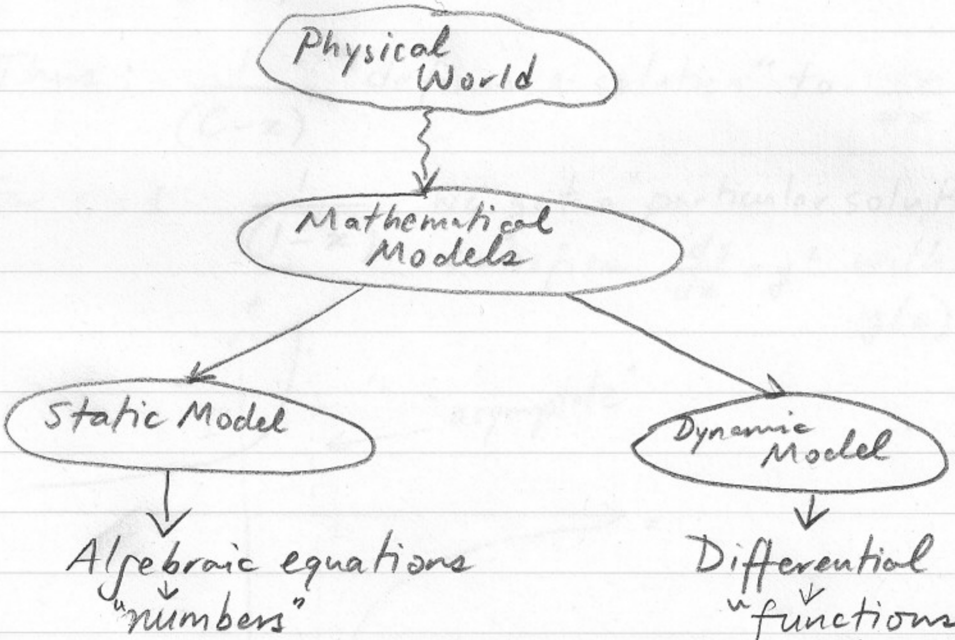
MATH 225, FALL '07

Linear Algebra and Differential Equations

Chapter 1: First-Order Differential Equations



ex's/ $\dfrac{dx}{dt} = x^2 + t^2$, $x(t)$: the unknown function of time

$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 7y = 0$, $y(x)$: the unknown function of x.

- The study of DE's has three pricipal goals:
  1. To discover the DE that describes a physical situation.
  2. To find an appropriate solution to that equations.
  3. To interpret the obtained solution.

ex/ $\dfrac{dy}{dx} = 2xy$ $\qquad \underbrace{y(x) = Ce^{x^2}}_{\text{family of solutions}} \Rightarrow \dfrac{dy}{dx} = 2xCe^{x^2} = 2xy.$

ex/ Let $P(t)$ be the size of a population with constant birth and death rates. Then:

$\dfrac{dP}{dt} = k \cdot P$ models population size as a function of time.

$p(t) = Ce^{kt}$ solves the DE where C is the size of population at $t=0$.

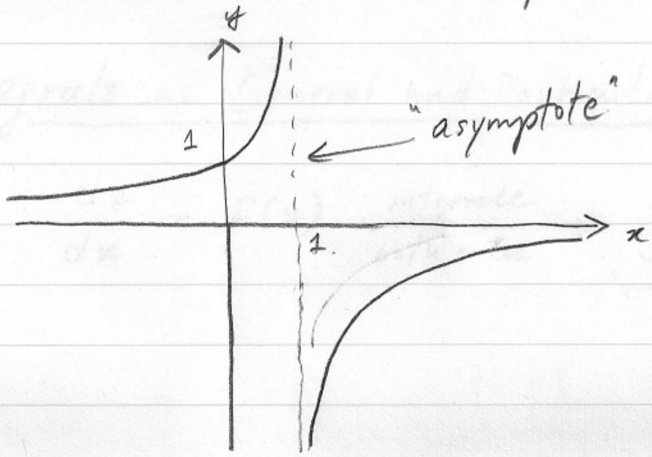
Examples and Terminology

1. If C is a constant and $y(x) = \dfrac{1}{(C-x)}$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2 \quad \text{if } x \neq C.$$

Thus: $\dfrac{1}{(C-x)}$ "defines a solution" to $\dfrac{dy}{dx} = y^2$ for $x \neq C$.

For $C=1$: $\dfrac{1}{(1-x)}$ we get a "particular solution", that satisfies $\dfrac{dy}{dx} = y^2$ with initial condition $y(0)=1$



2. $$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$
4^{th} Order DE

The general Form: $F(\underbrace{x, y, y', \cdots, y^{(n)}}_{(n+2)\text{ variables}}) = 0 \quad (\star)$

Solution: $u(x)$ solves $(\star)$ on an interval I if:
$u, u', \dots, u^{(n)}$ exist on I and
$F(x, u, u', \dots, u^{(n)}) = 0$ for $x \in I$.

