ENGINEERING MATHEMATICS I

FALL 2008

Introduction to Complex Numbers and Complex Algebra

☆ (Chapter 21, Sections upto and including 21.3.2)

Historically complex numbers have been introduced to have complete set of solutions for any polynomial equation such as:

$$x^2+1=0 \quad \text{or} \quad x^3-x^2-x-2=(x-2)(x^2+x+1)=0$$

- Real numbers do not form a large enough set to contain solutions to every polynomial equations.

Notation: $z \in \mathbb{C}$, $\underbrace{z = a+ib}_{\text{Mathematicians prefer this notation}}$ or $\underbrace{z = a+jb}_{\text{Electrical Engineers prefer this notation}}$

Real numbers.

a: the real part of z,
b: the imaginary part of z.
j: root to the $x^2+1=0$. $\Rightarrow (j)^2 \equiv -1$.
$a \equiv Re(a+jb)$; $b \equiv Im(a+jb)$

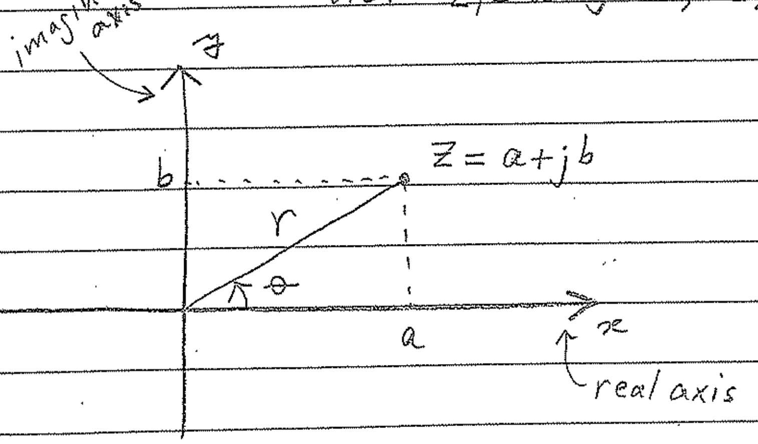
Note that: (i) imaginary part of z is also a real number.
(ii) $\mathbb{R} \subset \mathbb{C}$ because if $a \in \mathbb{R}$ then $a+j0 \in \mathbb{C}$.

Definitions:

(i) If $z = Re(z)$, z is called as "purely real"
i.e.: $z = 5 = 5+j0$

(ii) If $z = j\,Im(z)$, z is called as "purely imaginary"
i.e.: $z = j3 = 0+j3$

(iii) If $z_1 = z_2$, then $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$
i.e. $z_1 = 2+j5$, $z_2 = 2+j5$.



Complex z plane.

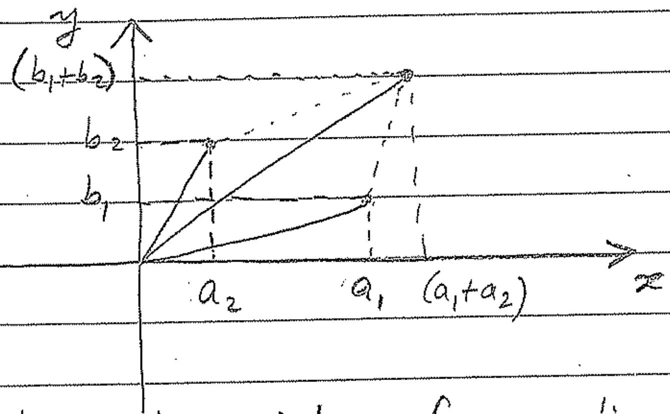
Algebraic Definitions

1. Summation Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$

Then: $z = z_1 + z_2 = a + jb$ where $a = a_1 + a_2$
$b = b_1 + b_2$

i.e.: $(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$

ex's: $z_1 = 2 + j3$, $z_2 = 5 - j4$, $z_1 + z_2 = (2+5) + j(3-4)$
$= 7 - j$



Complex z plane picture of summation.

Properties of Complex Summation: For any $z_1, z_2 \in \mathbb{C}$:

(i) $z_1 + z_2 = z_2 + z_1$; commutative.
(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$; associative
(iii) $z_1 + (0 + j0) = z_1$
(iv) for any $z_1 \in \mathbb{C}$, there exists a z_2 such that $z_1 + z_2 = 0$

proof

Let $z_1 = a + jb$ then choose $z_2 = -a - jb$, resulting
$z_1 + z_2 = (a - a) + j(b - b) = 0 + j0 = 0.$

(v) Additive inverse is unique.

proof Let $z_1 = a_1 + jb_1$ has two different additive inverses,
i.e., $z_1 + z_2 = 0$, $z_1 + z_3 = 0$ but $z_2 \neq z_3$.

Then: $z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$
$z_1 + z_3 = (a_1 + a_3) + j(b_1 + b_3)$

$$z_1 + z_2 = z_1 + z_3 \Rightarrow \left.\begin{array}{l} a_1 + a_2 = a_1 + a_3 \Rightarrow a_2 = a_3 \\ b_1 + b_2 = b_1 + b_3 \Rightarrow b_2 = b_3 \end{array}\right\} \Rightarrow z_2 = z_3$$

Contradiction

(vi) The unique additive inverse of a complex number $z = a + jb$ is $-a - jb \equiv -z$.

