Date: March 10, 2007, Saturday
Time: 13:00-15:00
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Math 206 Complex Calculus – Midterm Exam I – Solutions

Q-1) Find all the fourth roots of $\sqrt{3}\ i - 1$. Write the resulting numbers in rectangular form.

Answer: $\sqrt{3}\ i - 1 = 2\exp[i(\frac{2\pi}{3} + 2n\pi)],\ n \in \mathbb{Z}$. The fourth roots are $c_k = \sqrt[4]{2}\exp[i(\frac{\pi}{6} + \frac{k\pi}{2})]$ for $k = 0, 1, 2, 3.$, which gives: $c_0 = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$,

$c_1 = \sqrt[4]{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$,

$c_2 = \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$,

$c_3 = \sqrt[4]{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$.

Note for the curious: $c_0 = -.5946035575 + 1.029883572\ i$.

Q-2) Find all the values of $(-1-i)^{3+4i}$. Write the principal value in rectangular form.

Answer:

$$
\begin{aligned}
(-1-i)^{3+4i} &= \exp\left((3+4i)\log(-1-i)\right), \\
&= \exp\left((3+4i)(\ln\sqrt{2} + i[-\frac{3\pi}{4} + 2n\pi])\right),\ n \in \mathbb{Z}, \\
&= \exp\left((\frac{3}{2}\ln 2 + (3-8n)\pi) + i(2\ln 2 - \frac{9\pi}{4} + 6n\pi)\right).
\end{aligned}
$$

The principal value is obtained when $n = 0$ and it is given by

$$2\sqrt{2}e^{3\pi}\left(\cos(2\ln 2 - \frac{\pi}{4}) + i\sin(2\ln 2 - \frac{\pi}{4})\right).$$

Note here that $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ so we can use $\frac{\pi}{4}$. Again for the curious we note

$$\text{P.V.}(-1-i)^{3+4i} = 28909.33549 - 19815.99860i.$$

Q-3) Find a harmonic conjugate $v(x, y)$ for the function $u(x, y) = e^y \sin x + xy$ such that the analytic function $f(z) = u + iv$ takes the value $-2i$ at the origin.

Answer: u and v must satisfy the Cauchy-Riemann equations, $u_x = v_y$ and $v_x = -u_y$. Then

$$\begin{aligned}
v &= \int v_y\, dy = \int u_x\, dy = \int (e^y \cos x + y)\, dy = e^y \cos x + \frac{1}{2}y^2 + \phi(x). \\
v_x &= \frac{d}{dx}\left(e^y \cos x + \frac{1}{2}y^2 + \phi(x)\right) \\
&= -e^y \sin x + \phi'(x) = -u_y = -e^y \sin x - x. \\
\phi'(x) &= -x. \\
\phi(x) &= -\frac{1}{2}x^2 + C. \\
v(x,y) &= e^y \cos x + \frac{1}{2}y^2 - \frac{1}{2}x^2 + C. \\
f(0) &= i(1 + C) = -2i, \text{ so } C = -3.
\end{aligned}$$

Hence the required function is $v(x, y) = e^y \cos x + \frac{1}{2}y^2 - \frac{1}{2}x^2 - 3$.

Q-4) Let R be the rectangle whose corners are at the points $1 \pm i$, $6 \pm i$, and let ∂R denote the boundary of this rectangle, taken in the counterclockwise direction. Evaluate the following integral.

$$\int_{\partial R} \frac{1 + z + z^2}{(z-2)(z-5)^2}\, dz.$$

Answer: Cut the rectangle into two parts by a line perpendicular to the x-axis at $z = 3$, and call the left hand rectangle R_1 and the right hand rectangle R_2. Let $f_1(z) = \dfrac{1 + z + z^2}{(z-5)^2}$ and $f_2(z) = \dfrac{1 + z + z^2}{(z-2)}$. Using the Cauchy Integral Formula we have

$$\begin{aligned}
\int_{\partial R} \frac{1 + z + z^2}{(z-2)(z-5)^2}\, dz &= \int_{\partial R_1} \frac{1 + z + z^2}{(z-2)(z-5)^2}\, dz + \int_{\partial R_2} \frac{1 + z + z^2}{(z-2)(z-5)^2}\, dz \\
&= \int_{\partial R_1} \frac{f_1(z)}{(z-2)}\, dz + \int_{\partial R_2} \frac{f_2(z)}{(z-5)^2}\, dz \\
&= 2\pi i \left(f_1(2) + f_2'(5)\right) = 2\pi i \left(\frac{7}{9} + \frac{2}{9}\right) \\
&= 2\pi i.
\end{aligned}$$