3. $$y(x) = A\cos 3x + B\sin 3x$$
$$y'(x) = -3A\sin 3x + 3B\cos 3x$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x).$$

$\Rightarrow \underbrace{y'' + 9y = 0}_{F(x,y,y',y'')}$ or $\underbrace{y'' = -9y}_{y'' = G(x,y,y')}$

implicit form "explicit" or "normal Form"

4. If unknown function is a function of more than one variable typically we have "partial DE":

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2} \quad : \ 2^{nd} \text{ order PDE.}$$

5. $$\underbrace{\frac{dy}{dx} = G(x,y) \ : \text{ First-order DE with initial condition: } y(x_0) = y_0.}_{\text{Initial value problem.}}$$

Integrals as General and Particular Solutions

$$\frac{dy}{dx} = f(x) \xrightarrow[\text{both sides}]{\text{integrate}} \underbrace{\int \left(\frac{dy}{dx}\right) dx}_{y(x)} = \int f(x)\,dx$$

$$y(x) = \underbrace{\int f(x)\,dx + C}_{\text{general solution.}}$$

If $G(x) = \int f(x)\,dx$, then $y(x) = G(x) + C$.

ex/ $\dfrac{dy}{dx} = 2x+3$, $y(1) = 2$

$y(x) = \int (2x+3)\,dx + C = x^2 + 3x + C$

$y(1) = 1 + 3 + C = 2 \Rightarrow C = -2$

Hence: $y(x) = x^2 + 3x - 2$ is the particular solution to the given initial value problem.

ex/ $$\frac{d^2y}{dx^2} = g(x) \xrightarrow{\text{Integrate}} \frac{dy}{dx} = \underbrace{\int g(x)\,dx}_{G(x)} + C_1 \xrightarrow{\text{Integrate}} y(x) = \int G(x)\,dx + C_1 x + C_2$$

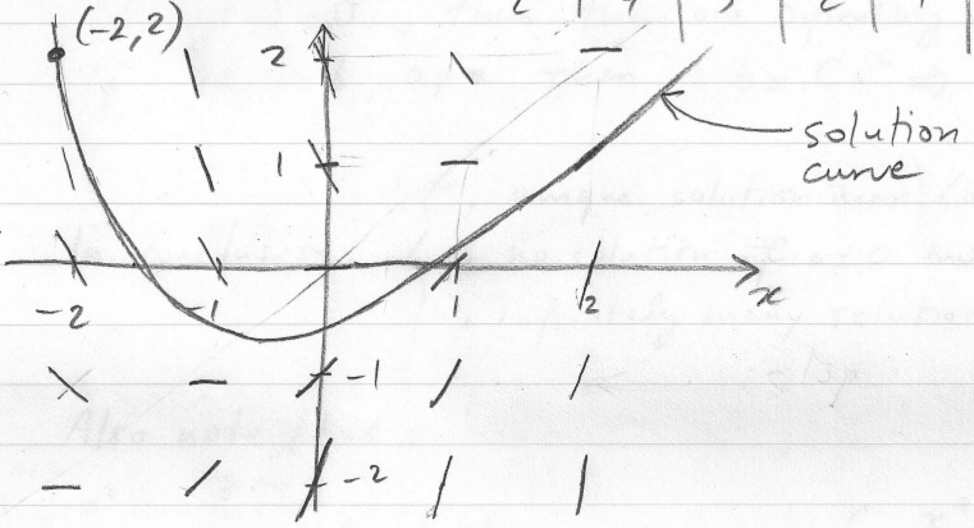
Slope Fields and Solution Curves

$\frac{dy}{dx} = f(x,y)$ specifies the derivative of $y(x)$ at (x,y) coordinate:

ex/ $\frac{dy}{dx} = x - y$

$x \backslash y$	-2	-1	0	1	2
-2	0	-1	-2	-3	-4
-1	1	0	-1	-2	-3
0	2	1	0	-1	-2
1	3	2	1	0	-1
2	4	3	2	1	0

slope field



if the solution passes from $(-2,2)$ i.e. $y(-2) = 2$.
(Note that the solution is: $y(x) = x - 1 + C_1 e^{-x}$.)

Existence and Uniqueness of Solutions

Consider $\frac{dy}{dx} = f(x,y)$ and $y(a) = b$

If $\frac{\partial}{\partial y} f(x,y) = D_y f(x,y)$ and $f(x,y)$ are both continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior, then for some open interval I containing a, the initial value problem has one and only one solution on the interval I.

ex/ $x \dfrac{dy}{dx} = 2y$; $f(x,y) = 2\dfrac{y}{x}$

$$\frac{\partial f(x,y)}{\partial y} = \frac{2}{x}$$

Hence, if $x \neq 0$, both $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous

Therefore, the DE has a unique solution for any open interval not containing $x=0$.