(vii) Subtraction of two complex numbers z_1 and z_2 is defined as:

$$z_1 + (-z_2) \equiv z_1 - z_2 .$$

If $z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$,

$$z_1 - z_2 = (a_1 + jb_1) - a_2 - jb_2 = (a_1 - a_2) + j(b_1 - b_2)$$

2. Multiplication

Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then $z = z_1 \cdot z_2$ is defined as:

$$z = (a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

This definition can be motivated by:

$$(a_1 + jb_1) \cdot (a_2 + jb_2) = a_1 a_2 + ja_1 b_2 + jb_1 a_2 + j^2 b_1 b_2$$

and since $j^2 \equiv -1$:

$$(a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

example $z_1 = 2 + j3$, $z_2 = 5 - j4$

$$z = z_1 \cdot z_2 = (2 \cdot 5 - 3 \cdot (-4)) + j(2 \cdot (-4) + 3 \cdot 5)$$
$$= 22 + j7 .$$

Properties of Complex Multiplication

$\forall\ z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$; and $z_3 = a_3 + jb_3$

show all three

(i) $z_1 z_2 = z_2 z_1$; commutative property

(ii) $z_1 (z_2 z_3) = (z_1 z_2) z_3$; associative property

(iii) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$; distributive property

(iv) $1 \cdot z_1 = z_1$; identity element

(v) If $z_1 \neq 0$, there exist a z such that $z_1 \cdot z = 1$ inverse element

proof Choose $z = \dfrac{a_1}{(a_1^2 + b_1^2)} - j \dfrac{b_1}{(a_1^2 + b_1^2)}$

and show that $z_1 \cdot z = 1$.

(vi) If $z_1 \neq 0$, its inverse is unique:

proof Assume that there exists a $z_1 \neq 0$ which has two different inverses z and $\hat{z}$. Then:

$$z_1 z = 1$$

$$\Rightarrow \hat{z}(z_1 \cdot z) = \hat{z}\, 1$$

$$\underbrace{(\hat{z} z_1)} z = \hat{z}$$

$$\underbrace{1 \cdot z} = \hat{z}$$

$$z = \hat{z} \quad \text{contradiction} \blacksquare$$

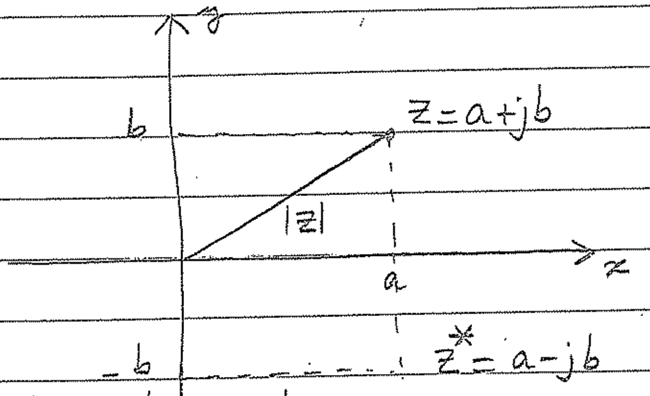
More very important Definitions.

For $z = a + jb$, its complex conjugate is defined as:

$$\bar{z} = a - jb = z^*$$

and its "modulus" or "magnitude" is defined as:

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}} = \sqrt{z \cdot z^*}$$



complex z plane

ex/ $z = 2 + j3$, $z^* = 2 - j3$, $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

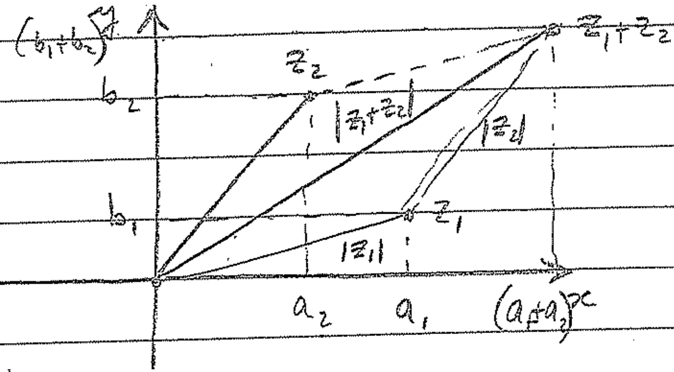
Properties of complex conjugation:

$$\left.\begin{array}{ll} (i) & (z^*)^* = z \\ (ii) & (z_1 + z_2)^* = z_1^* + z_2^* \\ (iii) & (z_1 \cdot z_2)^* = z_1^* z_2^* \end{array}\right\} \text{show all these.}$$

Triangle Inequality

$\forall$ z_1 and z_2, $\quad |z_1 + z_2| \le |z_1| + |z_2|$

Proof First look at the picture:



Since $|z_1+z_2|$, $|z_1|$ and $|z_2|$ form a triangle, $|z_1+z_2| \leq |z_1|+|z_2|$.

Redo this proof

Formally, $z_1+z_2 = (a_1+a_2)+j(b_1+b_2)$

Thus,
$$|z_1+z_2|^2 = (a_1+a_2)^2+(b_1+b_2)^2$$
$$= a_1^2+a_2^2+2a_1a_2+b_1^2+b_2^2+2b_1b_2$$
$$\leq a_1^2+a_2^2+2|a_1||a_2|+b_1^2+b_2^2+2|b_1||b_2|$$
$$= (a_1^2+b_1^2)+(a_2^2+b_2^2)+2\sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)} \quad \text{esit}$$
$$+2\left(|a_1||a_2|+|b_1||b_2|-\sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)}\right)$$

Since, $|a_1||a_2|+|b_1||b_2| \leq \sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)}$ (show this by squaring both sides)

