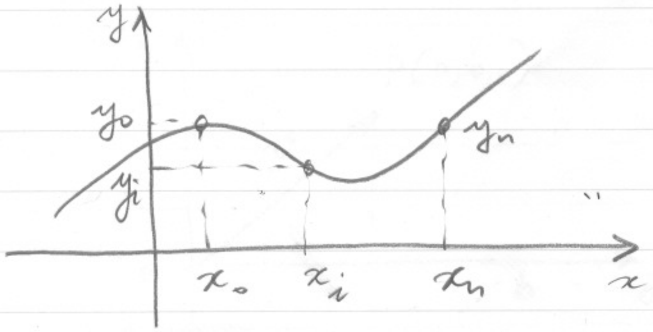
Linear Equations and Curve Fitting



Want to find a smooth function of x, that passes from the observed data points (x_i, y_i) $0 \le i \le n$.
In some applications we choose "the smooth function" as a polynomial of degree n:

$$a_0 + a_1 x + \cdots + a_n x^n = y$$

where we would like to choose $a_0, \ldots, a_n$ such that;

$$a_0 + a_1 x_0 + \cdots + a_n x_0^n = y_0$$
$$\vdots$$
$$a_0 + a_1 x_i + \cdots + a_n x_i^n = y_i$$
$$\vdots$$
$$a_0 + a_1 x_n + \cdots + a_n x_n^n = y_n$$

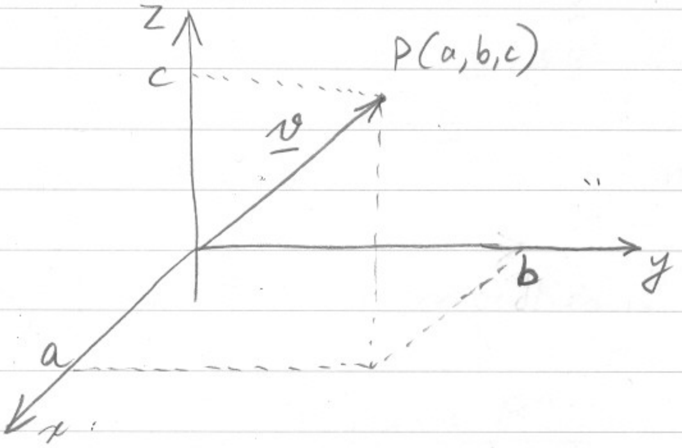
which can be represented as a linear system of equations in where the unknowns are the coefficients of the n^{th} order polynomial:

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{\underline{\underline{A}}\ (n+1)\times(n+1)} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

$\underline{\underline{A}}$ $(n+1)\times(n+1)$ is a Vandermonde matrix.

As investigated in the textbook (Problems 61-63 on page 214) Vandermonde matrix is invertible for $x_i \neq x_j$, $1 \le i < j \le n$.

Chapter 4: Vector Spaces

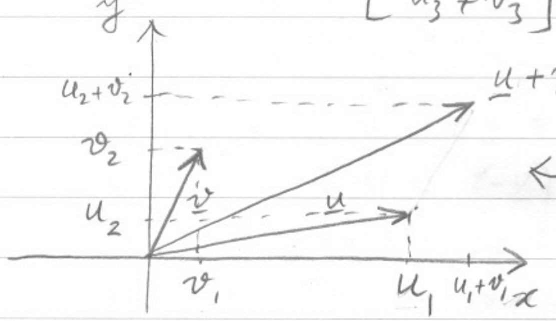


<u>Def'n</u> A vector in 3-Space $\mathbb{R}^3$ is an ordered triple

$\underline{v} = (a, b, c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ components.

<u>Def'n</u> Vector addition: $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$



parallelogram law of vector addition.

