is familiar from the continuous case [34, 35]. This constitutes further support for the strong structural analogy of this definition to the continuous case. Essentially equivalent results for the WD$^{\mathrm{m}}$ and auxiliary functions can also be deduced from [29].

3. Relationship between the two discrete WDs

Having shown that the definition of WD$^{\mathrm{m}}$, given by (5), can be written in the form of (4), it is now easy to show how WD$^{\mathrm{s}}$ and WD$^{\mathrm{m}}$ are related to each other:

$$W_f^{\mathrm{m}}[n,k] = \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{-j2\pi k\bar{n}/N}$$

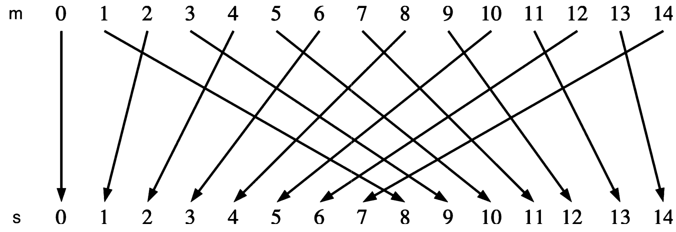

Fig. 2. The permutation of the values of $\mathrm{WD^m}$ and $\mathrm{WD^s}$ along the k axis for $N = 15$.

$$= \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}] f^*[n-\bar{n}] e^{-j2\pi(2\bar{n}k)/N}$$
$$= W_f^{\mathrm{s}}[n, 2k], \tag{14}$$

or equivalently

$$W_f^{\mathrm{s}}[n,k] = W_f^{\mathrm{m}}[n, 2^{-1}k], \tag{15}$$

where $2^{-1}k$ is again computed modulo N. We used the substitution $\bar{n} \to 2\bar{n}$ in passing to the second line of (14). This remarkably simple relationship means that the values of either of these WDs are obtained simply by rearranging (permuting) the values of the other along the frequency axis (Fig. 2). It is interesting to note that the resulting permutation is in the form of a perfect shuffle and is also related to decimation in frequency. This relationship also means that if we know the WD according to one of these definitions, we can quickly compute WD according to the other definition by simply rearranging the values.

As already noted, $\mathrm{WD^m}$ is usually considered to bear no relation to the continuous WD, in the sense that its values do not approximate the samples of the continuous WD. We have shown that $\mathrm{WD^m}$ is actually closely related to the continuous WD in that its values are mere permutations of a discrete WD which does approximate the samples of the continuous WD (at least over half the band if we work at the Nyquist rate).

To the best of our knowledge, $\mathrm{WD^s}$ has not been defined in group theoretical terms. However, its close relationship to $\mathrm{WD^m}$ implies that it too should possess some group theoretical structure. First, we note that $\rho[n,k,\tau] = \rho[n,k]e^{j2\pi\tau/N}$ gives a representation of the Heisenberg group and the operator $\rho[n,k]$ satisfies the following concatenation rule [28]:

$$\rho[n,k]\rho[n',k'] = e^{j2\pi[(nk'-kn')2^{-1}]/N}\rho[n+n', k+k'], \tag{16}$$

which can be shown from (6). Now, let us rewrite (3) by using (5) and (15):

$$W_f^{\mathrm{s}}[n,k] = W_f^{\mathrm{m}}[n,2^{-1}k] = \frac{1}{N}\sum_{\bar{n},\bar{k}=0}^{N-1} \langle \rho[\bar{n},\bar{k}]f, f\rangle e^{-j2\pi(n\bar{k}+2^{-1}k\bar{n})/N}$$
$$= \frac{1}{N}\sum_{\bar{n},\bar{k}=0}^{N-1} \langle \rho[2\bar{n},\bar{k}]f, f\rangle e^{-j2\pi(n\bar{k}+k\bar{n})/N}. \tag{17}$$

If we define

$$\varrho[\bar{n},\bar{k}] = \rho[2\bar{n},\bar{k}], \tag{18}$$

it can be shown that the operator $\varrho[\bar{n},\bar{k}]$ satisfies

$$\varrho[n,k]\varrho[n',k'] = e^{j2\pi(nk'-kn')/N}\varrho[n+n',k+k']. \tag{19}$$

This means that, $\varrho[n,k]$ also provides a projective unitary representation of the group of time and frequency shifts in finite phase space.

However, since the map between the two groups given by Equation (18) is not symplectic, it follows that $\varrho[n,k]$ is not an automorphism of the finite Heisenberg group [36].

4. Conclusion

In conclusion, we established a very simple connection between two different definitions of the discrete Wigner distribution, one widely studied in finite quantum mechanics and to a more limited extent in signal processing ($\mathrm{WD^m}$); and the other widely used in signal processing, especially for numerical and graphical purposes ($\mathrm{WD^s}$). In the process, we derived an expression for $\mathrm{WD^m}$ which is directly analogous to the continuous WD expression (1). We also showed how this definition can be related to the samples of the continuous WD, a result which seems to have escaped previous researchers. This is of considerable interest because this definition exhibits a high degree of structural similarity to the continuous definition in terms of the operational properties it satisfies. A number of other related results including the relation of $\mathrm{WD^m}$ to discrete ambiguity function and auxiliary functions were also presented. Our results were limited to the case of odd N, which is a common restriction in most work dealing with discrete Wigner distributions. It is an inconvenient reality in dealing with these types of entities that even and odd values of N often lead to very different behavior, with odd N usually leading to more desirable properties. One way to explain the difference is that 2^{-1} exists for odd N and not for even N [17]. (Recall that $2^{-1} = (N+1)/2$ in modulo N for odd N.) Number theoretical and topological reasons have also been discussed [29]. We refer the reader to the literature for more detailed discussion [17,29,37,38].

We finally note that the discussion of redundancy inherent in $\mathrm{WD^s}$ studied in [18] can be easily applied to $\mathrm{WD^m}$.

CRediT authorship contribution statement

Sayit Korkmaz: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Haldun M. Ozaktas:** Conceptualization, Formal analysis, Funding acquisition, Methodology, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Haldun M. Ozaktas reports financial support was provided by Turkish Academy of Sciences. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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