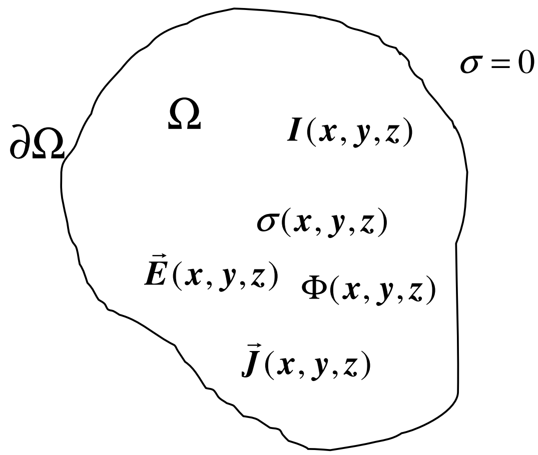
24-10-2007

VOLUME CONDUCTOR FIELDS

Consider a volume conductor defined by the closed and connected subdomain Ω of $\boldsymbol{R}^3$ with boundary $\partial\Omega$ and nonuniform conductivity distribution $\sigma(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$. (Figure below illustrates the problem in 2D however). In Ω there is a distributed volume current source $\boldsymbol{I}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ (units: A/cm^3). Note that integral of $\boldsymbol{I}$ over Ω must be zero; otherwise charge accumulates in Ω. Define $\vec{\boldsymbol{J}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ as the current density distribution in Ω (units: A/cm^2). Note that $\vec{\boldsymbol{J}}$ is a vector but $\boldsymbol{I}$ is a scalar.



The divergence of $\vec{\boldsymbol{J}}$ must be equal to $\boldsymbol{I}$, $\nabla\cdot\vec{\boldsymbol{J}} = \boldsymbol{I}$.

We also know from Ohm's Law that the electric field $\vec{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is related to $\vec{\boldsymbol{J}}$, by $\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$.

On the other hand the potential field $\Phi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is related by definition to the electric field by $\vec{\boldsymbol{E}} = -\nabla\Phi$.

Therefore $\nabla\cdot\vec{\boldsymbol{J}} = \nabla\cdot(\sigma\vec{\boldsymbol{E}}) = -\nabla\cdot(\sigma\nabla\Phi) = \boldsymbol{I}$.

Let us now revert to a special case where $\sigma(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \sigma$, that is σ is uniform in Ω.

Then, $\nabla\cdot(\sigma\nabla\Phi) = \sigma\nabla^2\Phi$ and

$\sigma\nabla^2\Phi = -\boldsymbol{I}$ in Ω

with the boundary condition

$\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{n}} = 0$ on $\partial\Omega$ where $\vec{\boldsymbol{n}}$ is the outward vector on $\partial\Omega$ normal to the boundary. This condition comes from the fact that current can only flow parallel to $\partial\Omega$ on $\partial\Omega$ because outside conductivity is zero. The boundary condition can be expressed in terms of Φ using $\vec{\boldsymbol{E}} = -\nabla\Phi$ and $\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$, as $-\sigma\nabla\Phi\cdot\vec{\boldsymbol{n}} = 0$ or as $\nabla\Phi\cdot\vec{\boldsymbol{n}} = 0$ or as

$\dfrac{\partial\Phi}{\partial\boldsymbol{n}} = 0$ on $\partial\Omega$ where $\dfrac{\partial}{\partial\boldsymbol{n}}$ operator means the derivative along the direction of the normal vector on $\partial\Omega$.

In summary we have

$\nabla^2\Phi = -\dfrac{\boldsymbol{I}}{\sigma}$ in Ω and

$$\frac{\partial \Phi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega .$$

We can solve this Partial differential equation for Φ and then obtain $\vec{\boldsymbol{E}} = -\nabla\Phi$, and $\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$.