<u>Def'n</u> scalar multiplication of a vector:

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad c\underline{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}, \quad c \in \mathbb{R}.$$

check this

<u>Def'n</u> length of a vector: $|\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $(\geq 0.)$

<u>Theorem 1</u> $\mathbb{R}^3$ is a vector space:

If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are vectors in $\mathbb{R}^3$, and r and s are real numbers:

(a) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$: commutativity of vector addition

(b) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$: associativity of vector addition

(c) $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$: zero element of vector addition

(d) $\underline{u} + (-\underline{u}) = \underline{0}$; additive inverse
(e) $r(\underline{u}+\underline{v}) = r\underline{u} + r\underline{v}$ } distributivity
(f) $(r+s)\underline{u} = r\underline{u} + s\underline{u}$
(g) $r(s\underline{u}) = (rs)\underline{u}$
(h) $1 \cdot \underline{u} = \underline{u}$ multiplicative identity.

ex/ Show all these properties!

The Vector Space $\mathbb{R}^n$

Def'n The n-dimensional space $\mathbb{R}^n$ is the set of all n-tuples: $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ of real numbers.

$\mathbb{R}^n$ is also a vector space and satisfies the vector space properties:

(a) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
(b) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
(c) $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
(d) $\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0}$
(e) $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$
(f) $(a+b)\underline{u} = a\underline{u} + b\underline{u}$
(g) $a(b\underline{u}) = (ab)\underline{u}$
(h) $1 \cdot \underline{u} = \underline{u}$

In $\mathbb{R}^n$: $\underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $-\underline{u} = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

Note that $\mathcal{F}$, set of all real valued functions, is also a vector space. But it is not a finite dimensional space.

Def'n Subspace
Let W be a nonempty subset of the vector space V. Then W is a subspace of V if;

(i) If $\underline{u}$ and $\underline{v}$ are in W, then $\underline{u}+\underline{v} \in W$.

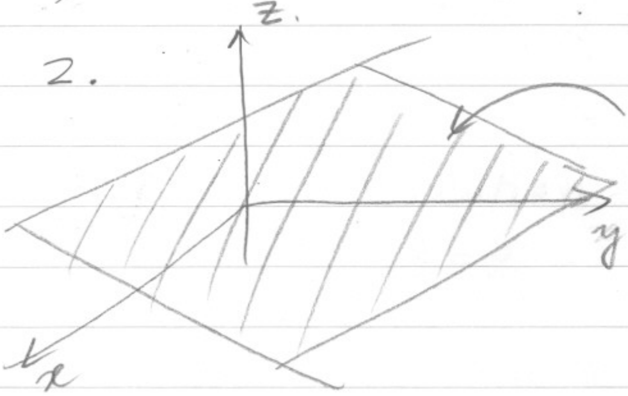
(ii) If $\underline{u} \in W$, then $\forall c \in \mathbb{R}$, $c\underline{u} \in W$.

examples.

1.

a line in $\mathbb{R}^2$, Q: Is it a subspace?
A: Not if it does not pass from the origin.

2.



a plane in $\mathbb{R}^3$ is a subspace if it passes from origin.

3. In vector space $\mathbb{R}^n$: $W = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : a_1 x_1 + \cdots + a_n x_n = 0 \right\}$

is a subspace.

proof

(i) $\underline{u} \in W \Rightarrow a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$

$\underline{v} \in W \Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

$\underline{u}+\underline{v} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$; $a_1(u_1+v_1) + a_2(u_2+v_2) + \cdots + a_n(u_n+v_n) = 0$ ✓

$\Rightarrow \underline{u}+\underline{v} \in W$.

(ii) $\underline{u} \in W \Rightarrow a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$

$c\underline{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$, $a_1 c u_1 + a_2 c u_2 + \cdots + a_n c u_n = c\underbrace{(a_1 u_1 + \cdots + a_n u_n)}_{0} = 0$ ✓

$\Rightarrow c\underline{u} \in W$.

Theorem 2 If $\underline{\underline{A}}_{m\times n}$ is a constant matrix, then the solution set of the homogeneous system:

$\underline{\underline{A}}\,\underline{x} = \underline{0}$ is a subspace of $\mathbb{R}^n$.

