ex/ The homogeneous system:

$$x_1 + 3x_2 - 15x_3 + 7x_4 = 0$$
$$x_1 + 4x_2 - 19x_3 + 10x_4 = 0$$
$$2x_1 + 5x_2 - 26x_3 + 11x_4 = 0$$

can be reduced to the following row echelon form:

$$\begin{bmatrix} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, x_1 and x_2 are leading variables and x_3 and x_4 are free variables. Therefore the parametric solution set is:

$$\left.\begin{aligned} x_4 &= t \\ x_3 &= s \\ x_2 &= 4s-3t \\ x_1 &= 3s+2t \end{aligned}\right\} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = s \cdot \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{x}_1} + t \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underline{x}_2}$$

$$\Rightarrow \quad \underline{x} = \underbrace{s\underline{x}_1 + t\underline{x}_2}_{\text{linear combination of } \underline{x}_1 \text{ and } \underline{x}_2.}.$$

Matrix Multiplication

$$\underline{\underline{A}} = [a_{ij}], \quad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \quad \equiv \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Hence:

$$\underbrace{\underline{\underline{A}}\,\underline{x}}_{\substack{\text{Matrix} \\ \text{Vector multiplication.}}} = \underline{b} \Rightarrow \sum_{j=1}^{n} a_{ij}x_j = b_i, \quad 1 \le i \le m.$$

$$\underbrace{\underline{\underline{A}}\,\underline{\underline{X}}}_{\substack{\text{matrix-matrix} \\ \text{multiplication}}} = \underline{\underline{A}}\,[\underset{\uparrow}{\underline{x}_1} \cdots \underset{\uparrow}{\underline{x}_k}] = \underbrace{[\underline{\underline{A}}\,\underline{x}_1, \underline{\underline{A}}\,\underline{x}_2 \cdots \underline{\underline{A}}\,\underline{x}_k]}_{k \text{ matrix-vector multiplication}}$$

vectors forming $\underline{\underline{X}}$

Formal Def'n of Matrix-Matrix Multiplication

Matrices $\underline{\underline{A}}_{m\times n}$ and $\underline{\underline{B}}_{p\times q}$ can be multiplied if $n=p$. Then the result is $\underline{\underline{C}}_{m\times q}$ where:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}, \quad 1\le i\le m,\ 1\le j\le q.$$

ex/

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 7 \\ 8 & 5 & 9 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 7 \\ 8 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 34 & 22 & 47 \\ 76 & 49 & 113 \end{bmatrix}$$

Note

$\underline{\underline{B}}\,\underline{\underline{A}}$ cannot be formed! $\underline{\underline{B}}$ and $\underline{\underline{A}}$ are not compatible.

Rules of Matrix Algebra

For $\underline{A}$, $\underline{B}$ and $\underline{C}$ with appropriate sizes:

$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}$; commutative law of addition

$(\underline{\underline{A}} + \underline{\underline{B}}) + \underline{\underline{C}} = \underline{\underline{A}} + (\underline{\underline{B}} + \underline{\underline{C}})$; associative law of addition

$\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}) = (\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{\underline{C}}$: associative law of multiplication.

$$\left.\begin{aligned} \underline{\underline{A}}\,(\underline{\underline{B}} + \underline{\underline{C}}) &= \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}} \\ (\underline{\underline{A}} + \underline{\underline{B}})\,\underline{\underline{C}} &= \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}} \end{aligned}\right\}$$ Distributive laws of multiplication over summation.

Proof for $\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}) = (\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{\underline{C}}$:

$$\underbrace{\underline{\underline{A}}_{m\times n}\,\underbrace{(\underline{\underline{B}}_{n\times p}\ \underline{\underline{C}}_{p\times q})}_{\underline{\underline{D}}_{n\times q}}}_{\underline{\underline{E}}_{m\times q}} :$$

$$d_{ij} = \sum_{k=1}^{p} b_{ik}\, c_{kj}$$

$$e_{lj} = \sum_{i=1}^{n} a_{li}\, d_{ij} = \sum_{i=1}^{n} a_{li} \left[\sum_{k=1}^{p} b_{ik}\, c_{kj} \right]$$

Interchange the order of the i and k sums to get:

$$e_{\ell j} = \sum_{k=1}^{p} \underbrace{\left[\sum_{i=1}^{n} a_{\ell i}\, b_{ik} \right]}_{f_{\ell k}} c_{kj}$$

where $\underline{\underline{F}} = [f_{\ell k}] = \underline{\underline{A}} \cdot \underline{\underline{B}}$

Hence: $e_{\ell j} = \sum_{k=1}^{p} f_{\ell k}\, c_{kj} \Rightarrow \underline{\underline{E}} = \underline{\underline{F}} \cdot \underline{\underline{C}} = (\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}}$. $\blacksquare$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 4 & 1 & -2 & 7 \\ 3 & 1 & -1 & 5 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 1 & 5 \\ 3 & -1 \\ -2 & 4 \\ 2 & -3 \end{bmatrix}$, $\underline{\underline{C}} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ -2 & 3 \\ 1 & -3 \end{bmatrix}$

$\underline{\underline{B}} \neq \underline{\underline{C}}$ but $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{A}}\,\underline{\underline{C}} = \begin{bmatrix} 25 & -10 \\ 18 & -5 \end{bmatrix}$

Hence : $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{A}}\,\underline{\underline{C}}$ does not imply $\underline{\underline{B}} = \underline{\underline{C}}$!