If $\Omega = \boldsymbol{R}^3$ then the problem is simplified if we also assume that $\boldsymbol{I}$ is confined to a finite volume:

$$\nabla^2\Phi = -\frac{\boldsymbol{I}}{\sigma} \text{ in } \boldsymbol{R}^3 \text{ and}$$

$$\Phi(\infty) = 0 .$$

In such a case the solution is

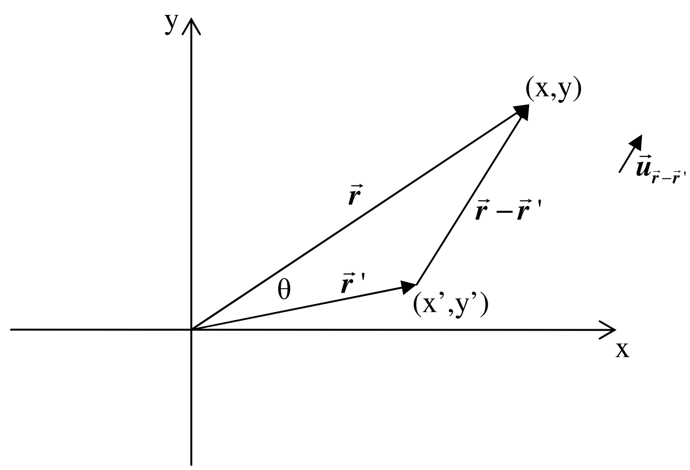
$$\Phi(\boldsymbol{x},\boldsymbol{y},z) = \frac{1}{4\Pi\sigma}\int_{\boldsymbol{R}^3}\frac{\boldsymbol{I}(\boldsymbol{x}',\boldsymbol{y}',z')}{\sqrt{(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(z-z')^2}}\boldsymbol{dx}'\boldsymbol{dy}'\boldsymbol{dz}' .$$

From this we can find the electric field by $\vec{\boldsymbol{E}} = -\nabla\Phi$ as

$$\begin{aligned}\vec{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{y},z) = \vec{\boldsymbol{u}}_x &\frac{1}{4\Pi\sigma}\int_{\boldsymbol{R}^3}\frac{\boldsymbol{I}(\boldsymbol{x}',\boldsymbol{y}',z')(\boldsymbol{x}-\boldsymbol{x}')}{[(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(z-z')^2]^{\frac{3}{2}}}\boldsymbol{dx}'\boldsymbol{dy}'\boldsymbol{dz}' \\ +\vec{\boldsymbol{u}}_y &\frac{1}{4\Pi\sigma}\int_{\boldsymbol{R}^3}\frac{\boldsymbol{I}(\boldsymbol{x}',\boldsymbol{y}',z')(\boldsymbol{y}-\boldsymbol{y}')}{[(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(z-z')^2]^{\frac{3}{2}}}\boldsymbol{dx}'\boldsymbol{dy}'\boldsymbol{dz}' \\ +\vec{\boldsymbol{u}}_z &\frac{1}{4\Pi\sigma}\int_{\boldsymbol{R}^3}\frac{\boldsymbol{I}(\boldsymbol{x}',\boldsymbol{y}',z')(z-z')}{[(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(z-z')^2]^{\frac{3}{2}}}\boldsymbol{dx}'\boldsymbol{dy}'\boldsymbol{dz}'\end{aligned}$$

where $\vec{\boldsymbol{u}}_x, \vec{\boldsymbol{u}}_y, \vec{\boldsymbol{u}}_z$ are unit vectors along x, y, z directions respectively.

To simplify the expressions let us define $\vec{\boldsymbol{r}} = \boldsymbol{x}\vec{\boldsymbol{u}}_x + \boldsymbol{y}\vec{\boldsymbol{u}}_y + z\vec{\boldsymbol{u}}_z$ as the field vector and $\vec{\boldsymbol{r}}' = \boldsymbol{x}'\vec{\boldsymbol{u}}_x + \boldsymbol{y}'\vec{\boldsymbol{u}}_y + z'\vec{\boldsymbol{u}}_z$ as the source vector, and their respective magnitudes as $\boldsymbol{r} = |\vec{\boldsymbol{r}}| = \sqrt{\boldsymbol{x}^2+\boldsymbol{y}^2+z^2}$ and $\boldsymbol{r}' = |\vec{\boldsymbol{r}}'| = \sqrt{\boldsymbol{x}'^2+\boldsymbol{y}'^2+z'^2}$.