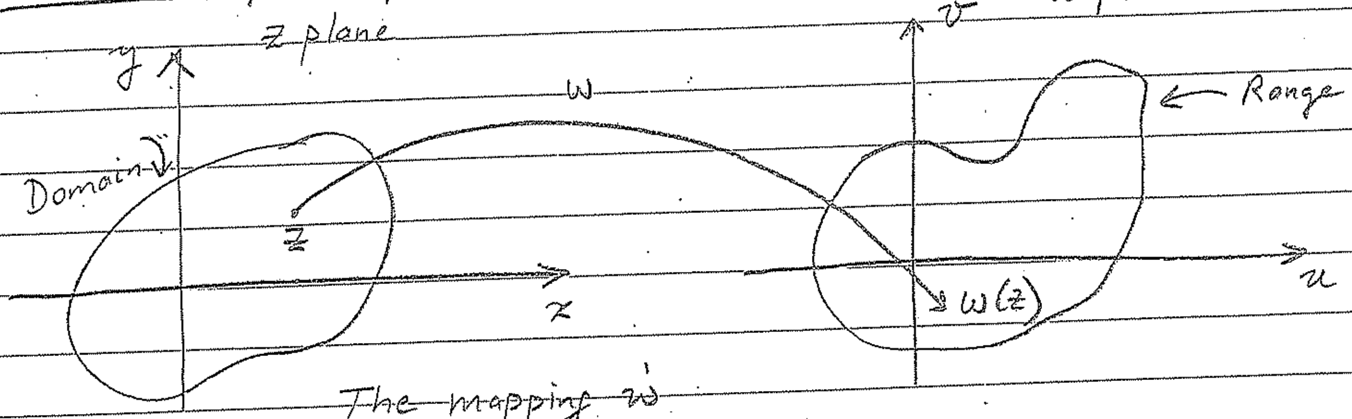
$$|z_1+z_2|^2 \leq (a_1^2+b_1^2)+(a_2^2+b_2^2)+2\sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)}$$
$$= \left(\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}\right)^2 \quad \text{esit}$$
$$= (|z_1|+|z_2|)^2$$
$$\Rightarrow |z_1+z_2| \leq |z_1|+|z_2|.$$

Def'n If z_1 and z_2 are two complex numbers where $z_2 \neq 0$, $\frac{z_1}{z_2} = z_1 \cdot z$ where z is the unique multiplicative inverse of z_2.

ex/
$$\frac{2+j3}{3-j4} = (2+j3)\cdot\frac{(3+j4)}{(3-j4)(3+j4)} = \frac{(2+j3)(3+j4)}{25}$$
$$= \frac{(6-12)+j(8+9)}{25} = -\frac{6}{25}+j\frac{17}{25}$$

$$\cancel{a_1^2a_2^2}+\cancel{b_1^2b_2^2}+2|a_1||a_2||b_1||b_2| \leq \cancel{a_1^2a_2^2}+a_1^2b_2^2+b_1^2a_2^2+\cancel{b_1^2b_2^2}$$
$$\Rightarrow 2|a_1||a_2||b_1||b_2| \leq (a_1^2b_2^2+b_1^2a_2^2)$$
$$\Rightarrow 0 \leq a_1^2b_2^2+b_1^2a_2^2-2|a_1||a_2||b_1||b_2|$$
$$0 \leq (a_1b_2-b_1a_2)^2$$

Elementary Complex Functions



~~The mapping w~~

$$w(z) = w(x+jy) = u(x,y) + j\,v(x,y)$$

ex/ $w(z) = z^2$, let Domain be the first quadrant in z plane, i.e., $0 < x < \infty$ and $0 < y < \infty$.

Then $w(x+jy) = (x+jy)^2 = (x+jy)(x+jy) = x^2 - y^2 + j2xy$

$\Rightarrow w(x+jy) = u(x,y) + j\,v(x,y)$
where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.
Note that the range of w is ; $-\infty < u < \infty$ and $0 < v < \infty$.

ex/ $w(z) = z^*$. Let the domain be the entire $\mathbb{C}$ plane.
Then $w(z) = w(x+jy) = x - jy = u(x,y) + j\,v(x,y)$
where $u(x,y) = x$, $v(x,y) = -y$, and the range is the $\mathbb{C}$ plane

ex/ $w(z) = e^z \equiv 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots$
↑ exponential function.

$w(z) = w(x+jy) = e^{x+jy} = e^x \cdot e^{jy}$
↑ will show this later.

$$e^{jy} = 1 + jy + \frac{(jy)^2}{2} + \cdots + \frac{(jy)^n}{n!} + \cdots$$

$$= \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \cdots + (-1)^k \frac{y^{2k}}{(2k)!} + \cdots\right) + j\left(y - \frac{y^3}{3!} + \cdots + (-1)^k \frac{y^{2k+1}}{(2k+1)!}\right.$$

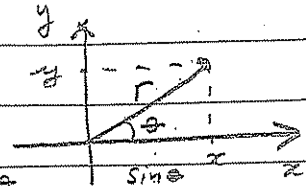
$$\boxed{e^{jy} = \cos y + j \sin y}$$ Euler's Formula.

Thus, $w(z) = e^{z} = e^{x} \cdot e^{jy} = e^{x}(\cos y + j\sin y)$

$$= e^{x}\cos y + je^{x}\sin y$$

$$= u(x,y) + j\,v(x,y)$$

Where $u(x,y) = e^{x}\cos y$, $v(x,y) = e^{x}\sin y$.



Note that for any z, $|z| < \infty$, $e^{z} \neq 0$.

* Polar representation $z = x + jy = \sqrt{x^2+y^2}\underbrace{\left(\overbrace{\frac{x}{\sqrt{x^2+y^2}}}^{\cos\theta} + j\overbrace{\frac{y}{\sqrt{x^2+y^2}}}^{\sin\theta}\right)}_{e^{j\theta}}$

$\Rightarrow$ $z = re^{j\theta}$, r: magnitude, θ: phase.

Systems of Linear Algebraic Equations

*(Chapter 8 in the textbook)

Modeling physical systems interms of linear systems of equations and obtaining solutions of these systems is of fundamental importance in engineering.

