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UNIVERSAL SEMICONSTANT REBALANCED PORTFOLIOS

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In this paper, we investigate investment strategies that can rebalance their target portfolio vectors at arbitrary investment periods. These strategies are called semiconstant rebalanced portfolios in Blum and Kalai and Helmbold et al. Unlike a constant rebalanced portfolio, which must rebalance at every investment interval, a semiconstant rebalanced portfolio rebalances its portfolio only on selected instants. Hence, a semiconstant rebalanced portfolio may avoid rebalancing if the transaction costs outweigh the benefits of rebalancing. In a competitive algorithm framework, we compete against all such semiconstant portfolios with an arbitrary number of rebalancings and corresponding rebalancing instants. We investigate this framework with and without transaction costs and demonstrate sequential portfolios that asymptotically achieve the wealth of the best semiconstant rebalanced portfolios whose number of rebalancings and instants of rebalancings are tuned to the individual sequence of price relatives.

KEY WORDS: portfolio, competitive, rebalancing times, universal.

1. INTRODUCTION

We address a sequential investment problem, by considering both how to invest as well as the best out of a large class of possible strategies and considering the practical issue of how and when to rebalance a given portfolio in the presence of transaction fees. Sequential portfolio investment strategies have been investigated in the information theory by Cover and Ordentlich (1996, 1998), Ordentlich and Cover (1998), and Cover (2004), in computational learning theory by Helmbold et al. (1998), Stoltz and Lugosi (2005), Vovk and Watkins (1998), Borodin, El-Yaniv, and Govan (2004), and recently in signal processing research literature by Kozat and Singer (2007). A problem extensively studied in this framework is to find sequential portfolios that asymptotically achieve the wealth of the best constant rebalanced portfolio (CRP) tuned to the individual sequence of price relative vectors. This amounts to finding a daily trading strategy that has the ability, without cheating, to do as well as the best asset diversified, constantly rebalanced portfolio. Note that such a portfolio could only be computed by an investor who has access to the entire sequence of daily outcomes in advance, that is, such an investor would need to know, before the first investment, exactly what the market was going to do over the duration of the market, a clearly impossible feat. A CRP is an investment strategy that keeps the same proportion of wealth among a set of stocks

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from one investment period to another. It has been shown that under mild stochastic assumptions on the sequence of price relatives, the portfolio that achieves the maximum wealth is a CRP (although, we make no such stochastic assumptions in this paper) in Cover and Ordentlich (1996) and Cover and Thomas (1991). Here, the market is modeled by a sequence of price relative vectors, $\boldsymbol{x}^n = \boldsymbol{x}[1], \ldots, \boldsymbol{x}[n]$, $\boldsymbol{x}[t] \in \mathbb{R}_+^m$, where $\mathbb{R}_+^m$ is the positive orthant. The jth entry $x_j[t]$ of a price relative vector $\boldsymbol{x}[t]$ represents the ratio of the closing price of the jth stock for the tth trading day to that from the $(t-1)$th trading day. An investment at investment period t is represented by the portfolio vector $\boldsymbol{b}[t]$, $\boldsymbol{b}[t] \in \mathbb{R}_+^m$ and $\sum_{j=1}^m b_j[t] = 1$ for all t. Each entry $b_j[t]$ corresponds to the portion of wealth invested in the stock $x_j[t]$ at investment period t. The wealth achieved after n investment periods is given by $W(\boldsymbol{x}^n \mid \boldsymbol{b}[t]) \triangleq \prod_{t=1}^n \boldsymbol{b}^T[t]\boldsymbol{x}[t]$. For a CRP, $\boldsymbol{b}[t]$ is fixed, that is, $\boldsymbol{b}[t] = \boldsymbol{b}$, for some $\boldsymbol{b}$ and all investment periods. For any n and $\{\boldsymbol{x}[t]\}_{t \geq 1}$, the best CRP is given by $\boldsymbol{b}^* = \arg\max_{\boldsymbol{b}} W(\boldsymbol{x}^n \mid \boldsymbol{b}) = \prod_{t=1}^n \boldsymbol{b}^T \boldsymbol{x}[t]$. Since, the best CRP, $\boldsymbol{b}^*$, depends on all $\boldsymbol{x}[t]$, $t = 1, \ldots, n$, it can only be chosen in hindsight, that is, one needs to know the future.

The market considered in Cover and Ordentlich (1996, 1998), and Vovk and Watkins (1998) is idealized where there are no transaction costs involved. The transaction cost is usually modeled by a fixed percent commission paid when trading (Davis and Norman 1990). Maintaining a CRP requires potentially significant trading due to rebalancing. As an example, if one starts with one dollar and invests with the CRP $\boldsymbol{b} = [b_1, \ldots, b_m]^T$, then at the end of the first period, the account would have $b_i x_i$ dollars in each stock $i = 1, \ldots, m$, where x_i is the relative price change of the ith stock. The new portfolio vector is given by $[b_1 x_1 / \sum_i (b_i x_i), \ldots, b_m x_m / \sum_i (b_i x_i)]^T$ (which can be significantly different from $\boldsymbol{b}$) and must be rebalanced to $\boldsymbol{b}$ at the next investment period. To illustrate the effect of transaction costs on the achieved wealth, we plot in Figure 1.1 the wealth achieved by several different CRPs on a historical stock pair, Kinark–Iroquois, from the New York Stock exchange collected over 22 years. Here, we plot the wealth of several CRPs under several different transactions costs with an initial investment of one dollar. Each value of c represents the portion of wealth trader spent on the transaction costs and the x-axis corresponds to the portion of Iroquois stock in the portfolio. As seen in Figure 1.1, the wealth achieved by the CRPs severely degrades with increasing transaction costs.

To avoid hefty transaction costs, semiconstant rebalanced portfolios have been suggested as good strategies in the presence of commissions by Helmbold et al. (1998) and Blum and Kalai (1998). Unlike a CRP, a semiconstant rebalanced portfolio (SCRP) rebalances only at selected instants and does not trade stock inbetween. Hence, a SCRP may avoid rebalancing if the transaction costs outweigh the benefits of rebalancing. In this paper, we approach this problem from a competitive algorithms perspective and compete against all such SCRPs with arbitrary numbers of rebalancing times and corresponding rebalancing instants.

For an arbitrary sequence of price relative vectors $\boldsymbol{x}^n$ and a given CRP, a competing SCRP with rebalancing times $\mathcal{R}_{k,n}$ with k rebalancings, represented by $(t_1, \ldots, t_k)$, divides $\boldsymbol{x}^n$ into $k+1$ segments such that $\boldsymbol{x}^n$ is obtained by the concatenation of

$$\{\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t_1 - 1]\}\{\boldsymbol{x}[t_1], \ldots, \boldsymbol{x}[t_2 - 1]\} \ldots \{\boldsymbol{x}[t_k], \ldots, \boldsymbol{x}[n]\}.$$

Given n and k, there exist $\binom{n}{k}$ such rebalancing paths $\mathcal{R}_{k,n}$. The SCRP for rebalancing path $\mathcal{R}_{k,n}$ only rebalances to $\boldsymbol{b}$ on these selected times, that is, $(t_1, \ldots, t_k)$, and does not rebalance otherwise. For notational simplicity we assume $t_0 = 1$ and $t_{k+1} = n + 1$. In