Then, $\Phi(\vec{r}) = \frac{1}{4\Pi\sigma}\int_{R^3}\frac{I(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dr'$ where $dr' = dx'dy'dz'$ by definition,

and

$$\vec{E}(\vec{r}) = -\nabla\left(\frac{1}{4\Pi\sigma}\int_{R^3}\frac{I(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dr'\right) = -\frac{1}{4\Pi\sigma}\int_{R^3} I(\vec{r}\,')\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dr'.$$

(Note that the del operator operates on the x,y,z coordinates: $\nabla = \frac{\partial}{\partial x}\vec{u}_x + \frac{\partial}{\partial y}\vec{u}_y + \frac{\partial}{\partial z}\vec{u}_z$)

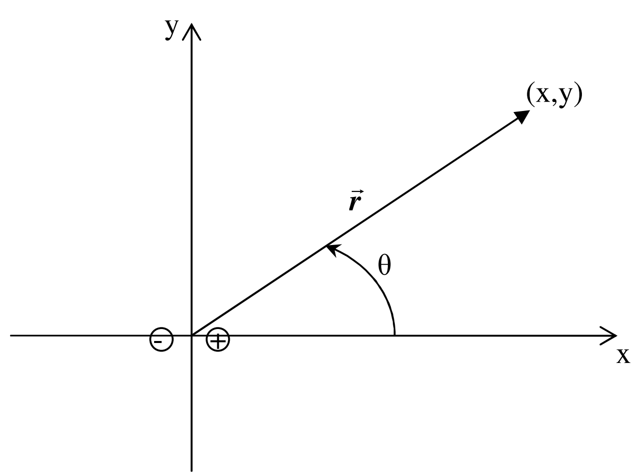
Since $\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2}$ where $\vec{u}_{\vec{r}-\vec{r}\,'}$ is the unit vector along the vector $\vec{r}-\vec{r}\,'$, that is it is the unit vector from $\vec{r}\,'$ to $\vec{r}$,

$$\vec{E}(\vec{r}) = \frac{1}{4\Pi\sigma}\int_{R^3}\frac{I(\vec{r}\,')\vec{u}_{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2}dr'.$$

Field of dipole sources:

Often the volume current sources can be expressed as dipole current source density per unit volume and the field is found from this source.

First let us consider a dipole current source located at the origin as shown below as an example. We assume $\Omega = R^3$. We have a monopole source of strength I at $x = \frac{d}{2}$ and a negative source of strength $-I$ at $x = -\frac{d}{2}$. In such a case we say that we have a dipole current source $\vec{P} = P\vec{u}_x$ *where* $P = Id$. Let us find its potential field.

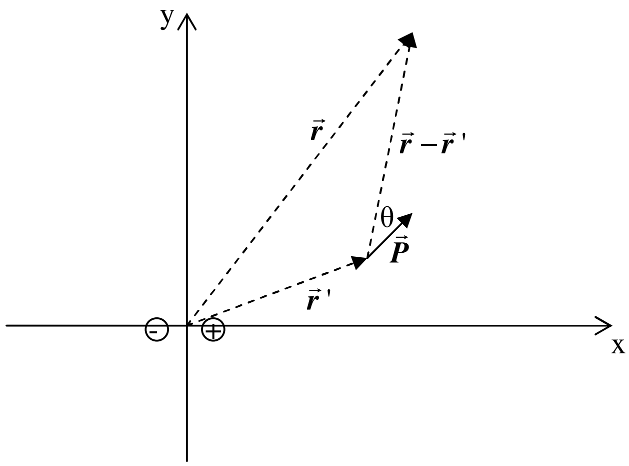


$$\phi(\vec{\mathbf{r}}) = \frac{I}{4\Pi\sigma}\left[\frac{1}{((x-\frac{d}{2})^2+y^2+z^2)^{\frac{1}{2}}} - \frac{1}{((x+\frac{d}{2})^2+y^2+z^2)^{\frac{1}{2}}}\right]$$

If d is small enough then

$$\phi(\vec{\mathbf{r}}) \approx \frac{I}{4\Pi\sigma}\frac{x}{r^3}d = \frac{I}{4\Pi\sigma}\frac{\vec{r}\cdot\vec{u}_x}{r^3}d = \frac{I}{4\Pi\sigma}\frac{\vec{u}_{\vec{r}}\cdot\vec{u}_x}{r^2}d = \frac{1}{4\Pi\sigma}\vec{P}\cdot\frac{\vec{u}_{\vec{r}}}{r^2}$$

If we generalize the problem to a general dipole source $\vec{P}$ at $\vec{r}\,'$, then the field becomes $\phi(\vec{\mathbf{r}}) = \frac{1}{4\Pi\sigma}\vec{P}\cdot\frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{\left|\vec{r}-\vec{r}\,'\right|^2}$



