HW MATH227/9 Solutions

1.
The set of all triplets of real numbers (x, y, z) with the operations $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$ and $k(x, y, z) = (0, 0, 0)$ is a not vector space. For any triplet of real numbers (x, y, z) we have $1(x, y, z) = (0, 0, 0) \neq (x, y, z)$ whenever $(x, y, z) \neq (0, 0, 0)$, hence axiom 10 ($1\mathbf{u} = \mathbf{u}$) does not hold.

2.
Let the set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ be denoted by V. Let $A_1 = \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix}$ then $A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & 2 \\ 2 & b_1 + b_2 \end{bmatrix}$ which does not belong to V, hence axiom 1 does not hold and V is not a vector space.

3.
(a) Let the set of all vectors of the form $(a, 0, 0)$ be denoted by W. W is a vector subspace since $(a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0) \in W$ and $k(a, 0, 0) = (ka, 0, 0) \in W$.
(b)Let the set of all vectors of the form $(a, 1, 1)$ be denoted by W. W is not a vector subspace since $k = 0$, $k(a, 1, 1) = (0, 0, 0) \notin W$.
(c) Let the set of vectors of the form (a, b, c), where $b = a + c$ be denoted by W. W is a vector space : $(a_3, b_3, c_3) = (a_1, b_1, c_1) + (a_2, b_2, c_2) \in W$ since $b_3 = a_3 + c_3$. Similarly we have $k(a, b, c) \in W$.
(d) Let the set of vectors of the form (a, b, c), where $b = a + c + 1$ be denoted by W. W is not a vector subspace since $(a_1, b_1, c_1) = 0 \times (a, b, c) = (0, 0, 0) \notin W$ as $b_1 \neq a_1 + b_1 + 1$.

4.
(a) Since $tr(A_1 + A_2) = tr(A_1) + tr(A_2) = 0$ and $tr(kA) = ktr(A) = 0$, the set of all $n \times n$ matrices A such that $tr(A) = 0$ is a subspace of $\mathbf{M}_{nn}$.
(b) $A_3^T = (A_1 + A_2)^T = A_1^T + A_2^T = -A_1 - A_2 = -A_3$ and $(kA)^T = kA^T = -kA$, therefore all $n \times n$ matrices A such that $A^T = -A$ is a vector subspace of $\mathbf{M}_{nn}$.
(c) The set of all $n \times n$ matrices A such that the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution is not a subspace $\mathbf{M}_{nn}$ since for any matrix A and $k = 0$ the linear system $kAx = 0$ has solutions other than the trivial solution.
(d) Let $A_3 = A_1 + A_2$, $A_3B = (A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2) = BA_3$ and $(kA)B = kBA = BkA$, therefore all all $n \times n$ matrices A such that $AB = BA$ for a fixed $n \times n$ matrix B is a subspace of $\mathbf{M}_{nn}$.

5.

(a) In order for $-9-7x-15x^2$ to be a linear combination of $\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3$, there must be scalars $k_1,\ k_2$ and k_3 such that $-9-7x-15x^2 = k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3$ that is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \\ -15 \end{bmatrix}.$$

Solving this system yields $k_1 = -2,\ k_2 = 1$, and $k_3 = -2$, hence $-9-7x-15x^2 = -2\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3$.

(b) $6+11x+6x^2 = 4\mathbf{p}_1 - 5\mathbf{p}_2 + \mathbf{p}_3$.

(c) $7+8x+9x^2 = -2\mathbf{p}_2 + 3\mathbf{p}_3$.

6.

(a) The vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ span $\mathbf{R}^3$ if and only if the matrix $A = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]$ has nonzero determinant. $det(A) = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -6$, so $\mathbf{v}_1,\ \mathbf{v}_2$, and $\mathbf{v}_3$ span $\mathbf{R}^3$.

(b)$det(A) = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, so $\mathbf{v}_1,\ \mathbf{v}_2$, and $\mathbf{v}_3$ do not span $\mathbf{R}^3$.

(c) Since we have four vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, and $\mathbf{v}_4$ we must check if there is a combination formed by any 3 of these vectors that spans $\mathbf{R}^3$. For $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, we have $det(A) = \begin{vmatrix} 3 & 2 & 5 \\ 1 & -3 & -2 \\ 4 & 5 & 9 \end{vmatrix} = 0$. For $\{\mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$, we have $det(A) = \begin{vmatrix} 2 & 5 & 1 \\ -3 & -2 & 4 \\ 5 & 9 & -1 \end{vmatrix} = 0$.

For $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\}$, we have $det(A) = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -3 & 4 \\ 4 & 5 & -1 \end{vmatrix} = 0$. For $\{\mathbf{v}_1,\ \mathbf{v}_3,\ \mathbf{v}_4\}$, we have $det(A) = \begin{vmatrix} 3 & 1 & 5 \\ 1 & 4 & -2 \\ 4 & -1 & 9 \end{vmatrix} = 0$, so $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ and $\mathbf{v}_4$ do not span $\mathbf{R}^3$.

(d) $det(A) = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 4 & 3 \\ 6 & 1 & 1 \end{vmatrix} = -39$, so $\mathbf{v}_1,\ \mathbf{v}_2$, and $\mathbf{v}_3$ span $\mathbf{R}^3$ and $\mathbf{v}_4$ can be written as a linear combination of $\mathbf{v}_1,\ \mathbf{v}_2$, and $\mathbf{v}_3$.