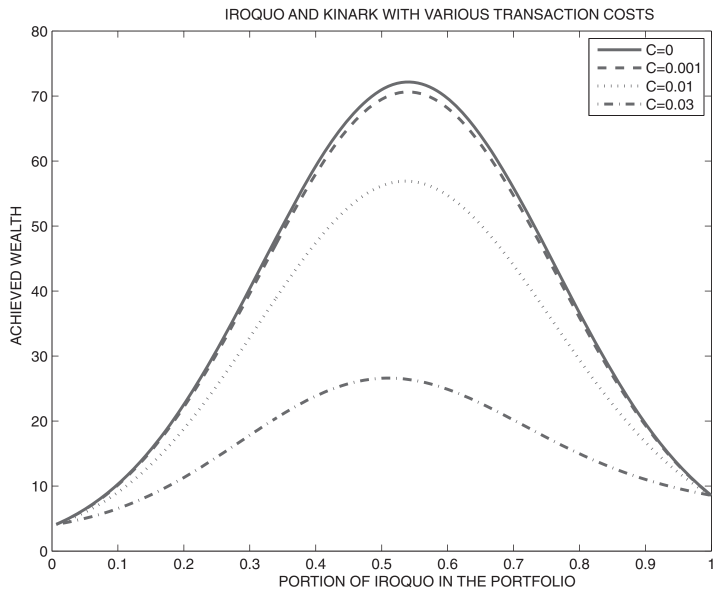

FIGURE 1.1. Wealth achieved by CRPs under several different transaction costs on Iroquois–Kinark pair, where each c represents the percent of wealth that should be spent as transaction costs, for example, $c = 0.01 = 1\%$. The x-axis represents the portion of Iroquois stock in the portfolio.

each segment, this SCRP will achieve the wealth $\sum_{j=1}^{m} b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t]$. Then, the total wealth achieved on $\boldsymbol{x}^n$ using $\mathcal{R}_{k,n}$ and $\boldsymbol{b}$ is

$$W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n}) \triangleq \prod_{i=1}^{k+1} \left(\sum_{j=1}^{m} b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t] \right).$$

This wealth can also be written as $W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n}) = \prod_{i=1}^{k+1} \boldsymbol{b}^T \boldsymbol{y}[i]$, where $\boldsymbol{y}[i] \triangleq [\prod_{t=t_{i-1}}^{t_i-1} x_1[t], \ldots, \prod_{t=t_{i-1}}^{t_i-1} x_m[t]]^T$, that is, a SCRP over n days with k rebalancings can be viewed as a CRP over $k+1$ days where price relatives of each segment are combined to yield $\boldsymbol{y}[i]$. We attempt to outperform all such portfolios, including the one that has been optimized by choosing the rebalancing path $\mathcal{R}_{k,n}$ and k based on observing the entire sequence $\boldsymbol{x}^n$ in advance. As such we try to minimize the following wealth ratio:

$$R_{\boldsymbol{b}}(n) \triangleq \sup_{\boldsymbol{x}^n} \frac{\sup_{t_1,\ldots,t_k \in \{2,\ldots,n\}_{0=t_0<t_1 \ldots t_k<t_{k+1}=n+1}} \prod_{i=1}^{k+1} \left(\sum_{j=1}^{m} b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t] \right)}{\prod_{t=1}^{n} \hat{\boldsymbol{b}}^T[t] \boldsymbol{x}[t]}, \tag{1.1}$$

where $\hat{\boldsymbol{b}}[t]$ is a strictly sequential portfolio assignment at time t, that is, $\hat{\boldsymbol{b}}[t]$ may be a function of $\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t-1]$ but cannot depend on the future, $\mathcal{R}_{k,n}$ is any rebalancing path representing $(t_1, \ldots, t_k)$ with an arbitrary k. We will show that we can construct a sequential portfolio for which the logarithm of this ratio is at most $k \ln(n) + O(k+1)$ for any $\mathcal{R}_{k,n}$, k and n, without knowledge of $\mathcal{R}_{k,n}$, k, or n a priori.

We next investigate this framework where we pay a fixed percentage in transaction costs. For an arbitrary sequence of price relative vectors $\boldsymbol{x}^n$, consider a portfolio selection algorithm with the rebalancing path $\mathcal{R}_{k,n}$, represented by $(t_1, \ldots, t_k)$, rebalancing to $\boldsymbol{b}$ at the start of each segment. This algorithm will pay a transaction cost at the start of each segment i to rebalance to $\boldsymbol{b}$ and no transaction costs within each segment. We define the wealth achieved by this algorithm as $W^c(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n})$ including commission costs c_{sell} and c_{buy}, where $c = c_{\text{sell}} + c_{\text{buy}}$. In determining the best algorithm in the competing class, we attempt to outperform all such portfolios, including the one that has been optimized by choosing the rebalancing path $\mathcal{R}_{k,n}$ and k based on observing the entire sequence $\boldsymbol{x}^n$ in advance, including the transaction costs. Here, we try to minimize the following wealth ratio:

$$R^c_{\boldsymbol{b}}(n) \triangleq \sup_{\boldsymbol{x}^n} \frac{\sup_{\mathcal{R}_{k,n}} W^c(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n})}{W^c(\boldsymbol{x}^n \mid \hat{\boldsymbol{b}})}, \tag{1.2}$$

where $W^c(\boldsymbol{x}^n \mid \hat{\boldsymbol{b}})$ is the wealth achieved by a strictly sequential portfolio $\hat{\boldsymbol{b}}[t]$, that is, $\hat{\boldsymbol{b}}[t]$ may be a function of $\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t-1]$ but cannot depend on the future, under transaction costs c_{sell} and c_{buy}, and $\mathcal{R}_{k,n}$ is any rebalancing path representing $(t_1, \ldots, t_k)$ with arbitrary k. We will show that we can construct a sequential portfolio for which the logarithm of this ratio is at most $k \ln(n) + O(k+1)$ for any of $\mathcal{R}_{k,n}$, k or n, without knowledge of $\mathcal{R}_{k,n}$, k, and n a priori.

In Cover and Ordentlich (1996), a sequential algorithm is presented that asymptotically achieves the wealth of the best CRP, that is, $W(\boldsymbol{x}^n \mid \boldsymbol{b}^*)$, for any sequence of price relative vectors, which can only be chosen in hindsight, since $\boldsymbol{b}^*$ is a function of $\boldsymbol{x}[1], \ldots, \boldsymbol{x}[n]$. Several different sequential algorithms have since been introduced achieving the performance of the best CRP, albeit either with different guaranteed performance bounds or different performance results on historical data in Helmbold et al. (1998), Vovk and Watkins (1998), Agarwal and Hazan (2006), Kalai and Vempala (2000), and Bianchi and Lugosi (2006). The framework of this problem involving transaction costs was investigated in Blum and Kalai (1998), where the authors demonstrated that a sequential algorithm using a similar weighting to that introduced in Cover and Ordentlich (1996) is also competitive under transaction costs, that is, asymptotically achieving the performance of the best CRP under transaction costs. However, as seen in Figure 1.1, even the performance of the best CRP is severely affected by transaction costs. Hence, it may not be enough to try to achieve the performance of the best CRP if the cost of rebalancing outweighs that which could be gained from rebalancing at every investment period. Several different approaches here also experimentally tested under different transaction costs in Helmbold et al. (1998) on historical data. In Singer (1998), the author introduced a switching portfolio that achieves the wealth of the best portfolio that switches between pure (i.e., single-stock) strategies where the switching pattern is tuned to the sequence of price relatives. The results in Singer (1998) were then extended to portfolios that switch between CRPs instead of a finite number of strategies, for example, pure strategies, in Kozat and Singer (2007). Although not introduced to compete against the best rebalancing times, application of the portfolio from Singer (1998) to the problem considered here, would yield a universal portfolio with an additional regret of $O(k \ln(m) + \frac{3}{2} k \ln(n))$ in the exponent, because the algorithm could be used to hedge against the switching among pure strategies in each segment, instead of hedging against the rebalancing instants. Although several results are introduced for tracking the best predictor or actions in Gyorgy, Linder, and Lugosi (2005, 2008), Takimoto and Warmuth (2002), and Bianchi and Lugosi (2006), these results cannot be generalized to here due to the unboundedness of the loss function considered.

2. SEMICONSTANT REBALANCED PORTFOLIOS

For an arbitrary sequence of price relative vectors $\boldsymbol{x}^n$, a semiconstant portfolio algorithm with a fixed portfolio vector $\boldsymbol{b}$ and a rebalancing path $\mathcal{R}_{k,n}$ with k rebalancing times, represented by $(t_1, \ldots, t_k)$, divides $\boldsymbol{x}^n$ into $k+1$ segments and only rebalances at times $t_1, \ldots, t_k$. For notational simplicity we assume $t_0 = 1$ and $t_{k+1} = n+1$. For this setting we have the following theorem.

