ex/
$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 + 8x_2 + 7x_3 = 20$$
$$2x_1 + 7x_2 + 9x_3 = 23$$

$$[\underline{\underline{A}}\ \underline{b}] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 3 \end{aligned}\right\} \text{unique solution.}$$

Back substitute x_3 in E_2 to get $x_2 = -2$, then back substitute x_2 & x_3 in E_1 to get $x_1 = 5$.

Theorem: If two equation systems obtained from one another by a finite sequence of elementary row operations, then their solution set are the same.

Def'n Echelon matrix:

$\underline{\underline{E}}$ is called an echelon matrix if:

(i) If $\underline{\underline{E}}$ has a row of zeros, it lies beneath every row that contains non-zero element(s).

(ii) A non zero row of $\underline{\underline{E}}$ has its first non zero element to the right of the first nonzero elements of the previous rows.

ex/
$$\underline{\underline{E}} = \begin{bmatrix} 0 & 3 & 0 & -1 & 4 & 0 \\ 0 & 0 & 2 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 4 & 3 & 1 & 0 \\ 2 & 5 & 0 & 2 \\ 10 & 0 & 3 & 4 \end{bmatrix}$$

Not Echelon

This entry should be zero

Def'n If a matrix is in Echelon form then:
(i) Those variables that correspond to columns containing leading entries are called leading variables
(ii) All other variables are called free variables.

Back Substitution Algorithm

Given $\underline{\underline{A}}\,\underline{x} = \underline{b}$,

S_1: Form augmented coefficient matrix $[\underline{\underline{A}} \,\vdots\, \underline{b}]$
S_2: By a sequence of elementary row operations reduce the augmented coefficient matrix into echelon form
S_3: Identify the free variables
S_4: Set each free variable equal to an arbitrary parameter
S_5: Solve the final non-zero equation for its leading variable
S_6: Substitute the result into the next-to the last equation and then solve for its leading variable
S_7: Work upward through the system of equations until all leading variables are determined

ex/
$$x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$$
$$x_3 + 2x_5 = -3$$
$$x_4 - 4x_5 = 7$$

$$[\underline{\underline{A}} \,\vdots\, \underline{b}] = \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

leading variables: x_1, x_3, x_4 , Free-variables: x_2, x_5.

Set $x_2 = s$, $x_5 = t$ (Arbitrary parameters)

Hence: $x_4 = 7 + 4x_5 = 7 + 4t$
$x_3 = -3 - 2x_5 = -3 - 2t$
$x_1 = 10 + 2x_2 - 3x_3 - 2x_4 - x_5$
$= 10 + 2s - 3(-3-2t) - 2(7+4t) - t = 5 + 2s - 3t$

Thus, the solution in parametric form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5+2s-3t \\ s \\ -3-2t \\ 7+4t \\ t \end{bmatrix},$$

for s and t are real valued free parameters. Any choice for s and t generates a solution to the system.

Gaussian Elimination Algorithm

1. Locate the first column of $\underline{\underline{A}}$ that contains a non-zero element.
2. If the first entry in this column is zero, interchange the first row of $\underline{\underline{A}}$ with a row in which the corresponding entry is nonzero.
3. Zero out the entries below this nonzero entry by elementary row operations
4. Repeat this procedure until an echelon form is obtained.

Ex/
$$x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$$
$$2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7$$
$$3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27$$

Augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{bmatrix} \xrightarrow{\text{Swap}(R_2,R_3)} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{bmatrix} \xrightarrow{-R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix}$$

x_1, x_3, x_4 leading variables x_2, x_5 free variables.
Same solution family as in the previous example.

Reduced Row-Echelon Matrices

The Gaussian elimination algorithm results in different echelon forms depending on the sequence of row operations used. Among all possible echelon forms, the reduced-row echelon matrix is unique for a given matrix.

Def'n Reduced echelon matrix

$\underline{\underline{E}}$ is a reduced row echelon matrix if:

1. It is an echelon matrix
2. Each leading entry is 1
3. Each leading entry of $\underline{\underline{E}}$ is the only nonzero entry in its column.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is not in reduced row echelon form. it violates property 2 and 3.

$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ is in reduced row echelon form.

Gauss-Jordan Elimination {to transform a matrix into its reduced row echelon form}

1. Transform $\underline{\underline{A}}$ into its echelon form by Gaussian elimination
2. Divide each nonzero row with its leading entry.
3. Use elementary row operations to zero out the remaining nonzero entries on the leading variable columns.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{-2R_1+R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix}$

$\xrightarrow{-3/2R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ Echelon Form.