General form of a system of linear algebraic equations is:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}\right\} m \text{ equations}$$

a_{ij} : coefficients (can be real or complex numbers)

c_i : Value coefficients or the "right hand side" coefficients

x_j : unknowns (can be real or complex numbers).

> If we replace x_1 with s_1, x_2 with s_2, ..., x_n with s_n in this system and satisfy all the equations, we say that $s_1, \ldots, s_n$ is a solution to the given system.

> If there exists one or more solutions, the system is consistent.

> If there is precisely one solution, the solution is called as unique

> If there are more than one solutions, the solution is called as nonunique

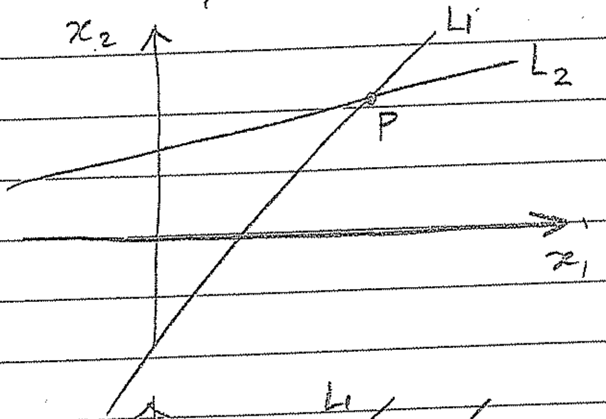
> If there are no solutions, the system is called as inconsistent.

> The collection of all solutions is called as the solution set

examples Consider $m=n=2$ case with real valued coefficients:

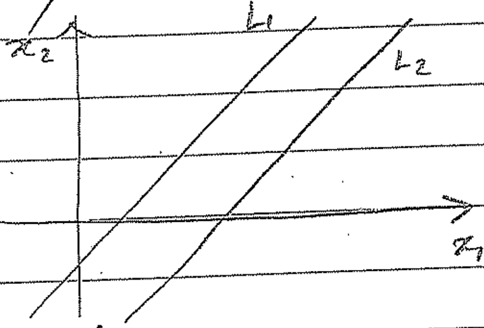
$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_1 + a_{22}x_2 = c_2$$

These equations individually define lines in 2-D space:



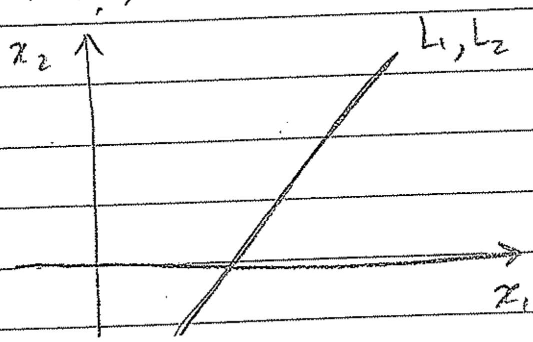
Consistent System with a unique solution.

$$2x_1 - x_2 = 5$$
$$x_1 + 3x_2 = -1$$



Inconsistent system.
No solution.

$$x_1 + 3x_2 = 1$$
$$x_1 + 3x_2 = 2$$



Consistent system with non unique solutions:

$$x_1 + 3x_2 = 1$$
$$3x_1 + 9x_2 = 3$$

In general, each equation defines a hyperplane in n-dimensional space. The system is consistent if these hyperplanes has a common intersection. If the intersection of these hyperplanes is just one point, then the solution is unique

Obtaining Solutions: Gauss or Gaussian Elimination

Consider: $m=n=3$

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 9 \\ x_1 - x_2 + 4x_3 &= 8 \end{aligned}\right\} \quad \xRightarrow{} \quad \begin{aligned} x_1 + x_2 - x_3 &= 1 \\ -2x_2 + 4x_3 &= 6 \\ -2x_2 + 5x_3 &= 7 \end{aligned}$$

Eliminate x_1 from 2^{nd} and 3^{rd} Equations by adding $-3 \times$ Eqn1 to Eqn2 and $-1 \times$ Eqn1 to Eqn3

Now, eliminate x_2 from the 3^rd Eqn. to obtain triangular System:

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ -2x_2 + 4x_3 &= 6 \\ x_3 &= 1 \end{aligned}\right\}$$

Normalize the leading coefficient of 2^nd Eqn to 1 by dividing Eqn. 2 with -2

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 - 2x_3 &= -3 \\ x_3 &= 1 \end{aligned}$$

Now $x_3 = 1$ can be identified as part of a solution. Then substitute $x_3 = 1$ into 2^nd Eqn to obtain $x_2 = 2x_3 - 3 = -1$.
Then substitute x_2 and x_3 into Eqn. 1 to obtain $x_1 = -x_2 + x_3 + 1 = 3$.
Thus: $x_1 = 3$, $x_2 = -1$, $x_1 = 1$ is the unique solution.

Important Observation:

$$\boxed{\text{Original System of Equations}} \xrightarrow[\text{Elimination}]{\text{Gauss}} \boxed{\text{Triangular System of Equations}} \xrightarrow[\text{Substitution}]{\text{Back}} \boxed{\text{Solution}}$$

<u>Question</u> Are the solution sets of the original and triangular systems identical?

<u>Answer</u> Yes! Because Gauss elimination utilizes elementary row operations defined next.

<u>Definition</u> Elementary row operations:

1. Addition of a multiple of one equation to another
   symbolically: $(eq.j) \longrightarrow (eq.j) + \alpha\,(eq.k)$
2. Multiplication of an equation by a nonzero constant
   symbolically: $(eq.j) \longrightarrow \alpha\,(eq.j)$
3. Interchange of two equations
   symbolically: $(eq.j) \longleftrightarrow (eq.k)$

<u>Thm 8.3.1</u> If one linear system is obtained from another by a finite number of elementary row operations, then the two systems are equivalent, i.e., they share the same solution set.