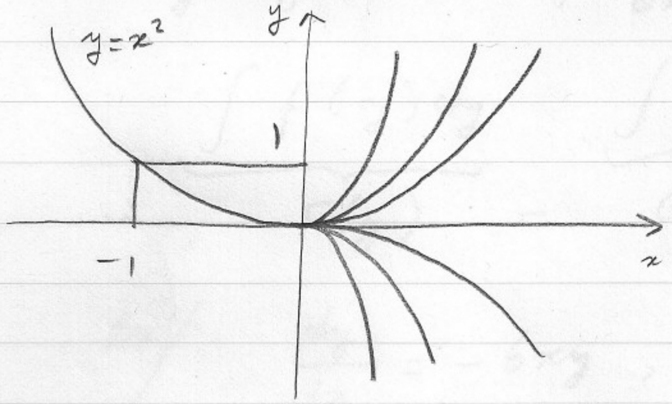
Actually $y(x) = Cx^2$ solves the DE.

If $y(0)=0$, then there are infinitely many solutions.

If $y(a)=b$, $a\neq 0$ then; $b = Ca^2 \Rightarrow C = \dfrac{b}{a^2} \Rightarrow y(x) = \dfrac{b}{a^2}x^2$ is the unique solution.

In conclusion:
- unique solution near (a,b) if $a\neq 0$
- no solution if $a=0$ but $b\neq 0$
- infinitely many solutions if $a=b=0$.

Also note that:



$$y(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ Cx^2 & \text{if } x > 0 \end{cases}$$

is continuous and satisfies $x\dfrac{dy}{dx} = 2y$ and $y(-1) = 1$.

Hence unique solution in the vicinity of $(-1,1)$ is guaranteed but the solution may not be unique in larger intervals.

Application Note

Check the course web page for computerized methods that generate slope fields and solution curves.

Separable Equations and Applications

$$\frac{dy}{dx} = H(x,y) \text{ is separable if } H(x,y) = g(x)h(y) = g(x)/f(y)$$

where $f(y) = \dfrac{1}{h(y)}$.

Separable equations can be solved easily as:

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \Rightarrow f(y)\frac{dy}{dx} = g(x)$$

integrating both sides yields:

$$\int f(y(x))\frac{dy(x)}{dx}\,dx = \int g(x)\,dx + C$$

$y = y(x) \Rightarrow dy = \frac{dy}{dx}dx$ gives:

$$\underbrace{\int f(y)\,dy}_{F(y)} = \underbrace{\int g(x)\,dx}_{G(x)} + C$$

$$F(y) = G(x) + C$$

ex/ $\dfrac{dy}{dx} = \underbrace{-6xy}_{\text{separable.}}$, $y(0) = 7$

$g(x) = -6x$, $f(y) = \dfrac{1}{y}$

$$\int \frac{1}{y}\,dy = \int -6x\,dx + C$$

$$\ln|y| = -3x^2 + C$$

$$|y| = e^{-3x^2} \cdot e^{C} = Ae^{-3x^2}$$

$y(0) = 7 \Rightarrow 7 = A \Rightarrow y(x) = 7e^{-3x^2}$.

Implicit Solutions and Singular Solutions

Def^{n} $K(x,y)=0$ is an implicit solution of a DE if a solution of DE satisfies $K(x,y)=0$.

ex/ $x + y\dfrac{dy}{dx} = 0$, $y(0)=2$; DE with initial condition

$y(x) = \sqrt{4-x^2}$ is the solution

$\Rightarrow$ $\underbrace{x^2+y^2-4}_{K(x,y)}=0$ is an implicit solution

Note that $y(x) = -\sqrt{4-x^2}$ is also a solution of $K(x,y)=0$ <u>but</u> it is not a solution to initial value problem.

ex/ $\dfrac{dy}{dx} = 6x\,(y-1)^{2/3}$, $y(1)=1$.

Note that this is a separable DE.

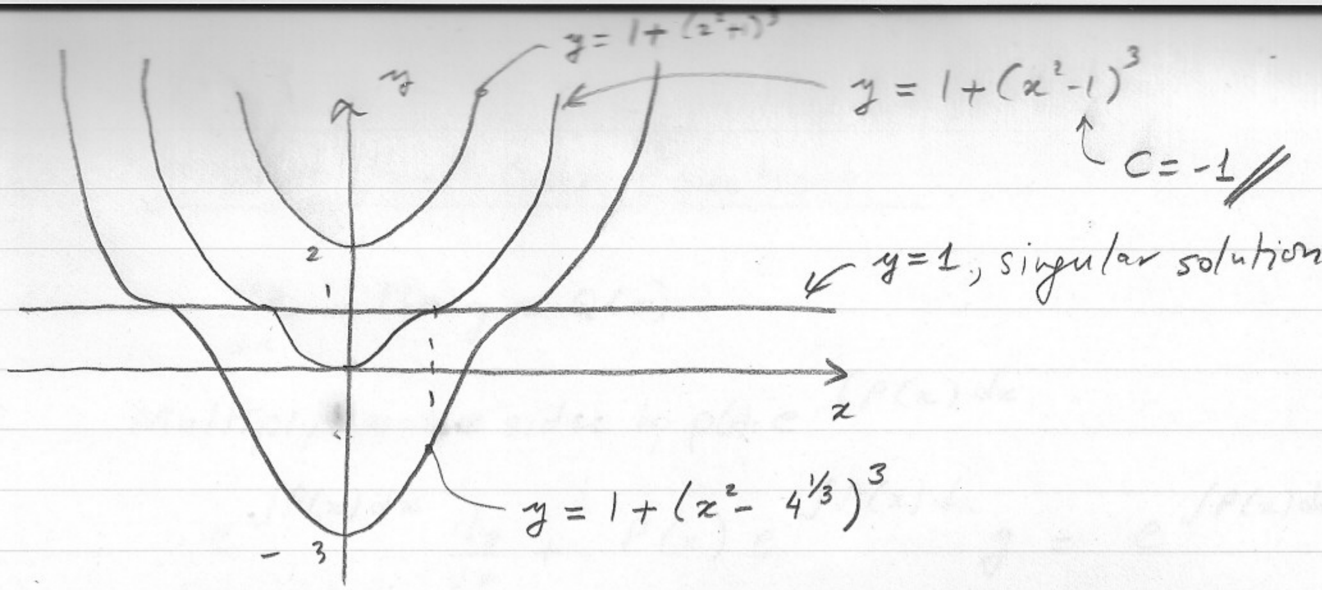
$$\frac{1}{(y-1)^{2/3}}\frac{dy}{dx} = 6x \xrightarrow[\text{both sides}]{\text{integrate}} \int \frac{1}{(y-1)^{2/3}}\,dy = \int 6x\,dx + C'$$