THEOREM 2.1. *Let* $\{\boldsymbol{x}[t]\}_{t\geq 1}$ *be an arbitrary sequence of price relative vectors such that* $\boldsymbol{x}[t] \in \mathbb{R}_+^m$ *for all t and where some components of* $\boldsymbol{x}[t]$ *can be zero. Then, for all* $\epsilon > 0$ *and given a portfolio vector* $\boldsymbol{b}$, $\boldsymbol{b} \in \mathbb{R}_+^m$ *and* $\sum_{j=1}^m b_j = 1$, *we can construct sequential portfolios* $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ *with complexity linear in t per investment period, such that, when applied to* $\{\boldsymbol{x}[t]\}_{t\geq 1}$ *for all k and n*

$$R_{u,\boldsymbol{b}}[n] = \frac{\sup\limits_{t_1,\ldots,t_k \in \{2,\ldots,n\}_{0=t_0<t_1\ldots t_k<t_{k+1}=n+1}} \prod_{i=1}^{k+1} \left(\sum_{j=1}^m b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t] \right)}{\prod_{t=1}^n \tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}^T[t] \boldsymbol{x}[t]} \tag{2.1}$$

satisfies

$$\frac{\ln R_{u,\boldsymbol{b}}[n]}{n} \leq (k+\epsilon)\frac{\ln(n)}{n} + \frac{1}{n}\left(\log(1+\epsilon) + k \log\frac{1}{\epsilon}\right) \tag{2.2}$$

and

$$\frac{\ln R_{u,\boldsymbol{b}}[n]}{n} \leq ((k+1)\epsilon + k)\,\frac{\ln(n/k)}{n} + \frac{1}{n}\left((k+1)\log\frac{1+\epsilon}{\epsilon} + \log\epsilon\right) \tag{2.3}$$

for any $\mathcal{R}_{k,n}$ *representing rebalancing times* $(t_1, \ldots, t_k)$ *and any k, such that* $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ *does not depend on* $\mathcal{R}_{k,n}$, *k or n.*

Theorem 2.1 states that given $\boldsymbol{b}$, the logarithm of the wealth ratio of the universal sequential portfolio $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ is within $O(k\ln(n))$ of the best batch SCRP with any k rebalancing times (tuned to the underlying sequence), uniformly, for every sequence of price relatives $\{\boldsymbol{x}[t]\}_{t\geq 1}$, for all n.

We next investigate this framework where we pay fixed percent transaction costs. For an arbitrary sequence of price relative vectors $\boldsymbol{x}^n$, consider a portfolio with the rebalancing path $\mathcal{R}_{k,n}$, represented by $(t_1, \ldots, t_k)$, rebalancing the investments to $\boldsymbol{b}$ at the start of each segment. This algorithm will pay a transaction cost at the start of each segment i to rebalance to $\boldsymbol{b}$ and no transaction costs within each segment. We define the wealth achieved by this algorithm as

$$W^c(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n})$$

including commission costs c_{sell} and c_{buy}, where $c = c_{\text{sell}} + c_{\text{buy}}$. We demonstrate that:

THEOREM 2.2. *Let* $\{\boldsymbol{x}[t]\}_{t\geq 1}$ *be an arbitrary sequence of price relative vectors such that* $\boldsymbol{x}[t] \in \mathbb{R}_+^m$ *for all t and where some components of* $\boldsymbol{x}[t]$ *can be zero. Then, for all* $\epsilon > 0$ *and given a portfolio vector* $\boldsymbol{b}$, $\boldsymbol{b} \in \mathbb{R}_+^m$ *and* $\sum_{j=1}^m b_j = 1$, *we can construct sequential portfolios* $\tilde{\boldsymbol{b}}^c_{u,\boldsymbol{b}}[t]$ *with complexity linear in t per investment period, such that when applied to* $\{\boldsymbol{x}[t]\}_{t\geq 1}$, *for any* $c = c_{\text{sell}} + c_{\text{buy}}$, *and for all n, k, the wealth ratio*

$$R^c_{u,\boldsymbol{b}}[n] = \frac{\sup_{t_1,\ldots,t_k\in\{2,\ldots,n\}_{0=t_0<t_1\ldots t_k<t_{k+1}=n+1}} W^c(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n})}{W^c(\boldsymbol{x}^n \mid \tilde{\boldsymbol{b}}^c_{u,\boldsymbol{b}})},$$

where $W^c(\boldsymbol{x}^n \mid \tilde{\boldsymbol{b}}^c_{u,b})$ *is the wealth achieved by the universal algorithm with commissions, satisfies*

$$\frac{\ln R^c_{u,\boldsymbol{b}}[n]}{n} \leq (k+\epsilon)\frac{\ln(n)}{n} + \frac{1}{n}\left(\log(1+\epsilon) + k\log\frac{1}{\epsilon}\right)$$

and

$$\frac{\ln R^c_{u,\boldsymbol{b}}[n]}{n} \leq ((k+1)\epsilon + k)\frac{\ln(n/k)}{n} + \frac{1}{n}\left((k+1)\log\frac{1+\epsilon}{\epsilon} + \log\epsilon\right)$$

for any $\mathcal{R}_{k,n}$ *representing rebalancing times* $(t_1,\ldots,t_k)$ *and any k, such that* $\tilde{\boldsymbol{b}}^c_{u,\boldsymbol{b}}[t]$ *does not depend on* $\mathcal{R}_{k,n}$, *k, or n.*

Theorem 2.2 states that given $\boldsymbol{b}$, the logarithm of the wealth ratio of the universal sequential portfolio $\tilde{\boldsymbol{b}}^c_{u,\boldsymbol{b}}[t]$ is within $O(k\ln(n))$ for all n of the best batch SCRP with k rebalancings (tuned to the underlying sequence), uniformly, for every sequence of price relatives $\{\boldsymbol{x}[t]\}_{t\geq 1}$ and c.

3. PROOF AND CONSTRUCTION OF THE PORTFOLIOS

Proof of Theorem 2.1. While proving Theorem 2.1, we use ideas from the proof of Theorem 1 in Kozat and Singer (2008). Hence, we study mainly the differences in here. We observe that given $\boldsymbol{b}$ and a possible rebalancing path $\mathcal{R}_{k,n}$, representing $(t_1,\ldots,t_k)$ with k rebalancings and data length n, a competing SCRP with $k+1$ segments can be constructed. For each such $\boldsymbol{b}$ and $\mathcal{R}_{k,n}$, this hypothetical sequential investment strategy achieves a wealth of

$$W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n}) = \prod_{i=1}^{k+1}\left(\sum_{j=1}^m b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t]\right). \tag{3.1}$$

There exists 2^{n-1} such possible rebalancing paths $\mathcal{R}_{k,n}$ for all $k = 1,\ldots,n-1$. Given any k and n, the algorithm with the best rebalancing times, that is, $\mathcal{R}^*_{k,n}$, achieves the largest wealth on $\boldsymbol{x}^n$, that is,

$$\begin{aligned} W\big(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}^*_{k,n}\big) &\triangleq \sup_{\mathcal{R}_{k,n}} W(\boldsymbol{x} \mid \boldsymbol{b}, \mathcal{R}_{k,n}) \\ &= \sup_{t_1,\ldots,t_k\in\{2,\ldots,n\}_{0=t_0<t_1\ldots t_k<t_{k+1}=n+1}} \prod_{i=1}^{k+1}\left(\sum_{j=1}^m b_j \prod_{t=t_{i-1}}^{t_i-1} x_j[t]\right). \end{aligned}$$

Our goal is to demonstrate a strongly sequential portfolio that has no prior knowledge of k, n, or the best rebalancing times, however, that achieves $W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}^*_{k,n})$. We will accomplish this result using a mixture approach as in Willems (1996) and Kozat and Singer (2008). First, we will show that a proper weighted combination of all sequential portfolios corresponding to each $\mathcal{R}_{q,n}$, $q = 0, \ldots, n-1$, a total of 2^{n-1} portfolios, achieves a wealth asymptotically as large as $W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}^*_{k,n})$ for any k. Next, we will show that the wealth of this weighted combination can be achieved by a universal sequential portfolio, hence the result.