<u>Proof</u>

Let the original system be labeled at LS_1 and the one obtained by using elementary row operations be labeled as LS_2.

We know that $LS_1 \xrightarrow{E_1} \cdots \xrightarrow{E_q} LS_2$
where $E_1, \ldots, E_q$ is a sequence of elementary row operations. Here, by using the method of induction, we will show that LS_2 and LS_1 shares the same solution set

For $q = 1$:

$$LS_1 \xrightarrow{E_1} LS_2$$

Now, E_1 can be any of the 3 elementary row operations. We will consider all of these cases individually:

(i) E_1: addition of a multiple of one equation to another:

$$(eq.\,j) \longrightarrow (eq.\,j) + \alpha \cdot (eq.\,k)$$

cause we have added really equal quantities

> Now, if $(s_1, \ldots, s_n) \in S_{LS_1}$ ← solution set of LS_1
then $(s_1, \ldots, s_n)$ satisfies all the equations including (eq. j) and (eq. k). Hence, it satisfies the jth eqn. of LS_2 which is (eq. j) + α(eq. k)

Therefore: $S_{LS_1} \subset S_{LS_2}$. → if S_{LS_2} is proved from S_{LS_1} ⇒ any soln of S_{LS_1} satisfies S_{LS_2}

> Now, if $(s_1, \ldots, s_n) \in S_{LS_2}$, then $(s_1, \ldots, s_n)$ satisfies all the equations including (eq. j) and (eq. k) of LS_2. Thus, it also satisfies (eq. j) $- \alpha$(eq. k) as well. But this is the (eq. j) of LS_1. Thus, $(s_1, \ldots, s_n)$ also satisfies all the equations in LS_1.
Hence: $(s_1, \ldots, s_n) \in LS_1$.

Therefore: $S_{LS_2} \subset S_{LS_1}$

Since: $S_{LS_1} \subset S_{LS_2}$ and $S_{LS_2} \subset S_{LS_1} \implies S_{LS_1} = S_{LS_2}$.

(ii) E_1: Multiplication of an equation by a non-zero constant.

$$(eq.\,j) \longrightarrow \alpha(eq.\,j)$$

> If $(s_1, \ldots, s_n) \in S_{LS_1}$ then it solves all the equations in LS_1 including the j^{th} eqn. The LS_2 differs from LS_1 in just its j^{th} equation. Therefore, to show that $(s_1, \ldots, s_n) \in S_{LS_2}$ we need to show that $(s_1, \ldots, s_n)$ also satisfies the (eq. j) of LS_2

If we show that $(s_1, \ldots, s_n)$ satisfies LS_2's j^{th} eqn. ⇓ $(s_1, \ldots, s_n) \in S_{LS_2}$

Since, (eq. j) of LS_2 is obtained by multiplying both sides of (eq. j) in LS_1, and $(s_1, \ldots, s_n)$ satisfies (eq. j) of LS_1, $(s_1, \ldots, s_n)$ also satisfies (eq. j) of LS_2. Therefore $(s_1, \ldots, s_n) \in S_{LS_2}$ as well.

Therefore, $S_{LS_1} \subset S_{LS_2}$.

> If $(s_1, \dots, s_n) \in S_{LS_2}$, then it satisfies all the equations in LS_2 including the jth eqn. Similar to the first case, to show that $(s_1, \dots, s_n) \in S_{LS_1}$, we have to show that $(s_1, \dots, s_n)$ also satisfies (eq. j) of LS_1. Since, $(s_1, \dots, s_n)$ satisfies (eq. j) of LS_2, it also satisfies $\frac{1}{\alpha}$ (eq. j) of LS_2. Note that since $\alpha \neq 0$, it is possible to divide (eq. j) of LS_2 by α. But $\frac{1}{\alpha}$ (eq. j) of LS_2 is identical to (eq. j) of LS_1. Thus, $(s_1, \dots, s_n) \in S_{LS_1}$. Hence, $S_{LS_2} \subset S_{LS_1}$.
Thus, $S_{LS_1} = S_{LS_2}$.

(iii) E_1: Interchange of two equations. (eq. j) $\longleftrightarrow$ (eq. k)
This case is the simplest one. If $(s_1, \dots, s_n) \in S_{LS_1}$ then it satisfies all the equations in LS_1 including (eq. j) and (eq. k). The LS_1 and LS_2 differs only in (eq. j) and (eq. k). Therefore, to show that $(s_1, \dots, s_n) \in S_{LS_2}$, we have to show that $(s_1, \dots, s_n)$ satisfies (eq. j) and (eq. k) of LS_2 as well. But, this is immediate, because (eq. j) of LS_2 is the (eq. k) of LS_1 which is satisfied by $(s_1, \dots, s_n)$; and (eq. k) of LS_2 is the (eq. j) of LS_1 which is also satisfied by $(s_1, \dots, s_n)$. Thus, $(s_1, \dots, s_n) \in S_{LS_2}$ as well. Therefore, $S_{LS_1} \subset S_{LS_2}$. The same argument by replacing roles of LS_1 and LS_2 will yield $S_{LS_2} \subset S_{LS_1}$. Thus, $S_{LS_1} = S_{LS_2}$.

This completes the proof of $q = 1$ case.

> Now assume that the claim is true for $q \geq 1$, i.e., $S_{LS_1} = S_{LS_q}$. We want to show that $S_{LS_1} = S_{LS_{(q+1)}}$ as well.