$$\Rightarrow 3(y-1)^{1/3} = 3x^2 + C' \longrightarrow (y-1)^{1/3} = x^2 + C$$

$$\Rightarrow y(x) = \underbrace{1+(x^2+C)^3}_{\text{General solution}}$$

$$y(1)=1 \Rightarrow 1+(1+C)^3 = 1 \Rightarrow C=-1$$

Hence $y(x) = 1+(x^2-1)^3$ is a particular solution.
But $y(x)=1$ is also a particular solution that is not member of the general solution family. Thus it is called as a <u>singular solution</u>.



$y(a) = b$ yields unique solution if :

$$1+(x^2+C)^3 > 1 \quad \text{for all } x$$
$$\Rightarrow \quad (x^2+C)^3 > 0 \quad \text{''}$$
$$\Rightarrow \quad x^2+C > 0 \quad \text{''}$$
$$\Rightarrow \quad C > 0$$

Hence : $1+(a^2+C)^3 = b$ should be satisfied by a $C>0$:

$$\Rightarrow \quad C = (b-1)^{1/3} - a^2 > 0$$

otherwise, there will be <u>infinitely many solutions!</u>

<u>Natural Growth and Decay</u>

ex's/ Compound Interest : $A(t)$: amount of money at time t in years.
r : annual interest rate.

Suppose that the interest is compound continuously ;

$$A(t+\Delta t) \simeq A(t) + r\Delta t \cdot A(t)$$

$$\lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = r A(t) \Rightarrow \frac{dA(t)}{dt} = r A(t)$$

Note that $A(t) = A(0)e^{rt}$ is the general solution.

Linear First-Order Equations

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Multiply both sides by $\rho(x) = e^{\int P(x)\,dx}$:

$$\underbrace{e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)\,e^{\int P(x)\,dx}\,y} = e^{\int P(x)\,dx}\,Q(x).$$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\,y\right] = Q(x)\,e^{\int P(x)\,dx}$$

Hence:

$$e^{\int P(x)\,dx}\,y(x) = \int Q(x)\,\rho(x)\,dx + C$$

$$y(x) = \underbrace{e^{-\int P(x)dx}\left[\int Q(x)\,\rho(x)\,dx + C\right]}_{\text{General Solution.}}$$

Comments ; GS is unique if $P(x)$ & $Q(x)$ are cont. on I, solution interval.

ex/ $\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}$, $y(0) = -1$.

$$P(x) = -1 \;, \quad Q(x) = \frac{11}{8}e^{-x/3}$$

$$\rho(x) = e^{\int P(x)\,dx} = e^{-\int dx} = e^{-x}.$$

$$\Rightarrow \quad \frac{d}{dx}\left[\rho(x)\,y\right] = \rho(x)\,Q(x) = \frac{11}{8}e^{-x/3}e^{-x} = \frac{11}{8}e^{-\frac{4x}{3}}$$

$$\Rightarrow \quad \rho(x)\,y(x) = -\frac{33}{32}e^{-4x/3} + C$$

$$y(x) = -\frac{33}{32}e^{-x/3} + C\,e^{x} \; : \text{General solution}$$

$$y(0) = -1 \Rightarrow -\frac{33}{32} + C = -1 \Rightarrow C = \frac{1}{32}$$

Finally: $y(x) = \frac{1}{32}\left[e^{x} - 33\,e^{-x/3}\right]$: particular solution.