For each $\mathcal{R}_{k,n}$ and given $\boldsymbol{b}$, we construct a hypothetical sequential portfolio as in (3.1). We next invest a fraction of our wealth, $F(\mathcal{R}_{k,n})$, in each such algorithm and then collect the total wealth at the end of n investment periods. This final combined total wealth achieved by all 2^{n-1} sequential portfolios (over all possible $\mathcal{R}_{k,n}$ and k) is given by

$$\tilde{W}_u(\boldsymbol{x}^n) \triangleq \sum_{k=0}^{n-1} \sum_{\mathcal{R}_{k,n}} F(\mathcal{R}_{k,n}) W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n}). \tag{3.2}$$

We next demonstrate a proper selection of $F(\mathcal{R}_{k,n})$ will yield a total wealth $\tilde{W}_u(\boldsymbol{x}^n)$ that is asymptotically as large as $W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}^*_{k,n})$.

For any rebalancing path $\mathcal{R}_{k,n}$, the fraction of the initial invested wealth, or weighting by Willems (1996), $F(\mathcal{R}_{k,n})$ should be nonnegative and should naturally satisfy $\sum_{k=0}^{n-1} \sum_{\mathcal{R}_{k,n}} F(\mathcal{R}_{k,n}) = 1$. Since $\tilde{W}_u(\boldsymbol{x}^n)$ is the total weighted wealth achieved by the class of all possible SCRP corresponding to all $\mathcal{R}_{k,n}$, this total wealth satisfies

$$\ln \tilde{W}_u(\boldsymbol{x}^n) \geq \ln F(\mathcal{R}_{k,n}) + \ln W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n}), \tag{3.3}$$

for any rebalancing path $\mathcal{R}_{k,n}$ including $\mathcal{R}^*_{k,n}$, since $\tilde{W}_u(\boldsymbol{x}^n) \geq F(\mathcal{R}_{k,n}) W(\boldsymbol{x}^n \mid \boldsymbol{b}, \mathcal{R}_{k,n})$. The fraction of the wealth invested on $\mathcal{R}_{k,n}$, $F(\mathcal{R}_{k,n})$, directly contributes to the log wealth ratio as $\ln(F(\mathcal{R}_{k,n}))$ over the best algorithm given any rebalancing path. Hence, it is desirable that the fraction of wealth invested on the "best rebalancing path" (i.e., the path with the largest wealth gain) be as large as possible and this fraction should also be sequentially constructable so that the overall weighting and the resulting portfolio can be sequentially computable. In here, we study three such initial wealth investments that were introduced as weight assignments in Willems (1996) and Shamir and Merhav (1999) for universal lossless source coding to assign probabilities to binary sequences (and then later used in a prediction context in Kozat and Singer 2008 to construct universal switching linear predictors under the square error loss). To use these weight assignments, for a given path $\mathcal{R}_{k,n}$, we construct a binary sequence such that each rebalancing instant is represented as a one and each instant without rebalancing as a zero, forming a sequence of length $n-1$. For any $\mathcal{R}_{k,n}$, there exist k ones and total $n-k-1$ zeros. Then, $F(\mathcal{R}_{k,n})$ is given as the weight assigned to this binary sequence generated from $\mathcal{R}_{k,n}$ using a sequential weight assignment (Willems 1996). For each such different weight assignments, where $F(\mathcal{R}_{k,n}) = F_i(\mathcal{R}_{k,n})$, $i = 1, 2, 3$, it can be shown that $\sum_{k=0}^{n-1} \sum_{\mathcal{R}_{k,n}} F_i(\mathcal{R}_{k,n}) = 1$ and these weight assignments yield the following bounds on $F(\mathcal{R}_{k,n})$

$$-\ln F_1(\mathcal{R}_{k,n}) \leq \frac{3k+1}{2} \ln(n/k) + O(k) \tag{3.4}$$

$$\text{and} - \ln F_2(\mathcal{R}_{k,n}) \leq (k+\epsilon)\ln(n) + \left(\log(1+\epsilon) + k \log \frac{1}{\epsilon}\right) \tag{3.5}$$

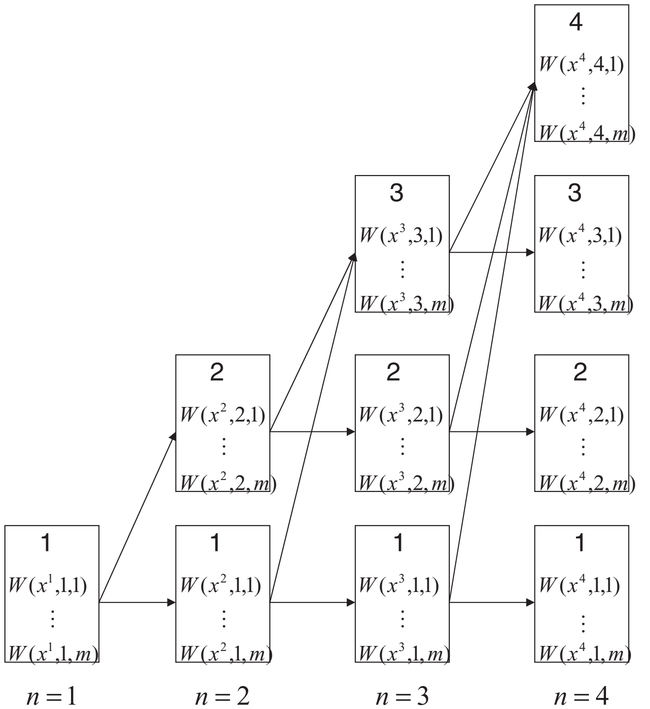

FIGURE 3.1. The rebalancing diagram for $n = 4$. Each box represents a state, where each number in the box is the time of the last rebalancing instant. In each box, we have accumulated wealth for each stock, $j = 1, \ldots, m$.

$$\text{(3.6)} \quad \text{and} - \ln F_3(\mathcal{R}_{k,n}) \leq (k + (k+1)\epsilon)\ln(n/k) + \left((k+1)\log\frac{1+\epsilon}{\epsilon} + \log\epsilon\right)$$

for all $\epsilon > 0$ and any $\mathcal{R}_{k,n}$, in Willems (1996) and Shamir and Merhav (1999). Hence, any such assignment $F_i(\mathcal{R}_{k,n})$, $i = 1, 2, 3$, can be used in place of $F(\mathcal{R}_{k,n})$. In the pseudo-code given in Figure 3.1 after the derivations, we implement our algorithms in a generic manner such that any of the weighting $i = 1, 2, 3$ can be used.

By using any of the bounds on $F(\mathcal{R}_{k,n})$, for example, such as the one from (3.5), that is, $F(\mathcal{R}_{k,n}) = F_2(\mathcal{R}_{k,n})$, we can provide a lower bound on $\ln(\tilde{W}_u(\boldsymbol{x}^n))$ in (3.3) as

$$\text{(3.7)} \quad \ln(\tilde{W}_u(\boldsymbol{x}^n)) \geq \ln W(\boldsymbol{x} \mid \boldsymbol{b}, \mathcal{R}_{k,n}) - (k+\epsilon)\ln(n) - \left(\log(1+\epsilon) + k\log\frac{1}{\epsilon}\right).$$

Hence, we now have a sequential strategy which invests $F(\mathcal{R}_{k,n})$ portion of wealth on each $\mathcal{R}_{k,n}$ and has a combined wealth asymptotically achieving, to the first order in the exponent, the same wealth as that achieved by any rebalancing path $\mathcal{R}_{k,n}$ as shown in (3.7). It still remains to find a sequential portfolio whose achieved wealth is as large as $\tilde{W}_u(\boldsymbol{x}^n)$.

