Hence $c_i \|\underline{v}_i\|^2 = 0 \Rightarrow c_i = 0$ because $\|\underline{v}_i\|^2 > 0$ (set of vectors are non-zero),
$\Rightarrow\Leftarrow$. Thus, the assumption leads to a contradiction.
Therefore : there exists no set of non-zero orthogonal vectors which are linearly dependent. $\blacksquare$

Orthogonal Complements

Def'n The vector $\underline{u}$ is orthogonal to a subspace $\mathcal{W}$ of vector space $\mathcal{V}$, provided that, $\langle \underline{u}, \underline{w} \rangle = 0$ for $\forall\, \underline{w} \in \mathcal{W}$.

Def'n The orthogonal complement $\mathcal{W}^{\perp}$ of $\mathcal{W}$ is the set of all vectors in $\mathcal{V}$ that are orthogonal to $\mathcal{W}$.
(Note that here $\mathcal{W}$ does not have to be a subspace!)

ex/ (i) $\mathcal{W}$ is a subspace of $\mathbb{R}^3$: $\mathcal{W} = \{ \underline{x} : \underline{x} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2, \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \}$

where $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

How can you find an orthogonal vector to $\mathcal{W}$?

Thm Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$.

(i) $\mathcal{W}^{\perp}$ is also a subspace
(ii) $\mathcal{W} \cap \mathcal{W}^{\perp} = \{ \underline{0} \}$
(iii) $(\mathcal{W}^{\perp})^{\perp} = \mathcal{W}$
(iv) S is a spanning set for $\mathcal{W}$, then: $\underline{u} \in \mathcal{W}^{\perp} \Longleftrightarrow \underline{u}$ is orthogonal to S.

Thm $\underline{\underline{A}}_{m\times n}$: $(\mathrm{Row}(\underline{\underline{A}}))^{\perp} = \mathrm{Null}(\underline{\underline{A}})$.

Proof : If $\underline{x} \in \mathrm{Null}(\underline{\underline{A}}) \Rightarrow \underline{\underline{A}}\,\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} \underline{r}_1^T \\ \vdots \\ \underline{r}_m^T \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow \underline{x} \perp \mathrm{span}\{\underline{r}_1, \dots, \underline{r}_m\} \Rightarrow \underline{x} \in (\mathrm{Row}(\underline{\underline{A}}))^{\perp}$

$\Rightarrow \mathrm{Null}(\underline{\underline{A}}) \subset (\mathrm{Row}(\underline{\underline{A}}))^{\perp}$

If $\underline{x} \in (\text{Row}(\underline{\underline{A}}))^{\perp} \Rightarrow \underline{x} \perp \underline{r}_i, \quad 1 \leq i \leq m$

$\Rightarrow \underline{\underline{A}}\,\underline{x} = \underline{0}$

$\Rightarrow \underline{x} \in \text{Null}(\underline{\underline{A}})$

$\Rightarrow (\text{Row}(\underline{\underline{A}}))^{\perp} \subset \text{Null}(\underline{\underline{A}})$

$$\Rightarrow \quad (\text{Row}(\underline{\underline{A}}))^{\perp} = \text{Null}(\underline{\underline{A}})$$

Chapter 6, Eigenvalues and Eigenvectors

Def'n

$$\underset{\text{square matrix}}{\underline{\underline{A}}}\,\underline{v} = \underset{\text{eigenvalue}}{\lambda}\,\underset{\text{eigenvector}}{\underline{v}}, \quad \underline{v} \neq \underline{0}.$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}, \quad \underline{\underline{A}} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \overset{\text{eigenvalue.}}{2} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Aslo ; $\underline{\underline{A}} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \overset{\text{eigenvalue.}}{1} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

The characteristic Equation

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v} = \lambda \underline{\underline{I}}\,\underline{v} \Rightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0}$$

A nonzero solution exists for $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0} \Leftrightarrow \underbrace{\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0}_{\text{characteristic Eqn.}}$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}, \quad \underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{bmatrix}$

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = (5-\lambda)(-4-\lambda) + 14 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

$$\Rightarrow |\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0 \Rightarrow \lambda = 3 \text{ or } \lambda = -2.$$

Eigenvector associated with $\lambda = -2$;

$$\begin{bmatrix} 5+2 & 7 \\ -2 & -4+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 7 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t . \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $t \in \mathbb{R}$. a non-zero eigenvector is: $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenvector associated with $\lambda = 3$;

$$\begin{bmatrix} 5-3 & 7 \\ -2 & -4-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t . \begin{bmatrix} -7/2 \\ 1 \end{bmatrix}$, can choose $\underline{v}_2 = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$.

Thm λ is an eigenvalue of $\underline{\underline{A}} \Leftrightarrow \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$.