But this case is straightforward. First, observe that $LS_{(q+1)}$ is obtained from LS_q by an elementary row operation. Therefore, as investigated in the case $q=1$, $S_{LS_{(q+1)}} = S_{LS_q}$. Since, we have assumed that $S_{LS_q} = S_{LS_1}$, this implies that:

$$S_{LS_{(q+1)}} = S_{LS_1}.$$

This concludes the proof.

examples

1. $$\left.\begin{aligned} 2x_1+3x_2-2x_3&=4\\ x_1-2x_2+x_3&=3\\ 7x_1\qquad -x_3&=2\end{aligned}\right\}\quad (eq.1)\to\tfrac{1}{2}(eq.1)$$

$$\left.\begin{aligned} x_1+\tfrac{3}{2}x_2-x_3&=2\\ x_1-2x_2+x_3&=3\\ 7x_1\qquad -x_3&=2\end{aligned}\right\}\quad \begin{aligned}(eq.2)&\to(eq.2)-(eq.1)\\(eq.3)&\to(eq.3)-7(eq.1)\end{aligned}$$

$$\left.\begin{aligned} x_1+\tfrac{3}{2}x_2-x_3&=2\\ -\tfrac{7}{2}x_2+2x_3&=1\\ -\tfrac{21}{2}x_2+6x_3&=-12\end{aligned}\right\}\quad \begin{aligned}(eq.2)&\to-\tfrac{2}{7}(eq.2)\\(eq.3)&\to(eq.3)+\tfrac{21}{2}(eq.2)\end{aligned}$$

$$\begin{aligned} x_1+\tfrac{3}{2}x_2-x_3&=2\\ x_2-\tfrac{4}{7}x_3&=-\tfrac{2}{7}\\ 0&=-15\end{aligned}$$

$\longleftarrow$ cannot be satisfied by any choice of x_1, x_2 and x_3

Thus, the system is inconsistent.

It would have been consistent if the original system has 17 on its right hand side. Then, the same elimination would yield:

$$\begin{aligned} \text{(p)}\quad\ \text{(p)}\qquad\ \text{(f)}\quad&\\ x_1+\tfrac{3}{2}x_2-x_3&=2\\ x_2-\tfrac{4}{7}x_3&=-\tfrac{2}{7}\\ 0&=0\end{aligned}$$

Here x_1 and x_2 are called as pivot variables and x_3 is a free variable. Then the solution set can be obtained as:

$x_3=\alpha$ (a free parameter)

$x_2=-\frac{2}{7}+\frac{4}{7}\alpha$

$x_1=\frac{11}{7}+\frac{1}{7}\alpha$

Therefore, the solution set can be expressed as:

$$S=\left\{(x_1,x_2,x_3):\ x_1=\tfrac{11}{7}+\tfrac{1}{7}\alpha,\ x_2=-\tfrac{2}{7}+\tfrac{4}{7}\alpha,\ x_3=\alpha,\ \alpha\in\mathbb{R}\right.$$

Note that, if number of pivot variables, r, is less than n, the number of unknowns, then if the system is consistent, the solution set can be expressed as a parametric set of $(n-r)$ variables. This type of a set is called as an "$(n-r)$-parameter family of solutions." Any choice of these parameters yields a "particular" solution.

Theorem 8.3.2 Existence / Uniqueness of Solutions for Linear Systems

Let the LS has the following form:

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= c_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= c_m \end{aligned}\right\} \text{- } m \text{ equations with } n \text{ unknowns.}$$

Apply Gauss elimination to obtain an upper triangular system with r pivot variables.

Then:

1. If $r = m$, the system is always consistent.
2. If $r < m$, then the system is consistent if the last $(m-r)$ equations have zero right hand sides.
3. If $r < m$, then the system is inconsistent if any of the last $(m-r)$ equations has a non-zero right hand side.
4. If the system is consistent and if $r = n$, then the solution is unique.
5. If the system is consistent and if $r < n$, then the solution set is an $(n-r)$-parameter family of solutions.

↓ How ?

Theorem 8.3.4 Existence and Uniqueness of solutions for $\underbrace{c_1 = \cdots = c_m = 0}_{\text{the homogeneous case}}$

$$\underbrace{\begin{array}{l} \text{① Always consistent.} \\ \text{② If } r < n \text{, the solution is non-unique.} \end{array}}_{\text{Special case of Thm. 8.3.2}}$$

Matrix Notation

$$\left.\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = c_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = c_m \end{matrix}\right\} \longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}}_{\text{coefficient matrix}}, \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}_{\text{right hand sides}}$$

~~The vector of~~ right hand sides

It is easier to apply Gauss elimination on the following augmented coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{bmatrix}$$

Note that elementary row operations can be performed on the rows of the augmented coefficient matrix.

Def'n Two matrices are called row equivalent if one can be obtained from the other by finitely many elementary row operations.

Gauss-Jordan Reduction or Elimination

Another way to obtain the solution set for a given linear system. It is better suited for hand computations. Also, it provides more accurate results in finite precision computations.

ex/

$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 + 8x_2 + 7x_3 = 20$$
$$2x_1 + 7x_2 + 9x_3 = 23$$

Form the augmented coefficient matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3 \to R_3} \underbrace{\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Result of Gauss Elimination: Echelon form}} \xrightarrow{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[-2R_3+R_2 \to R_2]{3R_3+R_1 \to R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Result of Jordan elimination: Reduced row echelon form}}$$

ex/
$$x_1 + x_2 + x_3 + x_4 = 12$$
$$x_1 + 2x_2 \quad + 5x_4 = 17$$
$$3x_1 + 2x_2 + 4x_3 - x_4 = 31$$

Form the augmented coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 1 & 2 & 0 & 5 & 17 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix} \xrightarrow[-3R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1 \to R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \underbrace{\begin{bmatrix} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{reduced row echelon form}} \end{array}$$

pivot variables

pivots

x_1 and x_2 are pivot variables
x_3 and x_4 are free variables
The system is consistent.

set: $x_3 = \alpha$, $x_4 = \beta$

Then: $x_1 = -2\alpha + 3\beta + 7$

$x_2 = \alpha - 4\beta + 5$

The solution set can be expressed in vector form as:

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}, \beta \in \mathbb{R} \right\}$$

Partial Pivoting

ex/ Assume that in Gauss elimination we reached:

$$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

Now can continue with a regular procedure by interchanging the 2^{nd} and 3^{rd} rows

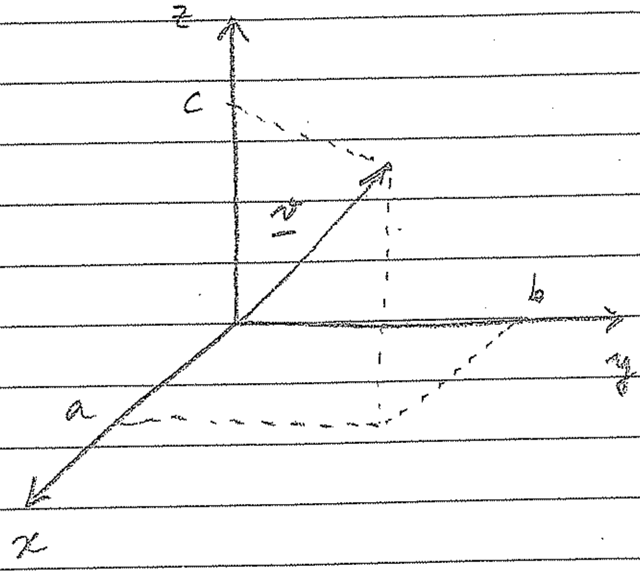
$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 5 & 8 \end{bmatrix}$$

This operation is called as partial pivoting.
If there exists more than one possible switchs, choose the one with the largest absolute value as the pivot.

go over [illegible] notes (15a) & (15b) – (15c)

Vector Spaces

*(Chapters 9 and 12 in the textbook.)

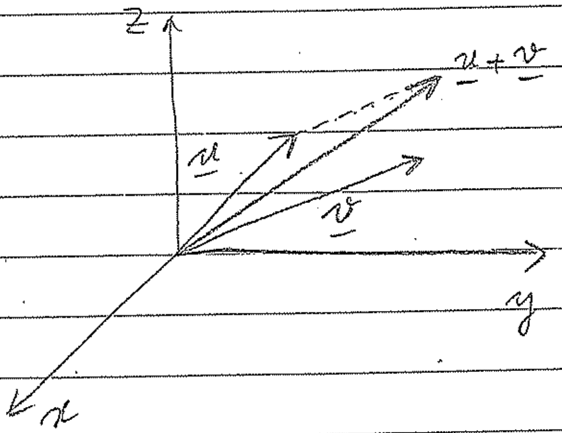


will use this notation

$$\underline{v} = (a, b, c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$ Components

Def'n Vector addition $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$$\underline{v} + \underline{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{bmatrix}$$ ← component wise addition



Def'n Scalar multiplication of a vector:

$$\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} \qquad c \cdot \underline{v} = \begin{bmatrix} c v_1 \\ \vdots \\ \vdots \\ c v_n \end{bmatrix}, \quad c \in \mathbb{R}.$$

Def'n The vector space. $\mathcal{V}$ is a vector space if $\forall \underline{u}, \underline{v}, \underline{w} \in$ and $r, s \in \mathbb{R}$

Axioms:
(a) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$; Commutativity of vector addition.
(b) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$; associativity of vector addition.

(c) $\underline{u}+\underline{0}=\underline{0}+\underline{u}=\underline{u}$; existence of a zero element
(d) $\underline{u}+(-\underline{u})=\underline{0}$; existence of an additive inverse
(e) $r(\underline{u}+\underline{v})=r\underline{u}+r\underline{v}$; scalar distribution over vector addition.
(f) $(r+s)\underline{u}=r\underline{u}+s\underline{u}$; vector distribution over scalar addition.
(g) $r(s\underline{u})=(rs)\underline{u}$; associativity of scalar multiplication.
(h) $1\cdot\underline{u}=\underline{u}$; multiplicative identity.

(see the remark!)

Theorem $\mathbb{R}^n=\left\{\underline{v}=\begin{bmatrix} v_1\\ \vdots \\ v_n\end{bmatrix} : v_i\in\mathbb{R},\ 1\le i\le n\right\}$

is a vector space with the vector summation and scalar multiplication defined earlier.

Proof show that all the conditions (a) to (h) of a vector space are satisfied for $\mathbb{R}^n$.

Also show that: → see remark2 & (16b)

(i) $\alpha\,\underline{0}=\underline{0}$
(ii) $\forall\,\underline{u}\in\mathbb{R}^n,\ -\underline{u}=(-1)\cdot\underline{u}$.
(iii) $0\cdot\underline{u}=\underline{0}$

Def'n "A function" from a vector space" to real numbers is a norm if, $\forall\,\underline{u}\in\mathcal{V},\ \underline{v}\in\mathcal{V}$ and $\alpha\in\mathbb{R}$:

(i) $f(\underline{u})>0$ if $\underline{u}\neq\underline{0}$
$f(\underline{u})=0$ if $\underline{u}=\underline{0}$

(ii) $f(\alpha\underline{u})=|\alpha|\,f(\underline{u})$

(iii) $f(\underline{u}+\underline{v})\le f(\underline{u})+f(\underline{v})$

examples In vector space $\mathbb{R}^n$, the following functions are norms:

1. $f(\underline{u})=\sqrt{u_1^2+u_2^2+\dots+u_n^2}=\left[\sum_{i=1}^{n}u_i^2\right]^{1/2}=\|\underline{u}\|_2$

2. $f(\underline{u})=\left[\sum_{i=1}^{n}u_i^p\right]^{1/p}=\|\underline{u}\|_p,\ p\ge 1$