$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{A}}\,\underline{\underline{C}} \Rightarrow \underline{\underline{A}}\,\underbrace{(\underline{\underline{B}} - \underline{\underline{C}})}_{\underline{\underline{D}}} = \underline{\underline{0}}$, $\underline{\underline{D}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Note that: $\underline{\underline{A}} \neq \underline{\underline{0}}$, $\underline{\underline{D}} \neq \underline{\underline{0}}$ but $\underline{\underline{A}}\,\underline{\underline{D}} = \underline{\underline{0}}$!.

Inverses of Matrices

Def'n For a matrix $\underline{\underline{A}}$, its inverse $\underline{\underline{B}}$, if it exists, satisfies: $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}} \leftarrow \left\{ \begin{matrix} \text{identity} \\ \text{matrix.} \end{matrix} \right\}$

Inverse of $\underline{\underline{A}}$, if it exists, shown as $\underline{\underline{A}}^{-1}$.

Observations (i) $\underline{\underline{A}}$, if it invertible, must be square.

proof Let $\underline{\underline{A}}_{m \times n}$. Then assume that $\underline{\underline{B}}$ is $\underline{\underline{A}}$ inverse. Then $\underline{\underline{A}}\,\underline{\underline{B}}$ is defined. Hence $\underline{\underline{B}}$ must have dimensions $n \times p$. But $\underline{\underline{B}}\,\underline{\underline{A}}$ is also defined, hence $\underline{\underline{B}}$ must be $n \times m$. Since , $\underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{B}} \Rightarrow m = n$. $\underline{\underline{A}}$ is square.

(ii) $$\underline{\underline{I}} = \begin{array}{c} \begin{matrix} \underset{\sim}{e_1} & \underset{\sim}{e_2} & \cdots & \underset{\sim}{e_n} \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & | \\ | & 0 & & | \\ | & | & & 0 \\ 0 & 0 & & 1 \end{bmatrix} \end{array} = \left[\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_n \right]$$

where $$\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xleftarrow{\quad} i\text{th entry.}$$

Note that $$\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}} \left[\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_n \right] = \left[\underline{\underline{A}}\,\underline{e}_1 \ \underline{\underline{A}}\,\underline{e}_2 \cdots \underline{\underline{A}}\,\underline{e}_n \right]$$

$$\underline{\underline{A}}\,\underline{e}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}}_{\underline{a}_1}$$

$$\underline{\underline{A}}\,\underline{e}_i = \underline{a}_i \ \swarrow i\text{th column of } \underline{\underline{A}}.$$

Hence: $$\underline{\underline{A}}\,\underline{\underline{I}} = \left[\underline{\underline{A}}\,\underline{e}_1 \ \underline{\underline{A}}\,\underline{e}_2 \cdots \underline{\underline{A}}\,\underline{e}_n \right] = \left[\underline{a}_1 \ \underline{a}_2 \cdots \underline{a}_n \right] = \underline{\underline{A}}.$$

Theorem If $\underline{\underline{A}}$ is invertible, then its inverse is unique.

proof Let there be a matrix $\underline{\underline{A}}$ with 2 different inverses:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}} \ , \quad \underline{\underline{A}}\,\underline{\underline{C}} = \underline{\underline{C}}\,\underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{\underline{C}}\,\underline{\underline{I}} = \underline{\underline{C}} = \underline{\underline{C}}\,(\underbrace{\underline{\underline{A}}\,\underline{\underline{B}}}_{\underline{\underline{I}}}) = (\underbrace{\underline{\underline{C}}\,\underline{\underline{A}}}_{\underline{\underline{I}}})\,\underline{\underline{B}} = \underline{\underline{I}}\,\underline{\underline{B}} = \underline{\underline{B}}$$

contradiction.
$\Rightarrow$ the inverse must be unique.

ex/ $$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{if } ad-bc \neq 0, \quad \underline{\underline{A}}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \frac{1}{(ad-bc)} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)} \begin{bmatrix} ad-bc & \overbrace{ab-ab}^{0} \\ \underbrace{cd-cd}_{0} & ad-bc \end{bmatrix}$$

$$= \underline{\underline{I}} \ //.$$

Def'n $\underline{\underline{A}}^0 = \underline{\underline{I}}$, $\underline{\underline{A}}' = \underline{\underline{A}}$, $\underline{\underline{A}}^{n+1} = \underline{\underline{A}}^n \cdot \underline{\underline{A}}$, $n \geq 1$.