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{3R_3+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$ Reduced Row Echelon Form.

ex/

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 12 \\ x_1 + 2x_2 \quad\quad + 5x_4 &= 17 \\ 3x_1 + 2x_2 + 4x_3 - x_4 &= 31 \end{aligned}$$

Augmented Coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 1 & 2 & 0 & 5 & 17 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Reduced row echelon form

x_1, x_2 are leading variables

x_3, x_4 are free variables

Set: $x_3 = s$, $x_4 = t$

Then: $x_1 = -2s + 3t + 7$

$x_2 = s - 4t + 5$

Hence, the solution set is of infinite size parametrized by s and t.

The Three Possibilities

Gauss-Jordan elimination based on reduced echelon form clearly indicates the number and type of solutions for a linear system of equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

> Apply Gauss-Jordan elimination to obtain reduced echelon form. Let us assume that we have $x_{j_1}, \ldots, x_{j_r}$ leading variables and $n-r$ free variables, $x_{f_1}, \ldots, x_{f_{n-r}}$. Then the reduced row echelon form will be:

$$\begin{aligned} x_{j_1} + \sum_{k=1}^{n-r} c_{1k}x_{f_k} &= d_1 \\ \vdots \qquad & \quad \vdots \\ x_{j_r} + \sum_{k=1}^{n-r} c_{rk}x_{f_k} &= d_r \\ 0 &= d_{r+1} \\ \vdots \\ 0 &= d_m. \end{aligned}$$

> This system is consistent iff $d_{r+1}, \ldots, d_m$ are all zero.

> If the system is consistent and there are no free variables, in other words, if $r = n$, then the solution is unique. $x_1 = d_1, \ldots, x_n = d_n$.

> If the system is consistent and $r < n$, then there are infinitely many solutions.

Theorem The three possibilities.

A linear system of equations either has a unique solution, has no solution or has infinitely many solutions.

Homogeneus Systems

$$\underline{\underline{A}}\,\underline{x} = \underline{0}$$

Observe that, this system is always consistent because $\underline{x} = \underline{0}$ is a solution. However, the solution may not be unique.

Theorem 3

Every homogeneous system with more variables than the equations has infinitely many solutions.

Proof

The augmented coefficient matrix is $[\underline{\underline{A}}\ \underline{0}]$.
By applying Gauss-Jordan elimination we can get its reduced echelon form:

$$\left.\begin{matrix}\text{At most}\\ m\text{ leading}\\ \text{variables}\end{matrix}\right\{\overbrace{\begin{bmatrix} 1 & & & 0\\ & 1 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0\end{bmatrix}}^{n+1\text{ columns}}$$

, since there are at most m leading variables there are at least $n-m$ free variables. Since $n>m$, there are more than 1 free variables. Hence the system has infinitely many solutions.

Example

$$x_1 - 2x_2 + x_3 - x_4 = 0$$
$$2x_1 - x_2 + 5x_3 - 5x_4 = 0$$
$$3x_1 - 5x_2 + 2x_3 + x_4 = 0$$

Have 3 equations and 4 unknowns, hence there are infinitely many solutions;
Application of Gauss-Jordan elimination will reveal this.

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 2 & -1 & 5 & -5 & 0\\ 3 & -5 & 2 & 1 & 0\end{bmatrix} \xrightarrow{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 0 & 3 & 3 & -3 & 0\\ 3 & -5 & 2 & 1 & 0\end{bmatrix} \xrightarrow{-3R_1+R_3\to R_3} \begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 0 & 3 & 3 & -3 & 0\\ 0 & 1 & -1 & 4 & 0\end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2\to R_2} \begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 1 & -1 & 4 & 0\end{bmatrix} \xrightarrow{-R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & -2 & 5 & 0\end{bmatrix} \xrightarrow{-\frac{1}{2}R_3\to R_3} \begin{bmatrix} 1 & -2 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 1 & -\frac{5}{2} & 0\end{bmatrix}$$

$$\xrightarrow{+2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 3 & -3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & 0 \end{bmatrix} \xrightarrow{-R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 3 & -3 & 0 \\ 0 & 1 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & 0 \end{bmatrix} \xrightarrow{-3R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{9}{2} & 0 \\ 0 & 1 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & 0 \end{bmatrix}$$

Hence, x_1, x_2 and x_3 are leading variables and x_4 is a free variable. Therefore, the solution set is:

$$\text{leading variables} \begin{cases} x_1 = -\frac{9}{2}t \\ x_2 = -\frac{3}{2}t \\ x_3 = \frac{5}{2}t \end{cases}$$

free variable $\to$ $x_4 = t$.

