MT-2 SOLUTIONS

Prob. 1 25 %

18% (a) Consider the following linear system $Ax = b$.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}$$

Solve for $x = [x_1 \quad x_2 \quad x_3]^T$ by $x = A^{-1}b$, (use $A^{-1} = \frac{adj(A)}{|A|}$.)

7% (b) Find the standard matrix for the started composition of the linear operations in R^2. (An orthogonal projection on the x-axis followed by a contraction with factor $k = \frac{1}{2}$.)

Solution: a) 18% $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \Rightarrow$ $M_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$, $M_{13} = \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}$, $M_{22} = \begin{vmatrix} 1 & 5 \\ -2 & 0 \end{vmatrix}$

$M_{12} = \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}$, $M_{21} = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix}$, $M_{23} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}$

$$\underbrace{M_{31} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}, \quad M_{32} = \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}, \quad M_{33} = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}}$$

$$\Downarrow$$

$$\det(A) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 \\ 0 & -6 & -9 \\ 0 & 7 & 10 \end{vmatrix} = \begin{vmatrix} -6 & -9 \\ 7 & 10 \end{vmatrix} = 3$$

$$adj(A) = \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$$

$$\Downarrow$$

$$A^{-1} = \frac{adj(A)}{\det(A)} \Leftarrow adj(A) = \begin{bmatrix} -1 & 5 & 3 \\ -2 & 10 & 9 \\ 2 & -7 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -2 & 10 & 9 \\ 2 & -7 & -6 \end{bmatrix} \Rightarrow x = A^{-1}b = \frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -2 & 10 & 9 \\ 2 & -7 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -5 \\ -4 \\ 4 \end{bmatrix}$$

b.) 7% For orthogonal projection on x-axis, transform matrix is: $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

For contraction by $\frac{1}{2}$, transform matrix is: $T_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Hence; $T = T_1 T_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}$

Prob. 2 25%

Consider the following linear system of the form Ax = b as

$$\begin{bmatrix} 1 & 3 & 3 & 2 & -1 \\ 2 & 1 & -3 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 3 & -1 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

10% (a) Evaluate the determinant of the A matrix.

10% (b) Determine the value of x_2.

5% (c) If $|B| = 2, |C| = 4$, calculate det $(B^T C^{-1} B^4)$ and det $(-2A)$.

Solution: 10% a.)

$$\begin{vmatrix} 1 & 3 & 3 & 2 & -1 \\ 2 & 1 & -3 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 3 & -1 & 2 & 0 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & -1 & 2 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 3 & 3 & 2 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -2 & 0 & 2 \end{vmatrix} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)$$

$$= -0.(3\cdot(-3)-2\cdot 2) + 1(3\cdot 2-(-2\cdot 2)) - (-1)(2\cdot 2-(-2\cdot(-3)))$$

$$= 10 - 2 = 8$$

$$\Rightarrow \det(A) = (-1)\cdot 8 \cdot (-1) = \underline{\underline{8}}$$

b.) Apply Cramer's Rule: $x_2 = \dfrac{|A_2|}{|A|}$, where $A_2 = \begin{bmatrix} 1 & 0 & 3 & 2 & -1 \\ 2 & 1 & -3 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 3 & 0 & 2 & 0 & 0 \end{bmatrix}$

10%

$$\Rightarrow |A_2| = \begin{vmatrix} 1 & 0 & 3 & 2 & -1 \\ 2 & 1 & -3 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 3 & 0 & 2 & 0 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 3 & 0 & 2 \\ 2 & 1 & -3 \\ -2 & 0 & 2 \end{vmatrix} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 10 \Rightarrow x_2 = \frac{|A_2|}{|A|} = \frac{10}{8} = \underline{\underline{\frac{5}{4}}}$$

$$\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)$$

(c) $\det(B^T C^{-1} B^4) = |B| \frac{1}{|C|} |B|^4 = 8$

$\det(-2A) = (-2)^5 |A| = 32 \cdot 8 = -256$

Prob. 3 25%

15% (a) The points $P_1(2, -1, 4)$, $P_2(3, -1, 3)$, $P_3(0, 2, 1)$ and $P_4(1, 3, 0)$ are given.

6% (i) Find the norms of $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$.

3% (ii) Determine if $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ are parallel.

6% (iii) Find the distance d between P_2 and P_3.

10% (b) Find the parametric equations for the line l passing through the points $P_1(2, 4, -2)$ and $P_2(5, 0, 7)$.

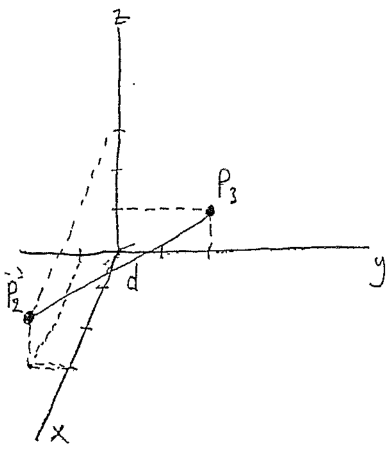
Solution: a.) 15% 6% i.) $\overrightarrow{P_1P_2} = (1, 0, -1)$, $\overrightarrow{P_3P_4} = (1, 1, -1)$

$\|\overrightarrow{P_1P_2}\| = \sqrt{2}$, $\|\overrightarrow{P_3P_4}\| = \sqrt{3}$

3% ii.) $\overrightarrow{P_1P_2} = (1, 0, -1)$ is not a scalar multiple of $\overrightarrow{P_3P_4} = (1, 1, -1)$. Hence they are not parallel.

6% iii.) $d = \|\overrightarrow{P_2P_3}\| = \|(-3, 3, -2)\| = \sqrt{9+9+4}$

$d = \sqrt{22}$



10% b.) $\overrightarrow{P_1P_2} = (3, -4, 9)$

$\overrightarrow{P_1P_2}$ is parallel to line l. Hence, parametric equations can directly be written as follows:

$x = 2 + 3t$
$y = 4 - 4t$
$z = -2 + 9t$

Prob. 4 25%

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

5% (a) Find the eigenvalues of A.

9% (b) Find the eigenvectors corresponding to these eigenvalues.

6% (c) Determine whether A is diagonalizable. If so find a matrix P and bring A to diagonal form.

5% (d) Evaluate A^6.

SOLUTION

6% a.) $|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda-3 \end{vmatrix} = \lambda(\lambda^2-3\lambda+2) = \lambda(\lambda-2)(\lambda-1) \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2. \end{matrix}$

9% b.) $[\lambda_1 I - A][x_1] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} \\ x_1 \\ \\ \end{bmatrix} = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[\lambda_2 I - A][x_2] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} \\ x_2 \\ \\ \end{bmatrix} = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$[\lambda_3 I - A][x_3] = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} \\ x_3 \\ \\ \end{bmatrix} = 0 \Rightarrow x_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

6% c.) $P = [x_1 | x_2 | x_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $A = PDP^{-1}$, where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix}$

5% d.) $A^6 = PD^6P^{-1} = [P] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^6 & 0 \\ 0 & 0 & 2^6 \end{bmatrix} [P^{-1}] = \begin{bmatrix} 0 & -30 & 31 \\ 0 & -62 & 63 \\ 0 & -126 & 127 \end{bmatrix}$