$\underline{\underline{A}}^{-n} = (\underline{\underline{A}}^{-1})^n$.

Theorem 3 Algebra of Inverse Matrices:

(a) $(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$ $\left(\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} \Rightarrow \underline{\underline{A}} = (\underline{\underline{A}}^{-1})^{-1} \right)$

(b) $(\underline{\underline{A}}^n)^{-1} = (\underline{\underline{A}}^{-1})^n$ $\left(\underline{\underline{A}}^n \cdot (\underline{\underline{A}}^{-1})^n = \underline{\underline{A}}^n \cdot \underbrace{\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}}^{-1} \cdots \underline{\underline{A}}^{-1}}_{n} \right)$

$$= \underbrace{\underline{\underline{A}} \cdot \underbrace{\underline{\underline{A}} \cdots \underbrace{\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1}}_{\underline{\underline{I}}} \underline{\underline{A}}^{-1} \cdots}_{\underline{\underline{I}}} \underline{\underline{A}}^{-1}}_{\underline{\underline{I}}}$$

(c) $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$:

$$(\underline{\underline{A}}\,\underline{\underline{B}}) \cdot (\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}) = \underline{\underline{A}}\,\underbrace{(\underline{\underline{B}}\,\underline{\underline{B}}^{-1})}_{\underline{\underline{I}}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}\,\underbrace{(\underline{\underline{I}}\,\underline{\underline{A}}^{-1})}_{\underline{\underline{A}}^{-1}} = \underline{\underline{I}} .$$

$$(\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1})(\underline{\underline{A}}\,\underline{\underline{B}}) = \underline{\underline{B}}^{-1}\underbrace{(\underline{\underline{A}}^{-1}\underline{\underline{A}})}_{\underline{\underline{I}}}\underline{\underline{B}} = \underline{\underline{B}}^{-1}\underbrace{(\underline{\underline{I}}\,\underline{\underline{B}})}_{\underline{\underline{B}}} = \underline{\underline{B}}^{-1}\underline{\underline{B}} = \underline{\underline{I}} \checkmark$$

Theorem 4 Inverse Matrix solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$:

If $\underline{\underline{A}}_{n \times n}$ is invertible, then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has the unique solution $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$.

Proof

(i) First, want to show that $\underline{\underline{A}}^{-1}\underline{b}$ is a solution:

$$\underline{\underline{A}}\,(\underline{\underline{A}}^{-1}\underline{b}) = \underbrace{(\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1})}_{\underline{\underline{I}}}\,\underline{b} = \underline{\underline{I}}\,\underline{b} = \underline{b} \quad \checkmark$$

(ii) Now, we will show that there exists no other solution.
Assume that $\underline{x}_1$ is another solution, $\underline{x}_1 \neq \underline{\underline{A}}^{-1}\underline{b}$.
Then:

$$\underline{\underline{A}}\,\underline{x}_1 = \underline{b} \quad , \quad \underline{\underline{A}}^{-1}(\underline{\underline{A}}\,\underline{x}_1) = \underline{\underline{A}}^{-1}\underline{b}$$

$$\underbrace{(\underline{\underline{A}}^{-1}\underline{\underline{A}})}_{\underline{\underline{I}}}\,\underline{x}_1 = \underline{\underline{A}}^{-1}\underline{b} \Rightarrow \underline{x}_1 = \underline{\underline{A}}^{-1}\underline{b}$$

contradiction!

ex/ $$\left.\begin{aligned} 4x_1 + 6x_2 &= 6 \\ 5x_1 + 9x_2 &= 18 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 6 \\ 18 \end{bmatrix}}_{\underline{b}}$$

$$\underline{\underline{A}}^{-1} = \begin{bmatrix} 9 & -6 \\ -5 & 4 \end{bmatrix} \cdot \frac{1}{(36-30)} = \begin{bmatrix} 3/2 & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

Then: $$\underline{x} = \underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} 3/2 & -1 \\ -5/6 & 2/3 \end{bmatrix} \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$$

How to find $\underline{\underline{A}}^{-1}$?

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}} = [\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_n]$$

Let $\underline{\underline{A}}^{-1} = [\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n]$. Then:

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = [\underline{\underline{A}}\,\underline{x}_1\ \ \underline{\underline{A}}\,\underline{x}_2 \cdots \underline{\underline{A}}\,\underline{x}_n] = [\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_n]$$

$$\Rightarrow \quad \underbrace{\underline{\underline{A}}\,\underline{x}_i = \underline{e}_i}, \quad 1 \le i \le n$$

a linear system of equation for each i.