- Note that $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ is a polynomial in λ of degree n for $\underline{\underline{A}}_{n \times n}$.
  Hence, we have n (possibly complex) roots for the characteristic eqn.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, $\det \begin{bmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0 \Rightarrow \underbrace{\lambda = 2}_{\text{double root}}$

$$(\underline{\underline{A}} - 2\underline{\underline{I}}) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, hence can choose only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is linearly independent in this solution set.

Eigenspaces

Solution set of $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0}$ for λ, an eigenvector, is called as the eigenspace of $\underline{\underline{A}}$ associated with eigenvalue λ.

Diagonalization of Matrices

If $\underline{\underline{A}}_{n \times n}$ has n linearly indendent eigenvectors, then:

$$\underline{\underline{A}} \cdot \underbrace{\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{bmatrix}}_{\underline{\underline{P}}} = \underbrace{\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{bmatrix}}_{\underline{\underline{P}}} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix}}_{\underline{\underline{D}}}$$

$$\Rightarrow \quad \underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{D}} \quad \Rightarrow \quad \underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1} \quad \text{or} \quad \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{D}}$$

($\underline{\underline{P}}^{-1}$ exists because its columns are linearly independent).

ex/ $\underline{\underline{A}} = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$, $\lambda_1 = 2 \Rightarrow \underline{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \Rightarrow \underline{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\underline{v}_1$ & $\underline{v}_2$ are linearly independent. Thus:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Def'n $\underline{\underline{A}}_{n \times n}$ and $\underline{\underline{B}}_{n \times n}$ are called as similar if:

$$\underline{\underline{B}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}.$$

Criterion for diagonalizability:

$\underline{\underline{A}}_{n\times n}$ is diagonalizable $\iff$ it has n linearly independent eigenvectors.

Theorem

Let $\underline{v}_1, \ldots, \underline{v}_k$ be eigenvectors of $\underline{\underline{A}}$ associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then, $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent.

proof by Induction

$K=1$; $\underline{v}_1$ forms a linearly independent set because $\underline{v}_1 \neq \underline{0}$.

Assume that any set of $(k-1)$ eigenvectors are linearly independent if they are associated with distinct eigenvalues. Now, consider following linear combination.

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n = \underline{0} \qquad (\star)$$

Need to show that $c_1 = c_2 = \cdots = c_n = 0$.
Multiply both sides by $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})$ to get:

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})(c_1 \underline{v}_1 + \cdots + c_n \underline{v}_n) = \underline{0}$$

Note that $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}) \cdot \underline{v}_j = \begin{cases} 0 & \text{if } j=1 \\ (\lambda_j - \lambda_1)\, \underline{v}_j & \text{if } j \neq 1. \end{cases}$

Hence, we get:

$$c_2 (\lambda_2 - \lambda_1) \underline{v}_2 + \cdots + c_n (\lambda_n - \lambda_1) \underline{v}_n = \underline{0}$$

Since these n vectors are linearly independent:

$$c_j (\lambda_j - \lambda_1) = 0 \quad \text{for } j > 1.$$

But $\lambda_j \neq \lambda_1 \Rightarrow c_j = 0$ for $j > 1$.
Thus, in $\star$, the only non-zero coefficient can be c_1.
But this is a contradiction because:

$$c_1 \underline{v}_1 = 0 \Rightarrow c_1 = 0.$$

Hence, the only combination in $\star$ that results $\underline{0}$ is $c_1 = \cdots = c_n = 0$.
Thus, the eigenvectors associated with distinct eigenvalues are lin. independent.

<u>Thm</u> $\underline{\underline{A}}_{n\times n}$ has distinct eigenvalues $\Rightarrow$ $\underline{\underline{A}}$ is diagonalizable.

Have seen that if the eigenvectors of $\underline{\underline{A}}$ are linearly independent then $\underline{\underline{A}}$ is diagonalizable. If $\underline{\underline{A}}$ has distinct eigenvalues, then it has a linearly independent set of eigenvectors. Thus, it is diagonalizable.

<u>Thm</u> Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $\underline{\underline{A}}_{n\times n}$. Let S_i be a basis for the eigenspace associated with λ_i. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set of eigenvectors of $\underline{\underline{A}}$.

<u>Proof</u> To keep the notation simple, assume $k=3$.

$S_1 = \{\underline{u}_1, \ldots, \underline{u}_p\}$

$S_2 = \{\underline{v}_1, \ldots, \underline{v}_q\}$

$S_3 = \{\underline{w}_1, \ldots, \underline{w}_r\}$

Assume that the union set $S = S_1 \cup S_2 \cup S_3$ is dependent. Then, there exists nonzero coefficients such that:

$$\underbrace{a_1\underline{u}_1 + a_2\underline{u}_2 + \cdots + a_p\underline{u}_p}_{\underline{u}} + \underbrace{b_1\underline{v}_1 + b_2\underline{v}_2 + \cdots + b_q\underline{v}_q}_{\underline{v}} + \underbrace{c_1\underline{w}_1 + c_2\underline{w}_2 + \cdots + c_r\underline{w}_r}_{\underline{w}} = \underline{0}$$

where $\underline{u}$ is an eigenvector associated with λ_1
" $\underline{v}$ " " " " λ_2
" $\underline{w}$ " " " " λ_3.