We are now ready to find the actual universal sequential portfolio. By definition, we have

$$\tilde{W}_u(\boldsymbol{x}^n) = \prod_{t=1}^{n} \frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})}.$$

If we look at each term in the product closely,

$$\frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})} = \frac{\sum_{k=0}^{t-1}\sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t})\, W(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}_{k,t})}{\sum_{l=0}^{t-2}\sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1})\, W(\boldsymbol{x}^{t-1} \mid \boldsymbol{b}, \mathcal{R}_{l,t-1})}$$

we observe that this product can be written as

$$\begin{aligned}
\frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})} &= \frac{\sum_{k=0}^{t-1}\sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) \prod_{i=1}^{k+1}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)}{\sum_{l=0}^{t-2}\sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1}) \prod_{i=1}^{l+1}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)} \\
&= \frac{\sum_{k=0}^{t-1}\sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) \left(\prod_{i=1}^{k}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)\right)\left(\sum_{j=1}^{m} b_j \prod_{o=t_k}^{t} x_j[o]\right)}{\sum_{l=0}^{t-2}\sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1}) \prod_{i=1}^{l+1}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)} \\
&= \frac{\sum_{k=0}^{t-1}\sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) \left[\prod_{i=1}^{k}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)\right]\left[b_1 \prod_{o=t_k}^{t-1} x_1[o], \ldots, b_m \prod_{o=t_k}^{t-1} x_m[o]\right]^T \boldsymbol{x}[t]}{\sum_{l=0}^{t-2}\sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1}) \prod_{i=1}^{l+1}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)} \\
&= \left(\frac{\sum_{k=0}^{t-1}\sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) \left(\prod_{i=1}^{k}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)\right)\left[b_1 \prod_{o=t_k}^{t-1} x_1[o], \ldots, b_m \prod_{o=t_k}^{t-1} x_m[o]\right]}{\sum_{l=0}^{t-2}\sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1}) \prod_{i=1}^{l+1}\left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i-1} x_j[r]\right)}\right)^T \boldsymbol{x}[t],
\end{aligned}$$

where in the third line, we used that $(\sum_{j=1}^{m} b_j \prod_{o=t_k}^{t} x_j[o]) = [b_1 \prod_{o=t_k}^{t-1} x_1[o], \ldots, b_m \prod_{o=t_k}^{t-1} x_m[o]]^T \boldsymbol{x}[t]$. Hence the desired sequential universal portfolio is given by

$$\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t] \triangleq \frac{\sum_{k=0}^{t-1} \sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) \left[\prod_{i=1}^{k} \left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i - 1} x_j[r] \right) \right] \left[b_1 \prod_{o=t_k}^{t-1} x_1[o], \ldots, b_m \prod_{o=t_k}^{t-1} x_m[o] \right]^T}{\sum_{l=0}^{t-2} \sum_{\mathcal{R}_{l,t-1}} F(\mathcal{R}_{l,t-1}) \prod_{i=1}^{l+1} \left(\sum_{j=1}^{m} b_j \prod_{r=t_{i-1}}^{t_i - 1} x_j[r] \right)}.$$

In this form, the sequential algorithm requires 2^{n-1} different sequential algorithms to be explicitly run in parallel on the sequence of price relatives. We now demonstrate that this sequential portfolio can be calculated efficiently by using a linear rebalancing diagram after assigning appropriate weights to each branch, similar to that used in Willems (1996) as in Figure 3.1.

At each time t, we divide the set of all possible rebalancing paths $\mathcal{R}_{k,t}$, $k = 0, \ldots, t-1$ into t disjoint sets. We label each set by a state variable s_t representing the most recent rebalancing instant of a corresponding path within the period $1 \le l \le t$, as an example, for a $\mathcal{R}_{k,t}$, $s_t = t_k$. Given t, there can be at most t states $s_t = 1, \ldots, t$. As an example, at time t, all rebalancing paths with the same last rebalancing instant, $t_k = s$, are represented by the state $s_t = s$. We then define $W_t(\boldsymbol{x}^t, s_t, j)$ as the total wealth achieved on stock j by all sequential algorithms at state s_t at time t, where $j = 1, \ldots, m$. For example, $W_t(\boldsymbol{x}^t, s, j)$ is the weighted sum of all the wealth on stock j achieved by the sequential portfolios whose rebalancing paths ended up at state $s_t = s$. Since the states partition the set of paths $\mathcal{R}_{k,t}$,

$$\tilde{W}_u(\boldsymbol{x}^t) = \sum_{\mathcal{R}} F(\mathcal{R}) W(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}) = \sum_{s=1}^{t} \sum_{j=1}^{m} W_t(\boldsymbol{x}^t, s, j).$$

To obtain a closed form expression for $\frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})}$, we will show that $W_t(\boldsymbol{x}^t, s_t, j)$ can be calculated recursively by using the linear rebalancing diagram as in Figure 3.1. Each box in Figure 3.1 represents a state variable s_t with the corresponding wealth $W_t(\boldsymbol{x}^t, s_t, j)$, $j = 1, \ldots, m$. In this figure, any directed path represents a rebalancing path where a horizontal move denotes no rebalancing, while an upward move represents a rebalancing. As such, state s_t represents the most recent rebalancing instant within the period $1 \le l \le t$.

We now derive a recursive update for each $W_t(\boldsymbol{x}^t, s_t, j)$, $s_t = 1, \ldots, t$, $j = 1, \ldots, m$. From Figure 3.1, we see that there exist only two transitions from each state to form a new rebalancing path. At time $t-1$, all the paths that ended at state $s_{t-1} = s$, $\mathcal{R}_{(.,t-1):s_{t-1}=s}$, will end up in state $s_t = s$ if no rebalancing happens at time t, that is, a horizontal move in Figure 3.1. For any state $s_{t-1} = s$, $s = 1, \ldots, t-1$, $j = 1, \ldots, m$, to get $W_t(\boldsymbol{x}^t, s, j)$ when there is no rebalancing, we only need to adjust wealth for each stock $W_{t-1}(\boldsymbol{x}^{t-1}, s, j)$: first by multiplying each with the relative gain for each stock $x_j[t]$ (since there is no rebalancing) and second adjusting the path weights $F(\mathcal{R}_{(.,t-1):s_{t-1}=s})$ using the weight assignment algorithm, since the data length is increased by one. This yields, $W_t(\boldsymbol{x}^t, s, j) = F_{\text{tr}}(s_t = s \mid s_{t-1} = s) W_{t-1}(\boldsymbol{x}^{t-1}, s, j) x_j[t]$, where we define $F_{\text{tr}}(s_t = s \mid s_{t-1} = s)$ as the adjustment required to scale the path weights when there is no rebalancing. To get different weight assignments of (3.4), (3.5), or (3.6), $F_{\text{tr}}(. \mid .)$ should be selected accordingly as in Kozat and Singer (2008).

When there is a rebalancing at time t, that is, $s_t = t$, the corresponding wealth from each s_{t-1}, $W_{t-1}(\boldsymbol{x}^{t-1}, s_{t-1}, j)$ should be adjusted to yield $W_t(\boldsymbol{x}^t, t, j)$. When $s_t = t$, there exist $t-1$ possible rebalancings (to generate this new state) from each state s_{t-1},

$s_{t-1} = 1, \dots, t-1$, at time $t-1$ to form the state $s_t = t$ at time t. Since we create a new state, we need to rebalance to $\boldsymbol{b}$ at every state $s = 1, \dots, t-1$. Hence, each $W_{t-1}(\boldsymbol{x}^{t-1}, s, r)$, $s = 1, \dots, t-1$, $r = 1, \dots, m$, will contribute to each $W_t(\boldsymbol{x}^t, t, j)$, $j = 1, \dots, m$, proportional to $b_j x_j[t]$ scaled by $F_{\text{tr}}(s_t = t \mid s_{t-1} = s)$, where $F_{\text{tr}}(s_t = t \mid s_{t-1} = s)$ is the corresponding adjustment to path weights (when there is a rebalancing), yielding

$$W_t(\boldsymbol{x}^t, t, j) = \sum_{s=1}^{t-1} \left(\sum_{r=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, r) \right) F_{\text{tr}}(s_t = t \mid s_{t-1} = s) b_j x_j[t]. \tag{3.8}$$

We observe that each $W_t(\boldsymbol{x}^t, t, j)$ has a contribution from all $W_{t-1}(\boldsymbol{x}^{t-1}, s, r)$, $s = 1, \dots, t-1$, $r = 1, \dots, m$ since there is a rebalancing. Hence, the sequential update for $W_t(\boldsymbol{x}^t, s_t, j)$ when $s_t = t$.