Homogeneous Systems with Unique Solutions

$\underline{\underline{A}}\,\underline{x} = \underline{0}$ where $\underline{\underline{A}}$ is $m \times n$ dimensional, has infinitely many solutions if $n > m$. Therefore, can expect to have unique solution only if $n \leq m$.

Theorem $\underline{\underline{A}}\,\underline{x} = \underline{0}$ has unique solution iff the reduced row echelon form of $\underline{\underline{A}}$ has no free variables.

Thus if $m = n$: reduced row echelon form of $\underline{\underline{A}}$ should be:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \underline{\underline{I}}_{n\times n}.$$

If $m > n$, the reduced row echelon form should be:

$$\begin{bmatrix} \underline{\underline{I}}_{n\times n} \\ \hline \begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array} \end{bmatrix} \quad \begin{array}{l} \\ (m-n) \text{ rows of zeros.} \end{array}$$

Matrix Operations

Def'n Matrix $\underset{=}{A}_{m\times n}$, and matrix $\underset{=}{B}_{p\times q}$ are equal if $m=p$, $n=q$ and $a_{ij}=b_{ij}$ for $1\le i\le m$, $1\le j\le n$.

Def'n Matrix addition:

$$\underset{=}{A}_{m\times n}+\underset{=}{B}_{m\times n}=\underset{=}{C}_{m\times n}$$

where: $c_{ij}=a_{ij}+b_{ij}$, $1\le i\le m$, $1\le j\le n$.

Also denoted as: $\underset{=}{A}+\underset{=}{B}=[a_{ij}+b_{ij}]$.

ex/ $\underset{=}{A}=\begin{bmatrix}1&0&4\\2&5&6\end{bmatrix}$, $\underset{=}{B}=\begin{bmatrix}0&4&1\\2&5&0\end{bmatrix}$, $\underset{=}{C}=\begin{bmatrix}2&4\\1&-2\end{bmatrix}$

$\underset{=}{A}+\underset{=}{B}=\begin{bmatrix}1&4&5\\4&10&6\end{bmatrix}$ but $\underset{=}{A}+\underset{=}{C}$ and $\underset{=}{B}+\underset{=}{C}$ are not defined because $\underset{=}{A}$ and $\underset{=}{C}$ and $\underset{=}{B}$ and $\underset{=}{C}$ have different dimensions.

Def'n Multiplication of a matrix by a number (scalar)

$\underset{=}{A}=[a_{ij}]$, $c\in\mathbb{R}$, then $c\underset{=}{A}=[ca_{ij}]$.

we also write:

$(-1)\cdot\underset{=}{A}=-\underset{=}{A}$ and $\underset{=}{A}-\underset{=}{B}=\underset{=}{A}+(-\underset{=}{B})$.

ex/ $\underset{=}{A}=\begin{bmatrix}2&1&0\\1&3&2\end{bmatrix}$, $\underset{=}{B}=\begin{bmatrix}0&4&-2\\5&1&3\end{bmatrix}$

$$3\underset{=}{A}-\underset{=}{B}=\begin{bmatrix}6&3&0\\3&9&6\end{bmatrix}+\begin{bmatrix}0&-4&2\\-5&-1&-3\end{bmatrix}=\begin{bmatrix}6&-1&2\\-2&8&3\end{bmatrix}$$

Vectors A matrix of size $n\times 1$ is called as a vector or column vector

$\underline{a}=\begin{bmatrix}2\\4\\1\end{bmatrix}$, $\underline{b}=\begin{bmatrix}0\\2\\-3\end{bmatrix}$: $3\underline{a}+2\underline{b}=\begin{bmatrix}6\\12\\3\end{bmatrix}+\begin{bmatrix}0\\4\\-6\end{bmatrix}=\begin{bmatrix}6\\8\\-3\end{bmatrix}$.

A row vector is an $1\times n$ matrix.

ex/ The homogeneous system:

$$x_1 + 3x_2 - 15x_3 + 7x_4 = 0$$
$$x_1 + 4x_2 - 19x_3 + 10x_4 = 0$$
$$2x_1 + 5x_2 - 26x_3 + 11x_4 = 0$$

can be reduced to the following row echelon form:

$$\begin{bmatrix} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, x_1 and x_2 are leading variables and x_3 and x_4 are free variables. Therefore the parametric solution set is:

$$\left.\begin{aligned} x_4 &= t \\ x_3 &= s \\ x_2 &= 4s - 3t \\ x_1 &= 3s + 2t \end{aligned}\right\} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = s \cdot \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{x}_1} + t \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underline{x}_2}$$

$$\Rightarrow \quad \underline{x} = \underbrace{s\underline{x}_1 + t\underline{x}_2}_{\text{linear combination of } \underline{x}_1 \text{ and } \underline{x}_2}.$$