The field can also be written in terms of θ which is the angle between $\vec{r}-\vec{r}\,'$ and $\vec{P}$, and also the distance from source point to the field point, $R = \left|\vec{r}-\vec{r}\,'\right|$ as

$$\phi(\vec{\mathbf{r}}) = \frac{1}{4\Pi\sigma}\frac{P\cos\theta}{R^2}.$$

Now if we have a distributed source, that is, dipole source density per unit volume $\vec{p}(\vec{r}\,')$ confined to a volume V, then

$$\phi(\vec{\mathbf{r}}) = \frac{1}{4\Pi\sigma}\int_V \vec{p}(\vec{r}\,')\cdot\frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{\left|\vec{r}-\vec{r}\,'\right|^2}dr'.$$

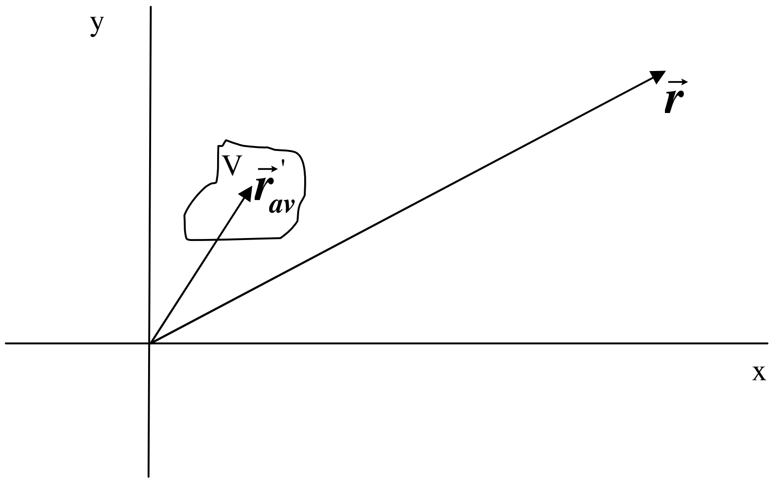
Note that $\vec{p}(\vec{r}\,')$ has units per volume and $\vec{p}(\vec{r}\,')dr'$ is the total dipole strength in the volume dr'.

Since $\nabla'\left(\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\right) = \frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{\left|\vec{r}-\vec{r}\,'\right|^2}$ where $\nabla' = \frac{\partial}{\partial \boldsymbol{x}'}\vec{\boldsymbol{u}}_x + \frac{\partial}{\partial \boldsymbol{y}'}\vec{\boldsymbol{u}}_y + \frac{\partial}{\partial \boldsymbol{z}'}\vec{\boldsymbol{u}}_z$, we can also express the field as

$$\phi(\vec{\mathbf{r}}) = \frac{1}{4\Pi\sigma}\int_V \vec{p}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\right)dr'.$$

Equivalent single dipole for a distributed dipole source density distribution:

Consider a case where the source is localized in V and the field point is far from the source as shown below:



Then

$$\frac{1}{|\vec{r}-\vec{r}'|} \approx \frac{1}{|\vec{r}-\vec{r}_{av}'|}$$

and does not depend on $\vec{r}'$, where $\vec{r}_{av}'$ is the average source location, and

$$\phi(\vec{\mathbf{r}}) = \frac{1}{4\Pi\sigma}\int_V \vec{p}(\vec{r}')\cdot\frac{\vec{u}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}dr' \approx \frac{1}{4\Pi\sigma}\int_V \vec{p}(\vec{r}')\cdot\frac{\vec{u}_{\vec{r}-\vec{r}_{av}'}}{|\vec{r}-\vec{r}_{av}'|^2}dr'$$

$$= \frac{1}{4\Pi\sigma}\frac{\vec{u}_{\vec{r}-\vec{r}_{av}'}}{|\vec{r}-\vec{r}_{av}'|^2}\cdot\int_V \vec{p}(\vec{r}')dr'$$

$$= \frac{1}{4\Pi\sigma}\vec{P}\cdot\frac{\vec{u}_{\vec{r}-\vec{r}_{av}'}}{|\vec{r}-\vec{r}_{av}'|^2}$$

where

$$\vec{\boldsymbol{P}} = \int_V \vec{\boldsymbol{p}}(\vec{\boldsymbol{r}}')d\boldsymbol{r}'$$

represents a single equivalent dipole.