A closer look at Figure 3.1 reveals that $\tilde{W}_u(\boldsymbol{x}^t) = \sum_{s=1}^{t} \sum_{j=1}^{m} W_t(\boldsymbol{x}^t, s, j)$ can be written as wealths coming from each $W_{t-1}(\boldsymbol{x}^{t-1}, s_{t-1}, j)$ as

$$\tilde{W}_u(\boldsymbol{x}^t) = \sum_{s=1}^{t-1} \sum_{j=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, j) \big\{ F_{\text{tr}}(s_t = s \mid s_{t-1} = s) x_j[t] + F_{\text{tr}}(s_t = t \mid s_{t-1} = s) \boldsymbol{b}^T \boldsymbol{x}[t] \big\}$$

since each $W_{t-1}(\boldsymbol{x}^{t-1}, s, j)$ contributes with two terms in the braces (i.e., two arrows on Figure 3.1). $\tilde{W}_u(\boldsymbol{x}^t)$ can also be written as,

$$\tilde{W}_u(\boldsymbol{x}^t) = \sum_{s=1}^{t-1} \sum_{j=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, j) \big\{ F_{\text{tr}}(s_t = s \mid s_{t-1} = s) \boldsymbol{e}_j + F_{\text{tr}}(s_t = t \mid s_{t-1} = s) \boldsymbol{b} \big\}^T \boldsymbol{x}[t], \tag{3.9}$$

where $\boldsymbol{e}_j \stackrel{\triangle}{=} [0, \dots, 0, 1, 0, \dots, 0]^T$, that is, a vector of all zeros except a single one at location j. Hence,

$$\frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})} = \frac{\sum_{s=1}^{t} \sum_{j=1}^{m} W_t(\boldsymbol{x}^t, s, j)}{\sum_{s=1}^{t-1} \sum_{r=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, r)}$$

$$= \frac{\sum_{s=1}^{n-1} \sum_{j=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, j) \left\{ F_{\text{tr}}(s_t = s \mid s_{t-1} = s) \boldsymbol{e}_j^T \boldsymbol{x}[t] + F_{\text{tr}}(s_t = t \mid s_{t-1} = s) \boldsymbol{b}^T \boldsymbol{x}[t] \right\}}{\sum_{s=1}^{n-1} \sum_{r=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s_{t-1}, r)}.$$

Then,

$$\frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})} = \sum_{s=1}^{t-1} \sum_{j=1}^{m} \sigma_{t-1}(s, j) \big\{ F_{\text{tr}}(s_t = s \mid s_{t-1} = s) \boldsymbol{e}_j^T \boldsymbol{x}[t] + F_{\text{tr}}(s_t = t \mid s_{t-1} = s) \boldsymbol{b}^T \boldsymbol{x}[t] \big\}, \tag{3.10}$$

where the weights $\sigma_{t-1}(s, j)$ are defined as

$$\sigma_{t-1}(s, j) \triangleq \frac{W_{t-1}(\boldsymbol{x}^{t-1}, s, j)}{\sum_{s=1}^{t-1} \sum_{r=1}^{m} W_{t-1}(\boldsymbol{x}^{t-1}, s, r)} \tag{3.11}$$

and are a form of performance-weighting for the states s_{t-1} and stocks $j = 1, \ldots, m$.

From (3.10) we conclude that,

$$\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t] = \sum_{s=1}^{t-1} \sum_{j=1}^{m} \sigma_{t-1}(s, j) \left\{ F_{\mathrm{tr}}(s_t = s \mid s_{t-1} = s) \boldsymbol{e}_j + F_{\mathrm{tr}}(s_t = t \mid s_{t-1} = s) \boldsymbol{b} \right\} \tag{3.12}$$

which gives the final portfolio at each time t with complexity $O(tm)$. Hence, combining 2^{t-1} sequential algorithms to obtain the portfolio in (3.12) only requires $O(tm)$ computations per investment period. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Proof of Theorem 2.2 follows from the proof of Theorem 2.1. Given the achieved wealth $W^c(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}_{k,t})$, we then define a weighted mixture of wealth from all paths

$$\tilde{W}_u^c(\boldsymbol{x}^t) \triangleq \sum_{k=0}^{t-1} \sum_{\mathcal{R}_{k,t}} F(\mathcal{R}_{k,t}) W^c(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}_{k,t}). \tag{3.13}$$

Wealth $\tilde{W}_u^c(\boldsymbol{x}^t)$ corresponds to investing in each $\mathcal{R}_{k,t}$, a portion $F(\mathcal{R}_{k,t})$ of the initial investment and then collecting the final wealth at time n. We note that each such sequential algorithm would pay the transaction costs separately. Since $\tilde{W}_u^c(\boldsymbol{x}^t)$ arises from a sum of terms,

$$\tilde{W}_u^c(\boldsymbol{x}^t) \geq F(\mathcal{R}_{k,t}) W^c(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}_{k,t})$$

for each $\mathcal{R}_{k,t}$. However, instead of explicitly running 2^{t-1} algorithms in parallel and paying transaction costs for each of them separately, one can even avoid many of the transaction costs by occasionally exchanging stocks among the sequential portfolios. As an example, since we have access to the portfolio from each path, if a sequential portfolio corresponding to a rebalancing path requires selling a particular stock and another sequential portfolio corresponding to some other rebalancing path requires buying the same stock, we can exchange that stock instead of going to the market and avoid the transaction costs. Hence, since in Figure 3.1 each state corresponds to a combination of sequential portfolios with rebalancing paths that end up in that state, the wealth achieved by that state will not be worse than the combined weighted wealth of all such portfolios due to such possible occasional savings. Finally, since the universal algorithm is a weighted combination of all such states, the wealth achieved by the universal algorithm will not be smaller than $\tilde{W}_u^c(\boldsymbol{x}^t)$. We next explain an efficient implementation of $\tilde{W}_u^c(\boldsymbol{x}^t)$, as in Theorem 2.1.

Instead of dividing the initial investment among all $\mathcal{R}_{k,t}$ and collecting the final wealth at the end, we can implement this investment strategy using the transition diagram in Figure 3.1, as in Theorem 2.1. We next define $W_t^c(s_t, \boldsymbol{x}^t, j)$ as the weighted sum of all the wealth in stock j from the sequential algorithms whose rebalancing paths ended up at

PSEUDO-CODE

```
% Initialization with b
for j = 1 : m do
    W_0(x^0, 0, j) = b_j
    b̃_{u,j}[t] = b_j
endfor
for each t ≥ 1:
    % Invest with b̃_u[t], then x[t] is revealed
    b̃_u^T[t]x[t] % The wealth gained
    % For paths with no rebalancing update W(·)
    for s = 1, ..., t − 1 do
        for j = 1, ..., m do
            W_t(x^t, s, j) = W_{t−1}(x^{t−1}, s, j)F_tr(s_t = s | s_{t−1} = s)x_j[t]
        endfor
    endfor
    % For paths with rebalancing at time t update W(·)
    for s = 1, ..., t − 1 do
        totalSum = 0
        for j = 1, ..., m do
            W_t(x^t, t, j) = 0
            totalSum = totalSum+W_{t−1}(x^{t−1}, s, j)
        endfor
        for j = 1, ..., m do
            W_t(x^t, t, j) = W_t(x^t, t, j)+ totalSum ×F_tr(s_t = t | s_{t−1} = s)b_j[t]x_j[t]
        endfor
    endfor
    b̃_u[t + 1] = Σ_{s=1}^{t} Σ_{j=1}^{m} σ_t(s, j)
    {F_tr(s_{t+1} = s | s_t = s)e_j + F_tr(s_{t+1} = t | s_t = s)b}
endfor
```

