MATH 225 Linear Algebra and Differential Equations

Fall 2007 MATLAB Homework 3

Due date: Friday, December 7, 17:30

This homework will help you to gain more information on row-column subspaces and rank of a matrix. Moreover, we will cover orthogonality, which is a fundamental subject in linear algebra. We will also continue to explore numerical issues confronted in real-life problems, such as noise in a matrix.

1. For this problem, we need a matrix with some special features. Let's prepare this **A** matrix with the following commands:

```
m=4; n=6; r=2;
[U,S,V]=svd(10*rand(m,n));
for i=r+1:m
    S(i,i)=0;
end
A=U*S*V';
```

You will soon learn more about SVD decomposition of a matrix. Now let's analyze the resulting **A** matrix. Remember that your answers and comments to the following questions should rely on your Matlab computations, unless stated otherwise.

(a) Are the rows of matrix **A** independent? Find a basis for the row subspace of **A**.

(b) Are the columns of matrix **A** independent? Find a basis for the column subspace of **A**.

(c) Find the null space of **A**. Show numerically that this space is orthogonal to the row space of **A**. That is, use a computational step to show this fact.

(d) Find a subspace that is orthogonal to the column subspace of **A**. Again verify your result with a computational step.

(e) Let's add a small noise to our matrix using the command

```
noisyA=A+1e-6*rand(m,n).
```

Note that you cannot see the difference between these matrices if you display them in Matlab. But what is the rank of this noisy matrix? What was the rank of matrix **A**? Repeat for `m=10, n=15, r=3`. Comment on the results.

Hints: Use the `rref` and `null` commands. `rank` can also be helpful.

2. Lets denote the columns of an $n \times n$ **A** matrix with $a_1, a_2, \ldots, a_n$. That is

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}.$$

In many applications we want to find an orthogonal set $\{q_1, q_2, \ldots, q_n\}$ that spans the same column space of **A**. In this question, we will learn how to construct such a basis.

(a) Let's start with first two vectors a_1 and a_2. Let's construct two other vectors as

$$q_1 = \frac{a_1}{|a_1|} \text{ and } q_2 = \frac{a_2 - (q_1{}^T a_2)q_1}{|a_2 - (q_1{}^T a_2)q_1|}.$$

Show *analytically* that

(i) $q_1 \perp q_2$

(ii) span$\{q_1, q_2\}$ =span$\{a_1, a_2\}$.

(b) Generalize this idea to *n* vectors and write a Matlab function `orthogonalizeA` that takes a matrix **A** and returns another matrix **Q** whose column space is an orthogonal set that spans the same column space of **A**. Use your routine to find an orthogonal column basis for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{bmatrix}.$$

Hint: You can notice that, when constructing q_2, we actually subtract from a_2 the portion of a_2 that lies in the span of q_1, and then we *normalize* it. Similarly, to obtain q_3, we should subtract the *portions* that lie in the span of q_1, q_2 from a_3, and then normalize the resulting vector. And so forth for the others. Use the `norm` function to find the length of a vector.