Proof Since the solution set consists of vectors in $\mathbb{R}^n$, which is a vector space, to show that the solution set is a subspace we have to prove that:

(i) If $\underline{x}_1$ and $\underline{x}_2$ are in the solution set, so is $\underline{x}_1+\underline{x}_2$

(ii) If $\underline{x}_1$ is in the solution set, so is $c\underline{x}_1$ for any $c \in \mathbb{R}$.

(i): $\underline{\underline{A}}\,\underline{x}_1 = \underline{0}$, $\underline{\underline{A}}\,\underline{x}_2 = \underline{0}$ $\Rightarrow$ $\underline{\underline{A}}(\underline{x}_1+\underline{x}_2) = \underline{\underline{A}}\,\underline{x}_1 + \underline{\underline{A}}\,\underline{x}_2 = \underline{0}$
$\Rightarrow$ $\underline{x}_1+\underline{x}_2$ is in the solution set

(ii) $\underline{\underline{A}}\,\underline{x}_1 = \underline{0}$ $\underline{\underline{A}}(c\underline{x}_1) = c(\underline{\underline{A}}\,\underline{x}_1) = c\cdot\underline{0} = \underline{0} \Rightarrow c\underline{x}_1$ is in the solution set.

Q Is the solution set of the non-homogeneous system
$\underline{\underline{A}}\,\underline{x} = \underline{b}$
a subspace?

A No! Because a subspace must have $\underline{0}$ as its member.
but: $\underline{\underline{A}}\,\underline{0} = \underline{0} \neq \underline{b}$ $\Rightarrow$ $\underline{0}$ is not in the solution set
$\Rightarrow$ Solution set of the non-homogeneous system is not a subspace!

ex/ (i) The smallest subspace: $\{\underline{0}\}$

(ii) The largest subspace of $\mathbb{R}^n$: $\mathbb{R}^n$ itself

Def'n A subspace which is not $\{\underline{0}\}$ or $\mathbb{R}^n$ is called as a proper subspace.

ex/ In $\mathbb{R}^n$: $S = \{\underline{x} : \underline{x} = c\cdot\underline{u}\}$ is a subspace

proof
(i) If $\underline{x}_1 \in S$, $\underline{x}_2 \in S$, then there exist c_1 and c_2 such that: $\underline{x}_1 = c_1\underline{u}$ and $\underline{x}_2 = c_2\underline{u}$. Hence:
$\underline{x}_1+\underline{x}_2 = (c_1+c_2)\underline{u} \in S$.

(ii) If $\underline{x} \in S$ and $\alpha \in \mathbb{R}$: $\alpha\underline{x} = \alpha(c\cdot\underline{u}) = (\alpha c)\underline{u} = c'\underline{u} \in S$.
$\Rightarrow \underline{x} = c\underline{u}$

Linear Combinations and Independence of Vectors

Def'n $\underline{w}$ is called as a linear combination of vectors $\underline{v}_1, \dots, \underline{v}_k$ if there exist scalars $c_1, \dots, c_k$ such that:

$$\underline{w} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_k \underline{v}_k .$$

Note that $\underline{w}$ can be expressed as a linear combination of $\underline{v}_1, \dots, \underline{v}_k$ if:

$$\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \underline{w}$$

has a solution.

ex/Q Is $\underline{w} = \begin{bmatrix} -7 \\ 7 \\ 11 \end{bmatrix}$ a linear combination of $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$?

$$\underbrace{\begin{array}{c} \\ A \end{array} \begin{array}{c} \begin{array}{ccc} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{array} \\ \begin{bmatrix} 1 & -4 & -3 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{array}{c} \underline{w} \\ \begin{bmatrix} -7 \\ 7 \\ 11 \end{bmatrix} \end{array}}$$