Hence, $\underline{u}$, $\underline{v}$ and $\underline{w}$ must be linearly independent if they are not all zeros. Thus: $\underline{u} = \underline{0}$, $\underline{v} = \underline{0}$ and $\underline{w} = \underline{0}$. But $\underline{u}$ can only be zero if $a_1 = \cdots = a_p = 0$ because $u_1, \ldots, \underline{u}_p$ forms a basis (hence they are linearly independent). Likewise, $b_1 = \cdots = b_q = 0$ and $c_1 = \cdots = c_r = 0$. Thus, we end up with a contradiction. Hence, the theorem has the right statement.

Matrix Powers

If $\underline{\underline{A}}_{n\times n}$ has linearly independent eigenvectors, then:

$$\underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}$$

Thus: $$\underline{\underline{A}}^k = \underbrace{\underline{\underline{A}} \cdots\cdots \underline{\underline{A}}}_{k \text{ times}} = (\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1})(\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1})\cdots(\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1})$$

$$= \underline{\underline{P}}\,\underline{\underline{D}}^k\,\underline{\underline{P}}^{-1}.$$

Since $\underline{\underline{D}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$\underline{\underline{D}}^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

The Cayley-Hamilton Thm

If $\underline{\underline{A}}_{n\times n}$ has the characteristic polynomial:

$$p(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

then: $$p(\underline{\underline{A}}) = (-1)^n\underline{\underline{A}}^n + c_{n-1}\underline{\underline{A}}^{n-1} + \cdots + c_1\underline{\underline{A}} + c_0\underline{\underline{I}} = \underline{\underline{0}}.$$

proof The theorem is valid for any $\underline{\underline{A}}_{n\times n}$. But we will provide here proof for $\underline{\underline{A}}$ which is diagonalizable: $\underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}$.

$$p(\underline{\underline{D}}) = (-1)^n\underline{\underline{D}}^n + c_{n-1}\underline{\underline{D}}^{n-1} + \cdots + c_1\underline{\underline{D}} + c_0\underline{\underline{I}}$$

$$= (-1)^n\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_n^n \end{bmatrix} + c_{n-1}\begin{bmatrix} \lambda_1^{n-1} & & \\ & \ddots & \\ & & \lambda_n^{n-1} \end{bmatrix} + \cdots + c_1\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} + \begin{bmatrix} c_0 & & \\ & \ddots & \\ & & c_0 \end{bmatrix}$$

$$= \begin{bmatrix} p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_n) \end{bmatrix} = \underline{\underline{0}}$$

$$p(\underline{\underline{A}}) = \underline{\underline{P}} \cdot p(\underline{\underline{D}})\, \underline{\underline{P}}^{-1} = \underline{\underline{P}} \cdot \underline{\underline{0}}\, \underline{\underline{P}}^{-1} = \underline{\underline{0}}. \quad \blacksquare$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$$p(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

Thus; $-\underline{\underline{A}}^3 + 7\underline{\underline{A}}^2 - 16\underline{\underline{A}} + 12\underline{\underline{I}} = \underline{\underline{0}}.$

$\Rightarrow \quad \underline{\underline{A}}^3 = 7\underline{\underline{A}}^2 - 16\underline{\underline{A}} + 12\underline{\underline{I}}.$

If we want to compute $\underline{\underline{A}}^4$:

$$\begin{aligned} \underline{\underline{A}}^4 = \underline{\underline{A}} \cdot \underline{\underline{A}}^3 &= 7\underline{\underline{A}}^3 - 16\underline{\underline{A}}^2 + 12\underline{\underline{A}} \\ &= 7(7\underline{\underline{A}}^2 - 16\underline{\underline{A}} + 12\underline{\underline{I}}) - 16\underline{\underline{A}}^2 + 12\underline{\underline{A}} \\ &= 33\underline{\underline{A}}^2 - 100\underline{\underline{A}} + 84\underline{\underline{I}}. \end{aligned}$$

Or if we want to compute $\underline{\underline{A}}^{-1}$:

$$\underline{\underline{A}}^{-1}(-\underline{\underline{A}}^3 + 7\underline{\underline{A}}^2 - 16\underline{\underline{A}} + 12\underline{\underline{I}}) = \underline{\underline{0}}$$

$\Rightarrow \quad -\underline{\underline{A}}^2 + 7\underline{\underline{A}} - 16\underline{\underline{I}} + 12\underline{\underline{A}}^{-1} = \underline{\underline{0}}$

$\Rightarrow \quad \underline{\underline{A}}^{-1} = \frac{1}{12}(\underline{\underline{A}}^2 - 7\underline{\underline{A}} + 16\underline{\underline{I}}) = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ -2 & 5 & -1 \\ -2 & 2 & 2 \end{bmatrix}.$