FIGURE 3.2. A pseudo-code for the universal algorithm, where $\sigma_t(s,\ j)$ is defined in (3.14).

s_n; that is, for all paths $\mathcal{R}'$ such that the last rebalancing instant was at $s_t = s$. Since the states partition the set of paths $\mathcal{R}_{k,t}$,

$$\tilde{W}_u^c(\boldsymbol{x}^t) = \sum_{\mathcal{R}} F(\mathcal{R}) W^c(\boldsymbol{x}^t \mid \boldsymbol{b}, \mathcal{R}) = \sum_{s=1}^{t} \sum_{j=1}^{m} W_t^c(\boldsymbol{x}^t, s, j).$$

We next calculate the following ratio

$$\frac{\tilde{W}_u^c(\boldsymbol{x}^t)}{\tilde{W}_u^c(\boldsymbol{x}^{t-1})} = \sum_{s=1}^{t-1} \sum_{j=1}^{m} \sigma_{t-1}^c(s,\ j) \left\{ F_{\rm tr}(s_t = s \mid s_{t-1} = s) \boldsymbol{e}_j^T \boldsymbol{x}[t] + F_{\rm tr}(s_t = t \mid s_{t-1} = s) \boldsymbol{b}^T \boldsymbol{x}[t] \right\},$$

where the weights $\sigma_{t-1}(s,\ j)$ are defined as

$$\sigma^c_{t-1}(s, j) \triangleq \frac{W^c_{t-1}(\boldsymbol{x}^{t-1}, s, j)}{\sum_{z=1}^{t-1}\sum_{r=1}^{m} W^c_{t-1}(\boldsymbol{x}^{t-1}, z, r)} \tag{3.14}$$

and are a form of performance-weighting for the states s_{t-1} and stocks $j = 1, \dots, m$. Hence, the universal sequential portfolio for Theorem 2.2 is given as

$$\tilde{\boldsymbol{b}}^c_u[t] = \sum_{s=1}^{t-1}\sum_{j=1}^{m} \sigma^c_{t-1}(s, j)\left\{F_{\text{tr}}(s_t = s \mid s_{t-1} = s)\boldsymbol{e}_j + F_{\text{tr}}(s_t = t \mid s_{t-1} = s)\boldsymbol{b}\right\},$$

which completes the proof of Theorem 2.2. □

In Figure 3.2, we present the pseudo-code of the introduced algorithms. We note that to get different weightings given in (3.4), (3.5), and (3.6), $F_{\text{tr}}(\cdot)$ terms should be set accordingly as given in Kozat and Singer (2008).

4. SIMULATIONS

In this section, we demonstrate the performance of the introduced algorithms with several different examples. The set of simulations includes application of different portfolio selection strategies to the historical data collected from the New York Stock Exchange over a 22-year period (Cover 1991). The total set includes 35 different stocks. Here, the investment period is every 5 days. We first randomly select pairs of stocks and invest using: universal algorithm introduced in Section 3 "Uni," Cover's universal portfolio (Cover and Ordentlich 1996) "Cover's," buy-and-hold portfolio "Buy&Hold," a portfolio that continuously rebalances to the target portfolio, that is, the CRP $\boldsymbol{b}$, "CRP," and portfolios that rebalance to the target portfolio in predefined intervals such as every 10, 20, and 30 investment periods "Swt-10," "Swt-20," "Swt-30," respectively. We have also included results for rebalancing strategies that employ no-trade zones around the target portfolio, that is, $\boldsymbol{b}$, including an investment strategy that rebalances to the target portfolio only when the absolute difference between the present portfolio and the target portfolio exceeds a predefined interval (Mulvey and Simsek 2002) "no-trade-1," an investment strategy that rebalances to the target portfolio only when the Euclidean distance between the present portfolio and the target portfolio exceeds a predefined value (Brandt, Santa-Clara, and Valkanov 2005) "no-trade-2," and an investment strategy that rebalances to the portfolio on the boundary of the no-trade-zone only when the Euclidean distance between the present portfolio and the target portfolio exceeds a predefine value (Brandt, Santa-Clara, and Valkanov 2005) "no-trade-3." In Table 4.1, we present the wealth achieved by these algorithms for an initial investment of 1 dollar, where the results are averaged over 50 independent trials, that is, over 50 independent stock pairs. In Table 4.1a, $\boldsymbol{b}$ is selected uniform, that is, $\boldsymbol{b} = [1/2 \;\; 1/2]^T$, and in Table 4.1b, $\boldsymbol{b}$ is selected randomly for each trial, where $\boldsymbol{b}$ is the target portfolio for all algorithms. The transaction costs are selected as $c = 0$, 0.001, 0.01, 0.02 (i.e., 0%, 0.1%, 1%, 2%) and applied on only for buying stocks in accordance with Blum and Kalai (1998). For the universal algorithm, the transition weights are chosen to give the weighting introduced in Willems (1996) to yield (3.4). However, to avoid excessive rebalancing, we reverse the probability of rebalancing and no-rebalancings such that the algorithm favors the paths with less rebalancings and reverse path weightings. For "no-trade-1" algorithm, the algorithm rebalances when the

TABLE 4.1. Average Wealth Gain for 50 Independent Trials, Where the Algorithms Invest on Pairs of Stocks Selected Randomly among 35 Stocks

	(a)				(b)				
	$c = 0$	$c = 0.001$	$c = 0.01$	$c = 0.02$		$c = 0$	$c = 0.001$	$c = 0.01$	$c = 0.02$
Uni	16.7092	16.6026	16.6980	16.6870	Uni	16.1823	16.1163	16.1754	16.1686
Cover's	14.9068	10.7354	14.3994	13.9151	Cover's	14.9068	10.7354	14.3994	13.9151
CRP	16.7322	5.7285	14.8885	13.2782	CRP	16.1915	8.0139	14.9622	13.8546
Buy&Hold	11.8733	11.8733	11.8733	11.8733	Buy&Hold	12.9792	12.9792	12.9792	12.9792
Swt-10	15.5308	11.2547	15.0121	14.5167	Swt-10	15.3007	12.3783	14.9590	14.6296
Swt-20	14.7975	11.8660	14.4588	14.1313	Swt-20	14.8071	12.7272	14.5716	14.3429
Swt-30	14.8744	12.4890	14.6046	14.3424	Swt-30	14.8327	13.1315	14.6431	14.4583
no-trade-1	15.4077	15.3872	15.2072	15.0164	no-trade-1	14.8610	14.8516	14.7683	14.6791
no-trade-2	15.2911	15.2806	15.1870	15.0851	no-trade-2	14.6779	14.6739	14.6379	14.5986
no-trade-3	13.9132	13.9096	13.8773	13.8419	no-trade-3	13.9038	13.9024	13.8902	13.8768

Here, c represents the ratio of transaction costs, where $c = 0$, 0.001, 0.01, 0.02. The sequential algorithms include: Universal algorithm "Uni"; Cover's portfolio "Cover's"; a CRP with $\boldsymbol{b}$; buy-and-hold with initial investment $\boldsymbol{b}$ for 2 stocks; a sequential portfolio that only rebalances to $\boldsymbol{b}$ on every fixed 10 investment periods "Swt-10," 20 investment periods "Swt-20,"; 30 investment periods "Swt-30"; no-trade zone algorithms described in the text. (a) Algorithms with $\boldsymbol{b} = [1/2 \;\; 1/2]^T$. (b) Algorithms with $\boldsymbol{b}$ selected randomly for each trial.