Reduced row echelon form is:
$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is consistent! Hence $\underline{w}$ is a linear combination of $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$:

$$c_3 = t, \quad c_2 = 3 - t, \quad c_1 = 5 - t. \quad \text{Thus:}$$

$$\underline{w} = (5-t)\underline{v}_1 + (3-t)\underline{v}_2 + t\underline{v}_3 \quad \text{for any } t \in \mathbb{R}.$$

Def'n Span / Spanning Set

Let $\underline{v}_1, \dots, \underline{v}_k$ be vectors in a vector space $\mathcal{V}$. Then, if every vector in $\mathcal{V}$ is a linear combination of $\underline{v}_1, \dots, \underline{v}_k$

then we say that the vectors $\underline{v}_1, \ldots, \underline{v}_k$ <u>span</u> the vector space $\mathcal{V}$. We can also say that $S = \{\underline{v}_1, \ldots, \underline{v}_k\}$ is a <u>spanning set</u>.

ex/ Unit vectors $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$S = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is a spanning set because:

$$\forall \underline{x} \in \mathbb{R}^3: \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3.$$

<u>Theorem 1</u>

Let $\underline{v}_1, \ldots, \underline{v}_k$ be vectors in the vectorspace $\mathcal{V}$. Then

$$W = \{\underline{x} : \underline{x} = \alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k \; ; \; \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$$

is a subspace of $\mathcal{V}$.

<u>Proof</u> (i) If $\underline{x}_1 \in W$ and $\underline{x}_2 \in W$, want to show that $\underline{x}_1 + \underline{x}_2 \in W$.

Since $\underline{x}_1 \in W$, there exists $\alpha_1, \ldots, \alpha_k$ such that

$$\underline{x}_1 = \alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k$$

Since $\underline{x}_2 \in W$, there exists $\beta_1, \ldots, \beta_k$ such that

$$\underline{x}_2 = \beta_1 \underline{v}_1 + \cdots + \beta_k \underline{v}_k$$

Hence $\underline{x}_1 + \underline{x}_2 = (\alpha_1 + \beta_1)\underline{v}_1 + \cdots + (\alpha_k + \beta_k)\underline{v}_k$

$$\Rightarrow \underline{x}_1 + \underline{x}_2 \in W.$$

(ii) We also want to show that if $\underline{x} \in W$ then for any $c \in \mathbb{R}$: $c\underline{x} \in W$.

Since $\underline{x} \in W \Rightarrow$ there exists $\alpha_1, \ldots, \alpha_k$ such that

$$\underline{x} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_k \underline{v}_k$$

$$c\underline{x} = c(\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_k \underline{v}_k)$$

$$= (c\alpha_1)\underline{v}_1 + (c\alpha_2)\underline{v}_2 + \cdots + (c\alpha_k)\underline{v}_k$$

$$\Rightarrow c\underline{x} \in W. \quad \blacksquare$$

We say that $W = \text{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$.

ex/ Show that span$\{\underline{v}_1, \ldots, \underline{v}_k\}$ is the smallest subspace that contains the vectors $\underline{v}_1, \ldots, \underline{v}_k$ in it.

Def'n Linear Independence:

$\underline{v}_1, \ldots, \underline{v}_k$ in vector space V are linearly independent if $c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k = \underline{0}$ implies that $c_1 = c_2 = \cdots = c_k = 0$.

ex/ $\underline{e}_1, \ldots, \underline{e}_n$ are linearly independent in $\mathbb{R}^n$.

proof

$$c_1 \underline{e}_1 + \cdots + c_n \underline{e}_n = \underline{0} \Rightarrow c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underline{0}$$

$$\Rightarrow c_1 = 0, c_2 = 0, \ldots, c_n = 0.$$

ex/ Determine whether $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 5 \end{bmatrix}$ are linearly independent in $\mathbb{R}^4$.

Linear independence of these vectors imply that:

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0}$$

has only one solution $c_1 = c_2 = c_3 = 0$.

This can be checked by finding the solution set to:

$$\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented coefficient matrix of this system is:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 2 & 4 & 7 & 0 \\ 1 & 1 & 5 & 0 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

all the variables c_1, c_2 and c_3 are leading variables.