Solve to obtain $\underline{x}_i$. Then: $\underline{\underline{A}}^{-1} = [\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n]$

ex/ Find the inverse of: $$\underline{\underline{A}} = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}$$

Form 3 linear systems of equations:

$$\underline{\underline{A}}\,\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\underline{A}}\,\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{\underline{A}}\,\underline{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Can solve each by Gaussian-Jordan elimination:

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 5 & 6 & 3 & 0 \\ 3 & 5 & 2 & 0 \end{array}\right] \xrightarrow{-5/4 R_1 + R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 0 & 9/4 & 1/2 & -\frac{5}{4} \\ 3 & 5 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{4}R_1+R_3\to R_3}\left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 0 & 9/4 & 1/2 & -\frac{5}{4} \\ 0 & 11/4 & 1/2 & -3/4 \end{array}\right] \xrightarrow{-\frac{11}{9}R_2+R_3\to R_3}\left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 0 & \frac{9}{4} & \frac{1}{2} & -\frac{5}{4} \\ 0 & 0 & -\frac{1}{9} & \frac{7}{9} \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_1\to R_1}\left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 1/4 \\ 0 & 9/4 & 1/2 & -\frac{5}{4} \\ 0 & 0 & -1/9 & 7/9 \end{array}\right] \xrightarrow{\frac{4}{9}R_2\to R_2}\left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 2/9 & -5/9 \\ 0 & 0 & -1/9 & 7/9 \end{array}\right]$$

$$\xrightarrow{-9R_3\to R_3}\left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 2/9 & -5/9 \\ 0 & 0 & 1 & -7 \end{array}\right] \xrightarrow{-3/4\,R_2+R_1\to R_1}\left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 2/9 & -5/9 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$$\xrightarrow{-1/3\,R_3+R_1\to R_1}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2/9 & -5/9 \\ 0 & 0 & 1 & -7 \end{array}\right] \xrightarrow{-2/9\,R_3+R_2\to R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{array}\right] \quad \underbrace{\begin{array}{c} 3 \\ 1 \\ -7 \end{array}}_{\underline{x}_1}.$$

Now, we should solve $\underline{\underline{A}}\,\underline{x}_2=\underline{e}_2$.

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 5 & 6 & 3 & 1 \\ 3 & 5 & 2 & 0 \end{array}\right]$$

Again Gauss-Jordan elimination can be used to solve for $\underline{x}_2$. Note that the steps of the Gauss-Jordan elimination is the same as $\underline{\underline{A}}\,\underline{x}=\underline{e}$.

Thus, we could use Gauss-Jordan elimination on the combined system: $[\underline{\underline{A}}\ \underline{e}_1\ \underline{e}_2\ \underline{e}_3]$ or $[\underline{\underline{A}}\,\vdots\,\underline{\underline{I}}]$:

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Exactly the Same Gauss-Jordan Elimination Steps}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right]$$

$$\underbrace{\qquad\quad}_{\underline{\underline{A}}}\ \underbrace{\qquad\quad}_{\underline{\underline{I}}} \qquad\qquad\qquad \underbrace{\qquad\quad}_{\underline{\underline{I}}}\ \underbrace{\qquad\quad}_{\underline{\underline{A}}^{-1}}$$

This idea can be formalized by using elementary matrices:

Def'n $n\times n$ matrix $\underline{\underline{E}}$ is called an elementary matrix, if it can be obtained by performing a single elementary row operation on the identity matrix $\underline{\underline{I}}$.

ex/ $\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{2R_1\to R_1} \begin{bmatrix}2&0\\0&1\end{bmatrix} = \underline{\underline{E}}_1$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{2R_1+R_3\to R_3} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = \underline{\underline{E}}_2$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\text{swap}(R_1,R_2)} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = \underline{\underline{E}}_3$

Elementary Matrices and Row Operations

Theorem 5 An elementary row operation on a matrix $\underline{\underline{A}}_{m\times n}$ is equivalent to $\underline{\underline{E}}\,\underline{\underline{A}}$ where $\underline{\underline{E}}$ is the elementary matrix obtained by performing the same operation on the $\underline{\underline{I}}_{m\times m}$.