In the case of initial value problems:

$$\frac{dy}{dx} + P(x)\, y = Q(x) \qquad y(x_0) = y_0$$

can choose $\quad \rho(x) = e^{\int_{x_0}^{x} P(x')\,dx'}, \quad \rho(x_0) = 1$

Then $\quad y(x) = \dfrac{1}{\rho(x)} \left[y_0 + \int_{x_0}^{x} \rho(x')\, Q(x')\, dx' \right]$

is the desired particular solution.

ex/ $\quad x^2 \dfrac{dy}{dx} + xy = x^3 \quad , \; y(1) = 2$

standard form $\quad \dfrac{dy}{dx} + \dfrac{1}{x} y = x \quad , \quad P(x) = \dfrac{1}{x}, \; Q(x) = x$

$$\rho(x) = e^{\int_1^x \frac{1}{x'} dx'} = e^{\ln x} = x$$

$$y(x) = \frac{1}{x}\left[2 + \int_1^x x' \cdot x' \, dx' \right]$$

$$= \frac{1}{x}\left[2 + \frac{1}{3}(x^3 - 1) \right]$$

$$= \frac{5}{3x} + \frac{x^2}{3} . \qquad \left(y(1) = \frac{5}{3} + \frac{1}{3} = 2 \right) \checkmark$$

Substitution Methods and Exact Equations

In case DE is not separable or linear, substitution methods might help to transform the DE into a new form that we know how to solve.

ex/ $\quad \dfrac{dy}{dx} = (x+y+3)^2 \quad , \quad$ let $v = x+y+3$

then $y = v - x - 3$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1,$$

$$\Rightarrow \text{DE} \longrightarrow \frac{dv}{dx} - 1 = v^2 \Rightarrow \overbrace{\frac{dv}{dx} = v^2+1}^{\text{separable}}$$

$$\Rightarrow \frac{1}{v^2+1}\frac{dv}{dx} = 1$$

$$\int \frac{1}{1+v^2}\,dv = \int dx + C$$

$$\tan^{-1} v = x + C \Rightarrow v = \tan(x+C).$$

$$\Rightarrow y(x) = \tan(x+C) - x - 3.$$

<u>Homogeneous Equations</u>

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \quad \text{use substitution: } v = \frac{y}{x}$$

$$\Rightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

Hence:

$$x\frac{dv}{dx} + v = F(v) \Rightarrow x\frac{dv}{dx} = F(v) - v$$

$$\Rightarrow \frac{1}{F(v)-v}\frac{dv}{dx} = \frac{1}{x}$$

separable DE in v. solve for v and find $y = vx$.

ex/ $$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

$$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = \overbrace{2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)}^{F\left(\frac{y}{x}\right)}$$

(Note that $F(v) = \frac{2}{v} + \frac{3}{2}v$)

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Then, DE becomes: $x\dfrac{dv}{dx} + v = \dfrac{2}{v} + \dfrac{3}{2}v$

$$\Rightarrow \quad x\frac{dv}{dx} = \underbrace{\frac{2}{v} + \frac{1}{2}v}_{\frac{4+v^2}{2v}} \quad \text{Separable form.}$$

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{1}{x}\,dx + C_1$$

$$\ln(4+v^2) = \ln|x| + C_1$$

$$4+v^2 = |x|e^{C_1}$$

Hence $\dfrac{y^2}{x^2} + 4 = e^{C_1}|x| \Rightarrow y^2 + 4x^2 = k\cdot x^3$, $k>0$ for $x>0$, $k<0$ for $x<0$

$$y(x) = \pm\sqrt{kx^3 - 4x^2} \quad \text{if} \quad kx > 4 \Rightarrow x > \frac{4}{k}, \; k>0$$

$$x < \frac{4}{k}, \; k<0.$$

ex/ $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}$, $y(x_0) = 0$, $x_0 > 0$.

if $x>0$: $\dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{1 - \left(\dfrac{y}{x}\right)^2}$ (Homogeneous)

Let $v = \dfrac{y}{x} \Rightarrow \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

Transformed Eqn:

$$x\frac{dv}{dx} + v = v + \sqrt{1-v^2}$$

Hence: $\displaystyle\int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx + C_1$

$$\Rightarrow \quad \sin^{-1}(v) = \ln|x| + C_1$$

$$v = \sin(\ln|x| + C_1)$$

$$y(x) = x \cdot \sin(\ln|x| + C_1)$$

$$y(x_0) = 0 \Rightarrow x_0 \sin(\ln|x_0| + C_1) = 0$$

$$C_1 = -\ln|x_0|$$

Hence: $y(x) = x \sin(\ln|x| - \ln|x_0|)$

$$y(x) = x \sin\left(\ln\left|\frac{x}{x_0}\right|\right).$$

Since both x & $x_0 > 0$;

$$y(x) = x \sin\left(\ln \frac{x}{x_0}\right).$$

Note that $\sin^{-1}(v) = \ln \frac{x}{x_0}$ and $|\sin^{-1}(v)| \le \frac{\pi}{2}$

Thus: $\left|\ln \frac{x}{x_0}\right| \le \frac{\pi}{2}$ or $e^{-\pi/2} \le \frac{x}{x_0} \le e^{\pi/2}$

Hence, the solution interval is:

$$x_0 e^{-\pi/2} \le x \le x_0 e^{\pi/2}.$$

Bernoulli Equations

$$\underbrace{\frac{dy}{dx} + P(x)y = Q(x)y^n}.$$

for $n = 0$ or $n = 1$ it is "linear DE"

otherwise it is non-linear DE. It is a generalization of linear DEs.