Therefore, the system has a unique solution of $c_1 = c_2 = c_3 = 0$.

<u>Def'n</u> If a set of vectors $\{\underline{v}_1, \dots, \underline{v}_k\}$ are not linearly independent then, it is called as a linearly dependent set.

<u>Claim</u> If $\{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly dependent then:

at least one of them can be represented as a linear combination of others:

<u>proof</u> Since $\underline{v}_1, \dots, \underline{v}_k$ are linearly dependent, then there exists $c_1, \dots, c_k$, not all zero, such that:

$$c_1 \underline{v}_1 + \dots + c_k \underline{v}_k = \underline{0}$$

Assume that $c_i \neq 0$, then:

$$\underline{v}_i = -\frac{1}{c_i}\left(c_1 \underline{v}_1 + \dots + c_k \underline{v}_k\right) = \sum_{j \neq i} \left(-\frac{c_j}{c_i}\right) \underline{v}_j \quad \blacksquare$$

is linear combination of $\{\underline{v}_1, \dots, \underline{v}_k\} \setminus \{\underline{v}_i\}$

<u>Thm</u> In $\mathbb{R}^n$ the set $\{\underline{v}_1, \dots, \underline{v}_n\}$ is a linearly independent set iff $\left|[\underline{v}_1 \cdots \underline{v}_n]\right| \neq 0$.

<u>Proof</u> $\{\underline{v}_1, \dots, \underline{v}_n\}$ linearly independent $\Longrightarrow$ $[\underline{v}_1 \cdots \underline{v}_n]\underline{c} = \underline{0}$ has a unique solution.

$[\underline{v}_1 \cdots \underline{v}_n]\underline{c} = \underline{0}$ has a unique solution

$\Longrightarrow$ $[\underline{v}_1 \cdots \underline{v}_n]$ is invertible

$\Longrightarrow$ $\left|[\underline{v}_1 \cdots \underline{v}_n]\right| \neq 0$. $\blacksquare$

<u>Def'n</u> Basis:

A finite set S of vectors in a vector space V is a basis for V if:

(a) The vectors in S are linearly independent

(b) The vectors in S span V.

ex/ The standard basis for $\mathbb{R}^n$:

$$S = \{ \underline{e}_1, \dots, \underline{e}_n \}$$

Have shown that S is linearly independent.

Any $\underline{x} \in \mathbb{R}^n$ $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{e}_1 + \dots + x_n \underline{e}_n$

Hence $\mathrm{span}(S) = \mathbb{R}^n$.

ex/ Any set of n linearly independent vectors is a basis for $\mathbb{R}^n$.

Proof: Assume that there exists a vector $\underline{x}$ in $\mathbb{R}^n$ that cannot be represented as a linear combination of the independent vectors $\underline{v}_1, \dots, \underline{v}_n$. Then the new set of vectors $\{\underline{v}_1, \dots, \underline{v}_n, \underline{x}\}$ is a set of $(n+1)$ linearly independent vectors in $\mathbb{R}^n$. This implies that

$$\begin{bmatrix} \underline{v}_1 & \underline{v}_2 \cdots \underline{v}_n & \underline{x} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a unique solution. But this system has n equations and $(n+1)$ unknowns. Thus cannot have a unique solution. Thus, there exists $\begin{bmatrix} c_1 \\ \vdots \\ c_{n+1} \end{bmatrix} \neq \underline{0}$ that solves the equation. Hence $\{\underline{v}_1, \dots, \underline{v}_n, \underline{x}\}$ is linearly dependent $\Rightarrow \Leftarrow$.

Thm: Let $S = \{\underline{v}_1, \dots, \underline{v}_n\}$ be a basis for the vector space V. Then any set of more than n vectors are linearly dependent.

Proof: Assume that there exists a set of m linearly independent vectors $\{\underline{w}_1, \dots, \underline{w}_m\}$, $m > n$.