Proof

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(i) Multiplication of the i^{th} row of $\underline{\underline{A}}$ by a constant $c \neq 0$:

$$\underline{\underline{I}} \xrightarrow{cR_i \to R_i} \underline{\underline{E}} = \begin{bmatrix} 1 & 0 & — & & 0 \\ 0 & 1 & 0 & — & 0 \\ 0 & \cdots & 0 \; c & 0 — & 0 \\ 0 & — & & 0 & 1 \end{bmatrix}$$

i^{th} row.

$$\underline{\underline{E}}\,\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & — & 0 \\ 0 & 1 & 0 — & 0 \\ 0 & — 0 & c & 0 — 0 \\ 0 & — & & 0\;1 \end{bmatrix} \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(ii) Swapping row i with row j:

$$\underline{\underline{I}} \xrightarrow{\text{swap}(R_i,R_j)} \underline{\underline{E}} \; ; \quad \text{ex}: \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\text{swap}(R_2,R_3)} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\underline{\underline{E}}\,\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i^{th}\text{ row} \\ \leftarrow j^{th}\text{ row} \\ \\ \end{matrix}$$

(iii) Adding a constant multiple of R_i to R_j :

$$\underline{\underline{I}} \xrightarrow{cR_i+R_j\to R_j} \underline{\underline{E}} \;:\; ex/ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{2R_2+R_3\to R_3} \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}}_{\underline{\underline{E}}\,.}$$

$$\underline{\underline{E}}\,\underline{\underline{A}} = \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ c\underline{e}_i^T + \underline{e}_j^T \\ \vdots \\ \underline{e}_n^T \end{bmatrix} \underline{\underline{A}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ ca_{i1}+a_{j1} & \cdots & ca_{in}+a_{jn} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \leftarrow j^{th}\text{ row.}$$

j^{th} row

<u>Theorem 6</u> Invertible matrices and elementary row operations:

$\underline{\underline{A}}_{n\times n}$ is invertible $\Leftrightarrow$ It is row equivalent to $\underline{\underline{I}}_{n\times n}$

<u>proof</u>

(i) $\Rightarrow$: $\underline{\underline{A}}$ is invertible $\Rightarrow$ $\underline{\underline{A}}\,\underline{x} = \underline{0}$ has a unique solution $\underline{\underline{A}}^{-1}\underline{0} = \underline{0}$.
Thus, Gauss-Jordan elimination will reduce $\underline{\underline{A}}$ to $\underline{\underline{I}}$. Therefore $\underline{\underline{A}}$ is row equivalent to $\underline{\underline{I}}$.

(ii) $\Leftarrow$: $\underline{\underline{A}}$ is row equivalent to $\underline{\underline{I}}$ implies that there is a sequence of elementary row operations that redue $\underline{\underline{A}}$ to $\underline{\underline{I}}$:

$$\underline{\underline{E}}_k\,\underline{\underline{E}}_{k-1}\cdots\underline{\underline{E}}_1\,\underline{\underline{A}} = \underline{\underline{I}}$$

$$\Rightarrow \quad \underline{\underline{A}} = \underline{\underline{E}}_1^{-1}\cdot\underline{\underline{E}}_2^{-1}\cdots\underline{\underline{E}}_k^{-1},$$

thus $\underline{\underline{A}}$ is invertible (it is a product of invertible matrices).

(Note that $\underline{\underline{E}}'s$ are all invertible.)

$$\underline{\underline{A}}^{-1} = \underline{\underline{E}}_k\,\underline{\underline{E}}_{k-1}\cdots\underline{\underline{E}}_1\,.$$

Matrix Equations

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{B}} \quad \text{or} \quad \underline{\underline{A}}[\underline{x}_1 \cdots \underline{x}_k] = [\underline{b}_1 \cdots \underline{b}_k]$$

Hence the matrix equation is equivalent to k linear system of equations: $\underline{\underline{A}}\,\underline{x}_i = \underline{b}_i$, $1 \le i \le k$.

If $\underline{\underline{A}}^{-1}$ exists: $\underline{\underline{X}} = \underline{\underline{A}}^{-1}\underline{\underline{B}}$ is the unique solution to the matrix equation.

Non-singular Matrices

Thm 7 The following properties of $\underline{\underline{A}}_{n\times n}$ are equivalent:

(a) $\underline{\underline{A}}$ is invertible

(b) $\underline{\underline{A}}$ is row equivalent to the $\underline{\underline{I}}_{n\times n}$

(c) $\underline{\underline{A}}\,\underline{x} = \underline{0}$ has a unique solution $\underline{x} = \underline{0}$.

(d) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a unique solution for any $\underline{b}$

(e) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is always consistent for any $\underline{b}$.

Proof

Thm 6 implies $(a) \Leftrightarrow (\underline{b})$

$(b) \Leftrightarrow (c)$ row equivalent to $\underline{\underline{I}}$ implies the existence of elementary row operations that reduces $\underline{\underline{A}}$ to $\underline{\underline{I}}$. Thus $\underline{\underline{A}}\,\underline{x} = \underline{0}$ will have unique solution $\underline{x} = \underline{0}$.

$(c) \Rightarrow (d)$: Since $\underline{\underline{A}}$ is invertible $(c \Rightarrow a)$, $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has unique solution for any $\underline{b}$

$(d) \Rightarrow (e)$: Since $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is always have a unique solution for any $\underline{b}$, it is also consistent for any $\underline{b}$.