Make the substitution $v = y^{1-n}$

Then:

$$\frac{dv}{dx} = (1-n)\, y^{-n} \cdot \frac{dy}{dx}$$

Hence: $$\frac{dy}{dx} = \frac{1}{(1-n)}\, y^{n} \frac{dv}{dx}$$

The transformed DE is:

$$\frac{1}{(1-n)}\, y^{n} \frac{dv}{dx} + P(x)\, y = Q(x)\, y^{n}$$

$$\Rightarrow \frac{1}{(1-n)} \frac{dv}{dx} + P(x) \cdot v = Q(x)$$ Linear DE in v.

Solve for v and backsubstitute to find y.

ex/ $x \frac{dy}{dx} + 6y = 3xy^{4/3}$ is Bernoulli with $n = 4/3$.

$$v = y^{1-n} = y^{1-4/3} = y^{-1/3}$$

$$\Rightarrow \frac{1}{1-4/3} \frac{dv}{dx} + \frac{6}{x} v = 3$$

$$\frac{dv}{dx} - \frac{2}{x} v = -1$$ linear DE in v

Use integrating factor $\rho = e^{\int -\frac{2}{x} dx} = e^{-2\ln|x|} = e^{-\ln x^2} = \frac{1}{x^2}$.

Hence: $$\frac{1}{x^2} \frac{dv}{dx} - \frac{2}{x^3} v = -\frac{1}{x^2}$$

or $$D_x\left[\frac{1}{x^2} v\right] = -\frac{1}{x^2}$$

$$\Rightarrow \frac{1}{x^2} v = \frac{1}{x} + C \Rightarrow v = x + Cx^2.$$

yielding $y = v^{-3} = \dfrac{1}{(x+Cx^2)^3}$.

ex/ $2x e^{2y} \dfrac{dy}{dx} = 3x^4 + e^{2y}$ non- $\left\{\begin{array}{l}\text{separable}\\ \text{linear}\\ \text{Bernoulli}\end{array}\right.$

let $v = e^{2y}$ then $\dfrac{dv}{dx} = 2\dfrac{dy}{dx} e^{2y}$

Hence, the transformed equation becomes:

$$x\frac{dv}{dx} = 3x^4 + v \text{ ; linear DE.}$$

$$\frac{dv}{dx} - \frac{1}{x} v = 3x^3$$

integrating factor $\rho = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$.

$$\Rightarrow \underbrace{\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2} v}_{D_x\left[\frac{1}{x} v\right]} = 3x^2$$

$$D_x\left[\frac{1}{x} v\right] = 3x^2$$

$$\Rightarrow \frac{1}{x} v(x) = x^3 + C_1 \Rightarrow v(x) = x^4 + x C_1$$

Hence $y(x) = \dfrac{1}{2} \ln\left|x^4 + xC_1\right|$.

Exact DEs

Solutions of 1st Order DEs can be implicitly given as:

$F(x, y(x)) = C_1$ Differentiating both sides wrt x

$$\Rightarrow \underbrace{\frac{\partial F}{\partial x}}_{M(x,y)} + \underbrace{\frac{\partial F}{\partial y}}_{N(x,y)}\frac{dy}{dx} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Thus if we can find F such that:

$$\frac{\partial F}{\partial x} = M \qquad \frac{\partial F}{\partial y} = N \quad \text{then}$$

$$F(x,y) = C$$

implicitly defines a general solution.

A necessary condition that exact DEs must satisfy:

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \implies \frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y).$$

ex/ $\underbrace{y^3}_{M(x,y)} dx + \underbrace{3xy^2}_{N(x,y)} dy = 0$

$$\frac{\partial M}{\partial y} = 3y^2 \quad , \quad \frac{\partial N}{\partial x} = 3y^2$$

Also $F(x,y) = xy^3 \longrightarrow M = \frac{\partial F}{\partial x} = y^3$

$$N = \frac{\partial F}{\partial y} = 3xy^2.$$

Hence: $xy^3 = C$ or $y = kx^{-1/3}$ is a general solution.

Theorem 1: Criterion for Exactness

$M(x,y)$ and $N(x,y)$ are continuous and have continuous first order partial derivatives in $R: a<x<b,\ c<y<d$. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact in R iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof

exact $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ✓

Need to show: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ exact.