Then; $c_1 \underline{w}_1 + \dots + c_m \underline{w}_m = \underline{0}$ has a unique solution of $c_1 = c_2 = \dots = c_m = 0$.

Since $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis; there exists $\underline{a}_1, \ldots, \underline{a}_m$ such that:

$$[\underline{v}_1 \cdots \underline{v}_n] \cdot \underline{a}_1 = \underline{w}_1$$

$$[\underline{v}_1 \cdots \underline{v}_n] \cdot \underline{a}_2 = \underline{w}_2$$

$$\vdots$$

$$[\underline{v}_1 \cdots \underline{v}_n] \, \underline{a}_m = \underline{w}_m$$

$$\Rightarrow \quad [\underline{v}_1 \cdots \underline{v}_n] \cdot [\underline{a}_1 \cdots \underline{a}_m] = [\underline{w}_1 \cdots \underline{w}_m]$$

Hence $[\underline{v}_1 \cdots \underline{v}_n] [\underline{a}_1 \cdots \underline{a}_m] \cdot \underline{c} = [\underline{w}_1 \cdots \underline{w}_m] \underline{c} = \underline{0}$

$$\Rightarrow \quad [\underline{a}_1 \cdots \underline{a}_m] \underline{c} = \underline{0}$$

This system cannot have a unique solution because it has n equations and $m > n$ unknowns. Thus, ~~there~~ exists other solutions to $[\underline{w}_1 \cdots \underline{w}_m] \underline{c} = \underline{0}$ than $\underline{c} = \underline{0}$. $\Rightarrow\Leftarrow$.

Thm Any two basis for a vector space consists of the same number of vectors which is called as the dimension of the vector space.

ex/ (i) Any basis for $\mathbb{R}^n$ has n-vectors in it.
(ii) The vector space of polynomials are infinite dimensional
(iii) The vector space of functions are also infinite dimensional

Thm $\mathcal{V}$, n-dimensional vector space, $S \subset \mathcal{V}$
(a) If S is linearly independent and has n-vectors it is a basis for $\mathcal{V}$.
(b) If S spans $\mathcal{V}$ and consists of n-vectors, it is a basis for $\mathcal{V}$.
(c) If S is linearly independent, then it is contained in a basis for $\mathcal{V}$.
(d) If S spans $\mathcal{V}$, it contains a basis for $\mathcal{V}$.

Bases for Solution Spaces (subspaces)

$\underline{\underline{A}}\,\underline{x} = \underline{0}$, $\underline{\underline{A}}_{m\times n}$. $\Rightarrow$ Solution set W is a subspace of $\mathbb{R}^n$

Want to find a basis for W.

Reduced Row Echelon form of $\underline{\underline{A}}$ provides the basis.

ex/

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_2 , x_4 and x_5 are free variables, x_1 and x_3 are leading variables.

Hence : $x_2 = r$, $x_4 = s$, $x_5 = t$, $x_1 = -2r + 2s - 3t$

$x_3 = s - 4t$.

$$W = \left\{ \underline{x} : \underline{x} = \begin{bmatrix} -2r+2s-3t \\ r \\ s-4t \\ s \\ t. \end{bmatrix} \right\}$$

$$= \left\{ \underline{x} : \underline{x} = r \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A basis for W is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Row and Column Spaces

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \underline{r}_1^T \\ \underline{r}_2^T \\ \vdots \\ \underline{r}_m^T \end{bmatrix} \quad \text{where } \underline{r}_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix} \in \mathbb{R}^n$$

row space : $\text{Row}(\underline{\underline{A}}) = \text{span}\{\underline{r}_1, \ldots, \underline{r}_m\}$

row rank $\equiv$ dimension $(\text{Row}(\underline{\underline{A}}))$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\underline{r}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \\ 3 \end{bmatrix}$, $\underline{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \\ 2 \end{bmatrix}$, $\underline{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 7 \end{bmatrix}$, $\underline{r}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathrm{span}\{\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4\} = \mathrm{span}\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$ row rank = 3.

