SOLUTIONS

Prob. 1 25 %

15% (a) Determine the solution of the following matrix equation $Ax = b$ given below.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & -1 & 1 \\ 0 & -1 & 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

10% (b) Determine for which values of α the system

$$\begin{bmatrix} 1 & 2 \\ 2 & \alpha^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

has

4% (i) Unique solution.
3% (ii) No solution.
3% (iii) Infinetely many solutions.

SOLUTION

a) $$\left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & -1 & 1 \\ 0 & -1 & 2 & 1 & 5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 0 \\ 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $x_5 = t \Rightarrow x_4 = t, \quad x_3 = -3t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) $$[A|b] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & \alpha^2 & \alpha \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & \alpha^2-4 & \alpha-2 \end{array}\right]$$

(i) $\alpha \neq \pm 2$
(ii) $\alpha = -2$
(iii) $\alpha = 2$

Prob. 2   25%

Consider the matrix given below.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

4% (a) Find the eigenvalues of A.
9% (b) Find the eigenvectors of A corresponding to the eigenvalues of A.
8% (c) Are the eigenvectors of A obtained in (b) linearly independent? If so find a nonsingular matrix P and show that $P^{-1}AP$ is diagonal.
4% (d) Evaluate A^{10}.

SOLUTION

(a) $$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda\left[\lambda^2 - 2\lambda + 1 - 1\right] = \lambda^2(\lambda-2) = 0$$

$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 2.$

(b) $$\left.\begin{matrix}(\lambda_1 I - A)x_1 \\ (\lambda_2 I - A)x_2\end{matrix}\right\} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(\lambda_3 I - A)x_3 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} x_3 = 0 \Rightarrow x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(c) $$P = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \Rightarrow |P| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 2 \Rightarrow$$ x_1, x_2, x_3 are lin. independent.

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

(d) $$= A^{10} = PD^{10}P^{-1} \Rightarrow \frac{1}{2}\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^{10} \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & 2^{10} \\ 0 & 0 & 0 \\ 0 & 0 & 2^{10} \end{bmatrix}}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^9 & 2^9 & 2^9 \\ 0 & 0 & 0 \\ 2^9 & 2^9 & 2^9 \end{bmatrix}$$

Prob. 3 25 %

13% (a) Determine if the following two lines intersect and if so find the point of intersection.

$$\left.\begin{array}{ll} x = 1 - 3t & x = -1 + s \\ y = 2 + 5t & y = 3 - 4s \\ z = 1 + t & z = 1 - s \end{array}\right\} \quad t \text{ and } s \text{ are parameters.}$$

6% (b) Let $u_1 = (1, 1, -1)$, $u_2 = (-1, 1, 2)$. If possible express $v = (1, 1, 1)$ as a linear combination of u_1 and u_2.

6% (c) Let $u_1 = (1, 2, 1)$, $u_2 = (1, 1, -2)$, $u_3 = (-1, 0, 1)$.
Determine if the set $S = \{u_1, u_2, u_3\}$ is a basis for R^3 or not.

SOLUTION

a.) Line l_1 is parallel to the vector $v_1 = (-3, 5, 1)$ and line l_2 is parallel to the vector $v_2 = (1, -4, -1)$. Since v_1 and v_2 are not parallel, l_1 and l_2 must intersect at a point, say $P(x_0, y_0, z_0)$, for specific values of t and s, say t_0 and s_0. So,

$$\left.\begin{array}{l} 1-3t_0 = x_0 = -1+s_0 \\ 2+5t_0 = y_0 = 3-4s_0 \\ 1+t_0 = z_0 = 1-s_0 \end{array}\right\} \begin{bmatrix} 1 & 1 \\ 5 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} s_0 = -1 \\ t_0 = 1 \end{array}}$$

So, the lines intersect at the point of intersection $P_0(-1, 7, 2)$.

b.)
$$k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \to \,!$$

Because of ! system is inconsistent. So v cannot be expressed as a linear combination of u_1 and u_2.

c.)
$$k_1u_1 + k_2u_2 + k_3u_3 = 0 \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix} = 4$$

Since $|A| \neq 0$, $S = \{u_1, u_2, u_3\}$ is a linear independent set and spans R^3. Hence, S is a basis for R^3.

Prob. 4 25 %

8% (a) Determine if the set of all nxn singular matrices is a subspace of M_{nn}.

10% (b) Consider the set S = $\{M_1, M_2, M_3, M_4\}$ such that

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad M_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix},\quad M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine if the set S = $\{M_1, M_2, M_3, M_4\}$ spans the set of all 2x2 matrices M_{22}.

7% (c) Determine if the set of functions $\{1-x, 2-x^2, 1+3x-x^2\}$ form a linearly independent set or not.

SOLUTION

a) Let W be the set of all nxn singular matrices, i.e, $|A|=0$ if $A \in W$.
Let $A_1 \in W$, $A_2 \in W$. Although $|kA_1|=0$, $|A_1+A_2| \neq 0$. As an example
let $A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. $|A_1| = |A_2| = 0$ but $|A_1+A_2| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \neq 0$.

Hence, $|A_1+A_2| \notin W$. W is not closed under addition. So W is not a subspace of M_{nn}.

b) $k_1 M_1 + k_2 M_2 + k_3 M_3 + k_4 M_4 = 0$

$$\begin{bmatrix} k_1-k_2 & k_2-k_3 \\ k_2+k_3 & k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad k_4 = d$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow Ak = b$$

$|A| = 2 \Rightarrow$ The set $S = \{M_1, M_2, M_3, M_4\}$ spans M_{22}.

c)
$$W(x) = \begin{vmatrix} 1-x & 2-x^2 & 1+3x-x^2 \\ -1 & -2x & 3-2x \\ 0 & -2 & -2 \end{vmatrix}$$

$$W(x) = (1-x)\left[4x+6-4x\right] + \left[-4+2x^2+2+6x-2x^2\right]$$
$$= 6-6x-2+6x = 4$$

Since $W(x) \neq 0$ for all x in the interval $(-\infty, +\infty)$,
the set $\{1-x, 2-x^2, 1+3x-x^2\}$ is a linearly independent set.