$(e) \Rightarrow (a)$: Choose $\underline{b}_1 = \underline{e}_1, \ldots, \underline{b}_n = \underline{e}_n$. Then:
$\underline{\underline{A}}\,\underline{x}_i = \underline{e}_i$, $1 \le i \le n$ has a solution, call $\underline{c}_i$.
Then: $\underline{\underline{A}}[\underline{c}_1 \cdots \underline{c}_n] = [\underline{\underline{A}}\,\underline{c}_1 \cdots \underline{\underline{A}}\,\underline{c}_n] = [\underline{e}_1 \cdots \underline{e}_n] = \underline{\underline{I}}$
Hence, $\underline{\underline{A}}\,\underline{\underline{C}} = \underline{\underline{I}}$. Now, want to show that $\underline{\underline{C}}$ is invertible. This can be concluded if $\underline{\underline{C}}\,\underline{x} = \underline{0}$ has a unique solution. Assume that $\underline{x}_1 \neq \underline{0}$ but $\underline{\underline{C}}\,\underline{x}_1 = \underline{0}$
Then: $\underline{\underline{A}}(\underline{\underline{C}}\,\underline{x}_1) = \underline{0} = \underbrace{(\underline{\underline{A}}\,\underline{\underline{C}})}_{\underline{\underline{I}}}\underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = \underline{0}$
contradiction.

Thus, $\underline{\underline{C}}\,\underline{x}=\underline{0}$ has a unique solution. Therefore $\underline{\underline{C}}$ is invertible.
Then $\underline{\underline{A}}\,\underline{\underline{C}}=\underline{\underline{I}}$ implies that

$$\underbrace{(\underline{\underline{A}}\,\underline{\underline{C}})\,\underline{\underline{C}}^{-1}}_{\underline{\underline{A}}\underbrace{(\underline{\underline{C}}\,\underline{\underline{C}}^{-1})}_{\underline{\underline{I}}} = \underline{\underline{A}} \;/\!/} = \underline{\underline{I}}\,\underline{\underline{C}}^{-1} = \underline{\underline{C}}^{-1}$$

Thus $\underline{\underline{A}}$ is $\underline{\underline{C}}^{-1}$. Thus $\underline{\underline{A}}$ is inverse of an invertible matrix. Hence it is also invertible.

Determinants

Def'n Minors and Cofactors

Let $\underline{\underline{A}} = [a_{ij}]$ be an $n \times n$ matrix. The ij^{th} minor of $\underline{\underline{A}}$ also called the minor of a_{ij} is the determinant M_{ij} of the $(n-1)\times(n-1)$ submatrix that remains after deleting the i^{th} row and j^{th} column of $\underline{\underline{A}}$. The ij^{th} cofactor A_{ij} of $\underline{\underline{A}}$ is:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Def'n $n \times n$ Determinants:

$$\det \underline{\underline{A}} = |a_{ij}| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

(Note that this definition reduces the det $\underline{\underline{A}}$ to computation of $(n-1)\times(n-1)$ order determinants. Thus we can recursively reduce it to 1×1 determinats.)

Def'n 1×1 Determinant: $\det \underline{\underline{A}} = \det [a_{11}] = a_{11}$.

examples

(i) $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12}$

$A_{11} = (-1)^{1+1} \cdot \det [a_{22}] = a_{22}$

$A_{12} = (-1)^{1+2} \det [a_{21}] = -a_{21}$

$\Rightarrow \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii)
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

Theorem 1 Cofactor Expansion of Determinants:
The determinant of an $n \times n$ matrix $\underline{\underline{A}}$ can be obtained by expansion along any row or column:

$$\det \underline{\underline{A}} = \sum_{j=1}^{n} a_{ij} A_{ij} \text{, expansion along the } i^{th} \text{ row.}$$

$$\det \underline{\underline{A}} = \sum_{i=1}^{n} a_{ij} A_{ij} \text{; expansion along the } j^{th} \text{ column.}$$

ex/ $\det \begin{bmatrix} 7 & 6 & 0 \\ 9 & -3 & 2 \\ 4 & 5 & 0 \end{bmatrix} = 2 \cdot A_{23} = 2 \cdot (-1)^{2+3} \begin{vmatrix} 7 & 6 \\ 4 & 5 \end{vmatrix} = \underbrace{-2 \cdot (35-24)}_{= -22.}$

↑ If expanded along this column, there will be only 1 term

ex/ $\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix} = 0$ (expand along the second row!)

Row and Column Properties

1. If $\underline{\underline{B}}$ is obtained from $\underline{\underline{A}}$ by multiplying a single row (or column) by a constant c, then $\det \underline{\underline{B}} = c \det \underline{\underline{A}}$

2. If $\underline{\underline{B}}$ is obtained from $\underline{\underline{A}}$ by interchanging two rows or columns, then $\det \underline{\underline{B}} = -\det \underline{\underline{A}}$.