Thm The non-zero row vectors of an echelon matrix are linearly independent and they form a basis for its row space.

Thm If two matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are row equivalent, then they have the same row space.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

row space $= \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -7 \end{bmatrix} \right\}$, row-rank = 3.

Column Space and Column rank

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [\underline{a}_1 \ \underline{a}_2 \cdots \underline{a}_n], \quad \underline{a}_i \in \mathbb{R}^m$$

$\mathrm{Col}(\underline{\underline{A}}) = \mathrm{span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$.

column rank $\equiv \dim(\mathrm{Col}(\underline{\underline{A}}))$.

ex/ $\underline{\underline{E}} = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathrm{Col}(\underline{\underline{E}}) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$

leading columns or "pivot" columns

claim Column rank of $\underline{\underline{A}}$ and its row echelon forms are the same.

hint Consider $\underline{\underline{A}}\,\underline{x} = \underline{0}$ and $\underline{\underline{E}}\,\underline{x} = \underline{0}$ which share the same solution set.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \underline{\underline{E}} = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

pivot columns of $\underline{\underline{E}}$.

$$\Rightarrow \mathrm{Col}(\underline{\underline{A}}) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

Note that

This way can extract a maximal linearly independent subset from a given set of vectors.

Rank and Dimension

Row-rank := # leading rows in $\underline{\underline{E}}$ = # of leading variables

column-rank = # pivot columns in $\underline{\underline{E}}$ = # of leading variables

$\Rightarrow$ Row-rank = Column-rank $\triangleq$ rank.

Def'n The solution space of $\underline{\underline{A}}\underline{x} = \underline{0}$ is called as the nullspace of $\underline{\underline{A}}$, denoted by $\mathrm{Null}(\underline{\underline{A}})$.

$\underline{\underline{A}}_{m\times n}$: If it has r leading variables, then it has $(n-r)$ free variables.

Hence $\dim \mathrm{Null}(\underline{\underline{A}}) = n-r \Rightarrow \dim \mathrm{Null}(\underline{\underline{A}}) + \mathrm{rank}(\underline{\underline{A}}) = n$ //

Non-homogeneous Linear Systems

$\underline{\underline{A}}\,\underline{x} = \underline{b}$ is consistent $\Rightarrow \underline{b} \in \mathrm{Col}(\underline{\underline{A}})$.

If $\underline{x}_0$ is a particular solution, i.e., :

$$\underline{\underline{A}}\,\underline{x}_0 = \underline{b}$$

then all the solutions can be found as:

$$\underline{x} = \underline{x}_0 + \underline{x}_h \leftarrow \text{any vector from } \mathrm{Null}(\underline{\underline{A}}).$$

Orthogonal Vectors in $\mathbb{R}^n$

Def'n Inner Product:

In a vector space V, inner product of two vectors $\underline{u}$ and $\underline{v}$ is a scalar $<\underline{u},\underline{v}>$ such that:

(i) $<\underline{u},\underline{v}> = <\underline{v},\underline{u}>$

(ii) $<\underline{u},\underline{v}+\underline{w}> = <\underline{u},\underline{v}> + <\underline{u},\underline{w}>$

(iii) $<c\underline{u},\underline{v}> = c<\underline{u},\underline{v}>$

(iv) $<\underline{u},\underline{u}> \geq 0$, $<\underline{u},\underline{u}> = 0 \iff \underline{u} = \underline{0}$.

show that $<\underline{u},\underline{v}> = \underline{u}\cdot\underline{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ is an inner product.

length of $\underline{u}$: $\|\underline{u}\|_2 = \sqrt{\underline{u}\cdot\underline{u}} = \sqrt{u_1^2 + \cdots + u_n^2}$

Thm The Cauchy-Schwartz Inequality:

If $\underline{u}$ and $\underline{v}$ are vectors in $\mathbb{R}^n$, then:

$$|\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|_2 \cdot \|\underline{v}\|_2$$

