HW MATH227/7

1. Find $\mathbf{u} \cdot \mathbf{v}$ given that $||\mathbf{u}+\mathbf{v}|| = 1$ and $||\mathbf{u}-\mathbf{v}|| = 5$. (Hint: Use Theorem 4.1.6)

2. It can be proved that if A is a 2×2 matrix with $det(A) = 1$ and such that the column vectors of A are orthogonal and have length 1, then multiplication by A is a rotation through some angle θ. Verify that

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

satisfies the stated conditions and find the angle of rotation.

3. Find the standard matrix for the stated composition of linear operators in R^2
(a) A rotation of 90°, followed by a reflection about the line $y = x$.
(b) An orthogonal projection on the y-axis, followed by a contraction with factor $k = \frac{1}{2}$.
(c) A reflection about the x-axis, followed by a dilation with factor $k = 3$.

4. Find the standard matrix for the linear operator defined by the equations and use Theorem 4.3.4 to determine whether the operator is one-to-one

$$(a) \quad \begin{aligned} w_1 &= 8x_1 + 4x_2 \\ w_2 &= 2x_1 + x_2 \end{aligned}$$

$$(b) \quad \begin{aligned} w_1 &= 2x_1 - 3x_2 \\ w_2 &= 5x_1 + x_2 \end{aligned}$$

$$(c) \quad \begin{aligned} w_1 &= -x_1 + 3x_2 + 2x_3 \\ w_2 &= 2x_1 + 4x_3 \\ w_3 &= x_1 + 3x_2 + 6x_3 \end{aligned}$$

$$(d) \quad \begin{aligned} w_1 &= x_1 + 2x_2 + 3x_3 \\ w_2 &= 2x_1 + 5x_2 + 3x_3 \\ w_3 &= x_1 + 8x_3 \end{aligned}$$

5. Determine whether the linear operator $T : R^2 \rightarrow R^2$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2)$.

$$(a) \quad \begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$$

$$(b) \quad \begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$$

$$(c) \quad \begin{aligned} w_1 &= -x_2 \\ w_2 &= -x_1 \end{aligned}$$

$$(d) \quad \begin{aligned} w_1 &= 3x_1 \\ w_2 &= -5x_1 \end{aligned}$$

6. Determine whether the linear operator $T : R^3 \to R^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2)$.

$$(a) \quad \begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

$$(b) \quad \begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$$

$$(c) \quad \begin{aligned} w_1 &= x_1 + 4x_2 - x_3 \\ w_2 &= 2x_1 + 7x_2 + x_3 \\ w_3 &= x_1 + 3x_2 \end{aligned}$$

$$(d) \quad \begin{aligned} w_1 &= x_1 + 2x_2 + x_3 \\ w_2 &= -2x_1 + x_2 + 4x_3 \\ w_3 &= 7x_1 + 4x_2 - 5x_3 \end{aligned}$$

7. Use Theorem 4.3.2 to determine whether $T : R^2 \to R^2$ is a linear operator.
(a) $T(x, y) = (2x, y)$ (b) $T(x, y) = (x^2, y)$ (c) $T(x, y) = (-y, x)$ (d) $T(x, y) = (x, 0)$