Note that if

1. $F(x,y) = \int M(x,y)\,dx + g(y) \Longrightarrow \dfrac{\partial F}{\partial x} = M$

Want to choose $g(y)$ so that:

$$\frac{\partial F}{\partial y} = N = \frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{d}{dy}g(y)$$

$$\Rightarrow \frac{dg}{dy} = N - \frac{\partial}{\partial y}\int M(x,y)\,dx.$$

Want to show that if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then

$N - \dfrac{\partial}{\partial y}\int M(x,y)\,dx$ is a function of y alone

This can be concluded if:

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right] = 0 \text{ in } R.$$

$$= \frac{\partial}{\partial x}N(x,y) - \frac{\partial}{\partial y}M(x,y) = 0 \text{ by hypothesis.}$$

Thus:

$$F(x,y) = \int M(x,y)\,dx + \int\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right)dy.$$

is the desired function.

ex/ $\overbrace{(6xy - y^3)}^{M(x,y)}\,dx + \overbrace{(4y + 3x^2 - 3xy^2)}^{N(x,y)}\,dy = 0$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x} \implies \text{DE is exact in } \mathbb{R}^2.$$

$$\frac{\partial F}{\partial x} = M(x,y) = 6xy - y^3$$

$$F(x,y) = 3x^2y - xy^3 + g(y)$$

$$\frac{\partial F}{\partial y} = 3x^2 - 3xy^2 + \frac{d}{dy}g(y) = 4y + 3x^2 - 3xy^2$$

$$\implies \frac{d}{dy}g(y) = 4y \implies g(y) = 2y^2 + C_1$$

Thus: $F(x,y) = 3x^2 - 3xy^2 + 2y^2 = C$ in $\mathbb{R}^2$.

Reducible Second Order Eqn's

$F(x, y, y', y'') = 0$ General implicit form.

If either y or x is missing, can be reduced to 1st order.

case 1 y is missing:

$F(x, y', y'') = 0$, substitute $p = y' = \frac{dy}{dx}$; $y'' = \frac{dp}{dx}$

in p: $F(x, p, p')$ is first order.

Solve for p and obtain y from $\frac{dy}{dx} = p$.

ex/ $F(x,y,y',y'') = 0 = xy'' + 2y' - 6x$.

y is missing. $p = y' \longrightarrow F(x,p,p') = xp' + 2p - 6x$

$p' + \frac{2}{x} p = 6$ linear DE

integrating factor $\rho = e^{\int \frac{2}{x} dx} = x^2$.

Then: $\underbrace{x^2 p' + 2xp}_{} = 6x^2$

$D_x(x^2 p) = 6x^2$

$\Rightarrow x^2 p = 2x^3 + C_1 \Rightarrow p = 2x + \frac{C_1}{x^2}$

$\Rightarrow \frac{dy}{dx} = 2x + \frac{C_1}{x^2} \longrightarrow y = x^2 - \frac{C_1}{x} + C_2$

Case 2 Independent variable x is missing:

$F(y, y', y'') = 0$

Then: $p = y'$, $y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{p} = p\frac{dp}{dy}$

$F(y, p, p\frac{dp}{dy}) = 0$: First order DE in p as a function of y.

ex/ $yy'' = (y')^2$.

$p(y) = \frac{dy}{dx}$, $y'' = p\frac{dp}{dy}$, assume that $y > 0, y' > 0$

$\Rightarrow yp\frac{dp}{dy} = p^2$ (separable 1st order DE)

$\Rightarrow \int \frac{1}{p} dp = \int \frac{1}{y} dy + C_1 \Rightarrow \ln p = \ln y + C$

$p = C_1 y$, $(C_1 = e^C)$

Hence: $\frac{dy}{dx} = C_1 y$ (separable DE)

$$\int \frac{1}{y}\,dy = \int C_1\,dx + C_2$$

$$\ln y = C_1 x + C_2$$

$$y(x) = e^{C_1 x + C_2} = Ae^{Bx}.$$

Note that Ae^{Bx} satisfies the DE for any value of A & B. Therefore, we get the most general solution.

Chapter 3: Linear Systems and Matrices

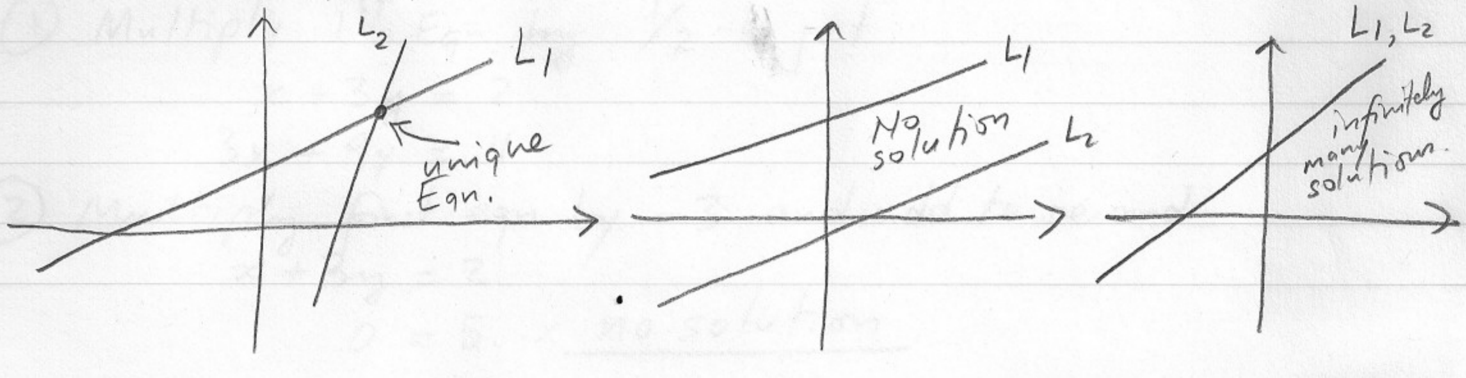
Interested in multiple variable algebraic equations of the form:

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\} \text{ linear in } x \text{ and } y$$

ex's/ 1. $\left.\begin{aligned} 2x - y &= 5 \\ x + 2y &= 0 \end{aligned}\right\}$ $x=2, y=-1$ solves the equation. It is consistent.

2. $\left.\begin{aligned} x + y &= 1 \\ 2x + 2y &= 3 \end{aligned}\right\}$ No solution! It is "inconsistent"

In case of systems with 2 variables (unknowns)



Method of Elimination

Use one of the equations to solve an unknown in terms of the other and then replace it in the second equation to find an equation with only one unknown.

ex/

$$5x+3y=1 \longrightarrow \underbrace{5(8+2y)+3y=1}$$

$$x-2y=8 \longrightarrow x=8+2y$$

$$13y=-39$$

$$y=-3$$

$$x=2$$

This procedure can be equivalently implemented by:

① interchange equations

$$x-2y=8$$
$$5x+3y=1$$

② Multiply the first eqn by -5 and add to the second:

$$x-2y=8$$
$$13y=-39$$

③ Solve for y: $y=-39/13=-3$

④ Substitute y in the first eqn to solve for x:

$$x+6=8 \Rightarrow x=2.$$

ex/

$$2x+6y=4$$
$$3x+9y=11$$

① Multiply 1st Eqn by ½ to get:

$$x+3y=2$$
$$3x+9y=11$$

② Multiply first eqn by -3 and add to second

$$x+3y=2$$
$$0=5 \quad \times \text{ no solution}$$

ex/ $\left.\begin{aligned} 2x+6y&=4 \\ 3x+9y&=6 \end{aligned}\right\} \to \left.\begin{aligned} x+3y&=2 \\ 3x+9y&=6 \end{aligned}\right\} \to \begin{aligned} x+3y&=2 \\ 0&=0. \end{aligned}$

Infinitely many solutions. Solutions lie on $x+3y=2$ line.

Can express the solutions in parametric form as:

$y=t, \quad x=2-3t, \quad t\in\mathbb{R}.$

parameter

ex/

$$\begin{aligned} 3x-8y+10z &= 22 \\ x-3y+2z &= 5 \\ 2x-9y-8z &= -11 \end{aligned}$$

① Interchange E1 & E2:

$$\begin{aligned} x-3y+2z &= 5 \\ 3x-8y+10z &= 22 \\ 2x-9y-8z &= -11 \end{aligned}$$

② Add $-3E_1$ to E_2 and $-2E_1$ to E_3:

$$\begin{aligned} x-3y+2z &= 5 \\ y+4z &= 7 \\ -3y-12z &= -21 \end{aligned}$$

③ To eliminate y from E3, add $3E_2$ to E_3

$$\begin{aligned} x-3y+2z &= 5 \\ y+4z &= 7 \\ \underbrace{0=0}_{} & \end{aligned}$$

consistent, but infinitely many solutions

parameter

④ $z=t$; $y=7-4t$; $x = 21-12t-2t+5$
$= 26-14t.$

Matrices and Gaussian Elimination

General form of Linear System of Equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Can conveniently represent the system as:

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

where
$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns} \leftarrow \text{number of unknowns}}\left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}}\right\} m \text{ rows} \leftarrow \text{number of Equations}$$

$m \times n$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Gaussian elimination can be operated on the "augmented" coefficient matrix: $[\underline{\underline{A}}\ \underline{b}]$.

ex/
$$\begin{aligned} 2x_1 + 3x_2 - 7x_3 + 4x_4 &= 6 \\ x_2 + 3x_3 - 5x_4 &= 0 \\ -x_1 + 2x_2 \qquad\quad - 9x_4 &= 17 \end{aligned}$$

$$[\underline{\underline{A}}\ \underline{b}] = \begin{bmatrix} 2 & 3 & -7 & 4 & 6 \\ 0 & 1 & 3 & -5 & 0 \\ -1 & 2 & 0 & -9 & 17 \end{bmatrix}$$

Elementary Row Operations

Elimination operations on System of Eqn's $\equiv$ Elementary row operations on $[\underline{\underline{A}}\ \underline{b}]$.

Def'n Elementary row operations:

1. Multiply a row of $[\underline{\underline{A}}\ \underline{b}]$ a nonzero constant
2. Interchange 2 rows of $[\underline{\underline{A}}\ \underline{b}]$
3. Add a constant multiple of a row of $[\underline{\underline{A}}\ \underline{b}]$ to another row.