Proof

Since $(x\underline{u}+\underline{v})\cdot(x\underline{u}+\underline{v}) \geq 0$ for any $x \in \mathbb{R}$

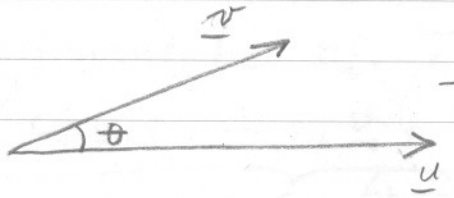
$$\Rightarrow \underbrace{x^2\underbrace{(\underline{u}\cdot\underline{u})}_{\|\underline{u}\|^2} + 2x\,\underline{u}\cdot\underline{v} + \underbrace{\underline{v}\cdot\underline{v}}_{\|\underline{v}\|^2}}_{2^{nd}\text{ order polynomial in } x.} \geq 0$$

$\Rightarrow$ Discriminant ≤ 0. $\Rightarrow (\underline{u}\cdot\underline{v})^2 - \|\underline{u}\|^2\cdot\|\underline{v}\|^2 \leq 0$

$$\Rightarrow (\underline{u}\cdot\underline{v})^2 \leq \|\underline{u}\|^2\cdot\|\underline{v}\|^2$$

$$\Rightarrow |\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|\cdot\|\underline{v}\|. \quad \blacksquare$$

Note that



$$\frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\cdot\|\underline{v}\|} = \cos\theta.$$

Def'n In vector space $\mathcal{V}$, $\underline{u}$ and $\underline{v}$ are orthogonal if $\langle \underline{u}, \underline{v} \rangle = 0$.

Def'n The distance between $\underline{u}$ and $\underline{v}$ is defined as:

$$d(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \|_2 = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$$

Thm The triangle inequality

If $\underline{u}$ and $\underline{v}$ are in $\mathbb{R}^n$:

$$\| \underline{u} + \underline{v} \| \leq \| \underline{u} \| + \| \underline{v} \|.$$

Proof

$$\begin{aligned}
\| \underline{u} + \underline{v} \|^2 &= (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) \\
&= \| \underline{u} \|^2 + \| \underline{v} \|^2 + 2\, \underline{u} \cdot \underline{v} \\
&\overset{\text{Cauchy-Scwartz Inequality}}{\leq} \| \underline{u} \|^2 + \| \underline{v} \|^2 + 2 \| \underline{u} \| \cdot \| \underline{v} \| \\
&= (\| \underline{u} \| + \| \underline{v} \|)^2
\end{aligned}$$

$$\Rightarrow \| \underline{u} + \underline{v} \| \leq \| \underline{u} \| + \| \underline{v} \|.$$

with equality if $\underline{u} \cdot \underline{v} = \| \underline{u} \| \cdot \| \underline{v} \|$ or $\cos\theta = 1$, $\theta = 0$

Also note that if $\underline{u} \cdot \underline{v} = 0$: $\left(\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \right)$

$$\| \underline{u} + \underline{v} \|^2 = \| \underline{u} \|^2 + \| \underline{v} \|^2 \quad \text{(Pythagorean Formula).}$$

Thm Orthogonality and linear Independence

If the non-zero vectors $\underline{v}_1, \ldots, \underline{v}_k$ are mutually orthogonal then they are linearly independent.

Proof

Assume that there exists an orthogonal set of nonzero vectors which are linearly dependent. Then for a non-zero set of scalars $c_1, \ldots, c_k$ we have:

$$c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k = \underline{0}$$

where at least one of the c's are non-zero. Let that non-zero coefficient be c_i. Then:

$$\underbrace{\underline{v}_i \cdot (c_1 \underline{v}_1 + \cdots + c_i \underline{v}_i + \cdots + c_k \underline{v}_k)}_{c_i \cdot (\underline{v}_i \cdot \underline{v}_i)} = 0$$