HW MATH227/7 Solutions

1. $\mathbf{u}.\mathbf{v} = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2 = -6.$

2. $det(A) = 1$, let $\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ be the column vectors of A, then $||\mathbf{v}_1|| = 1$, $||\mathbf{v}_2|| = 1$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Hence multiplication by A is a rotation through an angle $\theta = \frac{3\pi}{4}$.

3. (a)

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b)

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

(c)

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

4.

(a) $A = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$, $det(A) = 0$, so A is not invertible and T_A is not one-to-one.

(b) $A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$, $det(A) \neq 0$, so A is invertible and T_A is one-to-one.

(c) $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}$, $det(A) = 0$, so A is not invertible and T_A is not one-to-one.

(d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, det(A)=-1, so A is invertible and T_A is one-to-one.

5.

(a) T is one to one, the standard matrix of the inverse oprator is $T^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$;
$T^{-1}(w_1, w_2) = (\frac{1}{3}w_1 - \frac{2}{3}w_2, \frac{1}{3}w_1 + \frac{1}{3}w_2)$

(b) The linear operator T is not one-to-one since $det(T) = 0$.

(c) T is one to one, the standard matrix of the inverse oprator is $T^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$;
$T^{-1}(w_1, w_2) = (-w_2, -w_1)$

(d) The linear operator T is not one-to-one.

6.

(a) T is one to one, the standard matrix of the inverse oprator is $T^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$;
$T^{-1}(w_1, w_2, w_3) = (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3w_3, -w_1 + 3w_2 - 5w_3)$

(b) T is one to one, the standard matrix of the inverse oprator is $T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{5}{14} & \frac{5}{14} & \frac{3}{14} \\ -\frac{1}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix}$;
$T^{-1}(w_1, w_2, w_3) = (\frac{w_1+w_2-w_3}{2}, \frac{-5w_1+5w_2+3w_3}{14}, \frac{-w_1+w_2+w_3}{7})$

(c) T is one to one, the standard matrix of the inverse oprator is $T^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$;
$T^{-1}(w_1, w_2, w_3) = (\frac{-3w_1-3w_2+11w_3}{2}, \frac{w_1+w_2-3w_3}{2}, \frac{-w_1+w_2-w_3}{2})$

(d) The linear operator is not one-to-one since $det(T) = 0$

7. Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$.

(a) T is a linear transformation since $T(\mathbf{u}+\mathbf{v}) = (2x_1+2x_2, y_1+y_2) = (2x_1, y_1)+(2x_2, y_2) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = (2cx_1, cy_1) = c(2x_1, y_1) = cT(\mathbf{u})$.
(b) T is not a linear transformation since $T(\mathbf{u} + \mathbf{v}) = (x_1^2 + 2x_1x_2 + x_2^2, y_1 + y_2) \neq (x_1^2 + x_2^2, y_1 + y_2) = T(\mathbf{u}) + T(\mathbf{v})$.
(c) T is a linear transformation since $T(\mathbf{u} + \mathbf{v}) = (-y_1 - y_2, x_1 + x_2) = (-y_1, x_1) + (-y_2, x_2) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = (-cy_1, cx_1) = c(-y_1, x_1) = cT(\mathbf{u})$.
(d) T is a linear transformation since $T(\mathbf{u} + \mathbf{v}) = (x_1 + x_2, 0) = (x_1, 0) + (x_2, 0) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = (cx_1, 0) = c(x_1, 0) = cT(\mathbf{u})$.