TABLE 4.2. Average Wealth Gain for 50 Independent Trials, Where the Algorithms Invest on Sets of Stocks Selected Randomly among 35 Stocks

(a)					(b)				
	$c = 0$	$c = 0.001$	$c = 0.01$	$c = 0.02$		$c = 0$	$c = 0.001$	$c = 0.01$	$c = 0.02$
Uni	20.3469	20.1939	20.3309	20.3150	Uni	19.6917	19.5718	19.6792	19.6667
Cover's	18.4275	13.5928	17.8480	17.2927	Cover's	18.4275	13.5928	17.8480	17.2927
CRP	20.3804	5.8004	17.7823	15.5550	CRP	19.7091	6.4547	17.4683	15.5157
Buy&Hold	13.6651	13.6651	13.6651	13.6651	Buy&Hold	14.4480	14.4480	14.4480	14.4480
Swt-10	18.1605	12.5058	17.4592	16.7932	Swt-10	17.9811	12.9595	17.3719	16.7900
Swt-20	17.3227	13.4081	16.8625	16.4195	Swt-20	17.7218	14.0681	17.2976	16.8880
Swt-30	17.4272	14.2238	17.0593	16.7031	Swt-30	17.4546	14.6145	17.1330	16.8205
no-trade-1	18.9758	18.9466	18.6916	18.4228	no-trade-1	18.4338	18.4109	18.2105	17.9980
no-trade-2	17.2156	17.2065	17.1260	17.0383	no-trade-2	17.4061	17.3989	17.3347	17.2647
no-trade-3	15.5034	15.5006	15.4751	15.4472	no-trade-3	16.2336	16.2310	16.2077	16.1822

Here, c represents the ratio of transaction costs, where $c = 0, 0.001, 0.01, 0.02$. The sequential algorithms include: Universal algorithm "Uni"; Cover's portfolio "Cover's"; a CRP with $\boldsymbol{b}$; buy-and-hold with initial investment $\boldsymbol{b}$ for three stocks; a sequential portfolio that only switches on every fixed 10 investment periods "Swt-10," 20 investment periods "Swt-20,"; 30 investment periods "Swt-30"; no-trade zone algorithms described in the text. (a) Algorithms with $\boldsymbol{b} = [1/3 \;\; 1/3 \;\; 1/3]^T$. (b) Algorithms where $\boldsymbol{b}$ is selected randomly for each trial.

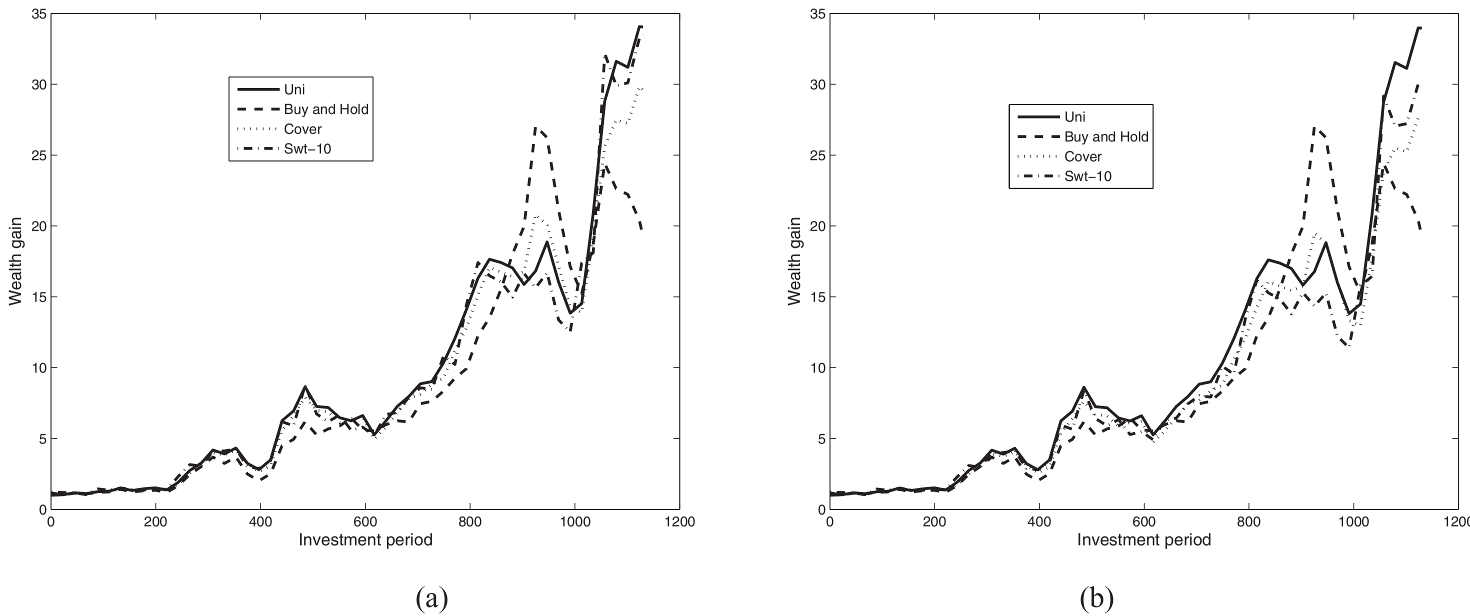

FIGURE 4.1. Wealth gain of sequential algorithms over a random selection of three stocks (Iroquois–Ford–Schlum) with respect to investment periods. The algorithms include: Universal algorithm "Uni"; Cover's portfolio "Cover's"; buy-and-hold with initial investment of $[1/3 \ \ 1/3 \ \ 1/3]^T$; a sequential portfolio that only switches on every fixed 10-day periods "Swt-10." (a) $c = 0$ and (b) $c = 0.01$.

difference between any entry of the current portfolio, say $\boldsymbol{b}[t]$, and the target portfolio $\boldsymbol{b}$ exceeds a predefined threshold $l \in \mathbb{R}^+$, that is, rebalance if any $|b_j - b_j[t]| > l$, $j = 1, \ldots, m$. For l, 50 uniform values are chosen in the range [0.01, 0.5] and the average wealth achieved by these 50 different algorithms for each l is presented. For "no-trade-2" and "no-trade-3," the algorithms rebalance if $\sum_j (b_j - b_j[t])^2 > \tau^2$, $\tau \in \mathbb{R}^+$. For τ, 50 uniform values in the range [0.1, 0.5] are chosen and the average wealth achieved by these 50 different algorithms for each k is presented. For all algorithms, we have used the rebalancing strategy under transaction costs introduced in Blum and Kalai (1998).

We next continue to simulate the performance of these algorithms applied to combination of three stocks. In Table 4.2, we present the achieved wealth of several algorithms with an initial wealth of 1 dollar over random sets of three stocks, where the results are averaged over 50 trials. In Table 4.2a, $\boldsymbol{b}$ is uniform, that is, $\boldsymbol{b} = [1/3 \;\; 1/3 \;\; 1/3]^T$ and in Table 4.2b, $\boldsymbol{b}$ is selected randomly for each trial.

As the final set of experiments, to demonstrate the progress of wealth gain, we have randomly selected a set of three stocks, for example, Iroquois–Ford–Schlum, and applied several different algorithms. In Figure 4.1, we demonstrate the wealth gain during a 22-year period for Cover's portfolio, Universal portfolio, buy-and-hold, and "Swt-10." For Figure 4.1a, $c = 0$ and for Figure 4.1b, $c = 0.01$.

5. CONCLUSION

In this paper, we investigated SCRPs under a competitive framework. We first provided a sequential universal portfolio whose achieved wealth is asymptotically as large as the wealth achieved by the best SCRP that is tuned to the individual sequence of the price relatives, which could only have been chosen in hindsight. We then extended this framework to the case when there are fixed transaction costs involved and presented a universal sequential portfolio achieving the wealth of the best SCRP with transaction costs. For both cases, the logarithm of the wealth ratio of these algorithms over the performance of the best SCRPs is at most $O(\ln(n))$. The universal portfolios are strongly sequential such that they do not require the knowledge of k or n a priori. We also provided explicit implementations of these algorithms with complexity linear in the data length n and compared the performance of the introduced algorithms with the performance of several well-known investment strategies.

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