3. If two rows or columns of $\underline{\underline{A}}$ are the same, then $\det \underline{\underline{A}} = 0$.
(Interchange the identical rows or columns, then $\det \underline{\underline{A}} = -\det \underline{\underline{A}} \Rightarrow \det \underline{\underline{A}} = 0$)

4. $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are identical except their i^{th} row. Then:
$\det(\underline{\underline{A}}_1 + \underline{\underline{A}}_2) = \det \underline{\underline{A}}_1 + \det \underline{\underline{A}}_2$.
(can show this by expanding the determinants along the i^{th} rows.)

5. If $\underline{\underline{B}}$ is obtained from $\underline{\underline{A}}$ by adding a constant multiple of one row of $\underline{\underline{A}}$ to another row of $\underline{\underline{A}}$, then $\det \underline{\underline{B}} = \det \underline{\underline{A}}$.

Proof

$$\underline{\underline{A}} = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_i^T \\ \vdots \\ \underline{a}_j^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_i^T \\ \vdots \\ c\,\underline{a}_i^T + \underline{a}_j \\ \vdots \\ \underline{a}_n^T \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{a}_1^T \\ \underline{a}_i^T \\ \vdots \\ c\underline{a}_i^T \\ \underline{a}_n^T \end{bmatrix}}_{\underline{\underline{A}}_1} + \underbrace{\begin{bmatrix} \underline{a}_1^T \\ \underline{a}_i^T \\ \vdots \\ \underline{a}_j^T \\ \underline{a}_n^T \end{bmatrix}}_{\underline{\underline{A}}}$$

$$\det \underline{\underline{B}} = \det(\underbrace{\underline{\underline{A}}_1 + \underline{\underline{A}}}_{\text{only their } j^{th} \text{ rows differ}}) = \det \underline{\underline{A}}_1 + \det \underline{\underline{A}} = c \cdot \underbrace{\det \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_i^T \\ \vdots \\ \underline{a}_i^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix}}_{0} + \det \underline{\underline{A}}$$

2 rows are the same

$\Rightarrow \det \underline{\underline{B}} = \det \underline{\underline{A}}$.

ex/
$$\det \begin{bmatrix} 2 & -3 & -4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{bmatrix} \xrightarrow[\text{elementary column operation}]{+2C_1 + C_3 \to C_3} = \det \begin{bmatrix} 2 & -3 & 0 \\ -1 & 4 & 0 \\ 3 & 10 & 7 \end{bmatrix} = 7 \cdot \det \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= 7 \cdot (8-3) = 35.$$

6. Determinant of a triangular matrix is equal to multiplication of its diagonal entries.

ex/
$$\det \begin{bmatrix} 3 & 5 & 7 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 3 \cdot 1 \cdot (-1) \cdot 2 = -6.$$

Simply expand the determinant along the first column of corresponding matrices.

Transpose of a Matrix

$$\underline{\underline{A}}^T = \underline{\underline{B}}, \quad \underline{\underline{B}} = [b_{ij}] = [a_{ji}]$$

ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\underline{\underline{A}}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(i) $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$

(ii) $(\underline{\underline{A}}+\underline{\underline{B}})^T = \underline{\underline{A}}^T+\underline{\underline{B}}^T$

(iii) $(c\underline{\underline{A}})^T = c\underline{\underline{A}}^T$

(iv) $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\,\underline{\underline{A}}^T$ (show this!)

7. $\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$

Determinants and Invertibility

$\underline{\underline{A}}$ is invertible $\Longleftrightarrow$ $\det \underline{\underline{A}} \neq 0$.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det \underline{\underline{A}} = 4-6 = -2 \neq 0$.
$\underline{\underline{A}}$ is invertible.

Theorem 3

Determinants and matrix products:

$$|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{A}}||\underline{\underline{B}}|.$$

(see appendix B for the proof)

ex/ $\underline{\underline{A}}\cdot\underline{\underline{A}}^{-1} = \underline{\underline{I}}$ $\underbrace{\det(\underline{\underline{A}}\cdot\underline{\underline{A}}^{-1})}_{\det \underline{\underline{A}}\cdot\det \underline{\underline{A}}^{-1}} = \det(\underline{\underline{I}}) = 1$ $\Rightarrow \det \underline{\underline{A}}^{-1} = \dfrac{1}{\det \underline{\underline{A}}}$.

Theorem 4

Cramer's Rule

Consider nxn linear system: $\underline{\underline{A}}\,\underline{x} = \underline{b}$, $\underline{\underline{A}} = [\underline{a}_1 \cdots \underline{a}_n]$.
If $\det \underline{\underline{A}} \neq 0$, then the system is uniquely solvable and the i^{th} entry of the solution is:

$$x_i = \frac{\det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n]}{\det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{a}_i\ \underline{a}_{i+1} \cdots \underline{a}_n]}$$

Proof Since $\det \underline{\underline{A}} \neq 0$, $\underline{\underline{A}}$ is invertible and there exists a unique solution.

$$\underline{\underline{A}}\,\underline{x} = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_i \cdots \underline{a}_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_i\\\vdots\\x_n\end{bmatrix} = \underbrace{\underline{a}_1 x_1 + \underline{a}_2 x_2 + \cdots + \underline{a}_i x_i + \cdots + \underline{a}_n x_n}_{\underline{b}}$$

$$\det[\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n] = \det\left[\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_{i-1} \left(\sum_{k=1}^{n} \underline{a}_k x_k\right) \underline{a}_{i+1} \cdots \underline{a}_n\right]$$

$$= \sum_{k=1}^{n} x_k \underbrace{\det[\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_{i-1}\ \underline{a}_k\ \underline{a}_{i+1} \cdots \underline{a}_n]}_{\overset{\shortparallel}{0} \text{ if } k \neq i}$$

$$= x_i \cdot \underbrace{\det[\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_{i-1}\ \underline{a}_i\ \underline{a}_{i+1} \cdots \underline{a}_n]}_{\det \underline{\underline{A}}}$$

$$\Rightarrow \quad x_i = \frac{\det[\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n]}{\det \underline{\underline{A}}}.$$

ex/ Using Cramer's Rule solve:

$$\begin{aligned} x_1 + 4x_2 + 5x_3 &= 2 \\ 4x_1 + 2x_2 + 5x_3 &= 3 \\ -3x_1 + 3x_2 - x_3 &= 1 \end{aligned}$$

$$\det \underline{\underline{A}} = \det\begin{bmatrix}1 & 4 & 5\\4 & 2 & 5\\-3 & 3 & -1\end{bmatrix} = \det\begin{bmatrix}1 & 4 & 5\\0 & -14 & -15\\0 & 15 & 14\end{bmatrix} = -14\cdot 14 + 15\cdot 15 = 29.$$

$$x_1 = \frac{\det\begin{bmatrix}2 & 4 & 5\\3 & 2 & 5\\1 & 3 & -1\end{bmatrix}}{29} = \frac{33}{29}, \quad x_2 = \frac{\det\begin{bmatrix}1 & 2 & 5\\4 & 3 & 5\\-3 & 1 & -1\end{bmatrix}}{29} = \frac{35}{29}$$

$$x_3 = \frac{\det\begin{bmatrix}1 & 4 & 2\\4 & 2 & 3\\-3 & 3 & 1\end{bmatrix}}{29} = -\frac{23}{29}.$$

Inverses and Adjoint Matrix

Can find inverse of $\underline{\underline{A}}$ (if it exists) by solving:

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}} \quad \text{or} \quad [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n][\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n] = [\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_n]$$

Hence we have: $\underline{\underline{A}}\,\underline{x}_j = \underline{e}_j$, $1 \leq j \leq n$

By using the Cramer's rule, we get:

$$x_{ij} = \frac{|\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{e}_j\ \underline{a}_{i+1} \cdots \underline{a}_n|}{|\underline{\underline{A}}|}$$

<u>Hence</u> :

<u>Theorem 5</u>

$$\underline{\underline{A}}^{-1} = \frac{[A_{ij}]^T}{|\underline{\underline{A}}|} = \frac{\overset{\text{adjoint matrix}}{\text{adj}\ \underline{\underline{A}}}}{|\underline{\underline{A}}|}$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{bmatrix}$, $|\underline{\underline{A}}| = 29.$

$$\underline{\underline{A}}\,[\underline{x}_1\ \underline{x}_2\ \underline{x}_3] = [\underline{e}_1\ \underline{e}_2\ \underline{e}_3]$$

$$\underline{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

$$x_{11} = \frac{\det \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix}}{|\underline{\underline{A}}|} = \frac{A_{11}}{|\underline{\underline{A}}|}$$

$$x_{21} = \frac{\det \begin{bmatrix} 1 & 1 & 5 \\ 4 & 0 & 5 \\ -3 & 0 & -1 \end{bmatrix}}{|\underline{\underline{A}}|} = \frac{A_{12}}{|\underline{\underline{A}}|}.$$

$$x_{31} = \frac{\det \begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 0 \\ -3 & 3 & 0 \end{bmatrix}}{|\underline{\underline{A}}|} = \frac{A_{13}}{|\underline{\underline{A}}|}$$

$$\Rightarrow \quad \underline{\underline{X}} = \underline{\underline{A}}^{-1} = \frac{1}{|\underline{\underline{A}}|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{|\underline{\underline{A}}|} \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T}_{\text{adj}\ \underline{\underline{A}}.}.$$