(

<u>Def'n</u> The dot product of two vectors in $\mathbb{R}^n$ is defined as:

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i .$$

Note that $\|\underline{u}\|_2 = \sqrt{\underline{u} \cdot \underline{u}}$

move this after Schwartz inequality

<u>Def'n</u> The angle between two vectors in $\mathbb{R}^n$ is defined as:

$$\theta = \cos^{-1}\left[\frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|_2 \|\underline{v}\|_2}\right]$$

which yields:

$$\underline{u} \cdot \underline{v} = \|\underline{u}\|_2 \cdot \|\underline{v}\|_2 \cdot \cos\theta .$$

<u>The properties of the dot product</u>

(i) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ commutativity

(ii) $\underline{u} \cdot \underline{u} > 0$ for all $\underline{u} \neq \underline{0}$, $\underline{u} \cdot \underline{u} = 0$ for $\underline{u} = \underline{0}$.

(iii) $(\alpha \underline{u} + \beta \underline{v}) \cdot \underline{w} = \alpha(\underline{u} \cdot \underline{w}) + \beta(\underline{v} \cdot \underline{w})$; linearity.

<u>example</u> Expand the dot product $(3\underline{u} - \underline{v}) \cdot (5\underline{w} + \underline{t})$

$$\begin{aligned}(3\underline{u} - \underline{v}) \cdot (5\underline{w} + \underline{t}) &= (3\underline{u} - \underline{v}) \cdot 5\underline{w} + (3\underline{u} - \underline{v}) \cdot \underline{t} \\ &= 15(\underline{u} \cdot \underline{w}) - 5(\underline{v} \cdot \underline{w}) + 3(\underline{u} \cdot \underline{t}) - (\underline{v} \cdot \underline{t}).\end{aligned}$$

<u>Theorem</u> The Schwartz inequality:

$$\forall\, \underline{u}, \underline{v} \in \mathbb{R}^n, \quad |\underline{u} \cdot \underline{v}| \leq \|\underline{u}\|_2 \|\underline{v}\|_2$$

<u>Proof</u>

$J(\alpha) = (\underline{u} + \alpha \underline{v}) \cdot (\underline{u} + \alpha \underline{v}) \geq 0$ for any $\alpha \in \mathbb{R}$.

$$J(\alpha) = (\underline{u} + \alpha \underline{v}) \cdot (\underline{u} + \alpha \underline{v}) = \overbrace{(\underline{u} \cdot \underline{u}) + 2\alpha(\underline{u} \cdot \underline{v}) + \alpha^2(\underline{v} \cdot \underline{v})}^{\text{a parabolic function interms of } \alpha}$$

The minimum of $J(\alpha)$ can be found easily by equating $\frac{d}{d\alpha} J(\alpha)$ to zero:

$$\frac{d}{d\alpha} J(\alpha) = 2\, \underline{u} \cdot \underline{v} + 2\alpha \underbrace{(\underline{v} \cdot \underline{v})}_{\|\underline{v}\|_2^2} = 0 \Rightarrow \alpha = -\frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|_2^2}$$

Thus, the minimum value of $(\underline{u} + \alpha \underline{v}) \cdot (\underline{u} + \alpha \underline{v})$ is:

$$\left(\underline{u} - \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|^2} \underline{v}\right) \cdot \left(\underline{u} - \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|^2} \underline{v}\right) = \|\underline{u}\|^2 - \frac{2(\underline{u} \cdot \underline{v})^2}{\|\underline{v}\|^2} + \frac{(\underline{u} \cdot \underline{v})^2}{\|\underline{v}\|^2}$$

$$= \|\underline{u}\|^2 - \frac{(\underline{u} \cdot \underline{v})^2}{\|\underline{v}\|^2} \geq 0$$

$$\Rightarrow \|\underline{u}\|^2 \cdot \|\underline{v}\|^2 \geq (\underline{u} \cdot \underline{v})^2$$

$$\Rightarrow |\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \cdot \|\underline{v}\| \quad \blacksquare$$

Note that to show it yields the minimum value $\frac{d^2}{d\alpha^2} J(\alpha) > 0$

$\frac{d^2}{d\alpha^2} J(\alpha) = 2\|\underline{v}\|_2^2 > 0$ so α is a minim

Def'n (i) Two vectors are orthogonal if: $\underline{u} \cdot \underline{v} = 0$.

(ii) $\{\underline{u}_1, \ldots, \underline{u}_k\}$ is an orthogonal set if for any i and j $1 \leq i \neq j \leq k$, $\underline{u}_i \cdot \underline{u}_j = 0$.

Question Given a vector $\underline{u} \neq \underline{0}$, want to find another vector $\hat{\underline{u}}$ which is unit-norm and angle between $\hat{\underline{u}}$ and $\underline{u}$ is zero. How can we find it?

Answer Only vectors in the form $\underline{v} = \alpha \underline{u}$, $\alpha > 0$ has a zero angle between $\underline{u}$. Therefore, we want to find α so that $\|\alpha \underline{u}\| = 1$.

$$1 = \|\alpha \underline{u}\|^2 = (\alpha \underline{u}) \cdot (\alpha \underline{u}) = \alpha^2 (\underline{u} \cdot \underline{u}) = \alpha^2 \|\underline{u}\|^2.$$

$$\Rightarrow \alpha = \frac{1}{\|\underline{u}\|}.$$

Thus $\hat{\underline{u}} = \frac{\underline{u}}{\|\underline{u}\|}$. This is called vector normalization.

Def'n $\{\underline{u}_1, \dots, \underline{u}_k\}$ is called as a set of orthonormal vectors if:

$$\underline{u}_i \cdot \underline{u}_j = 0 \quad \text{for } 1 \le i \ne j \le k$$

and $\|\underline{u}_i\| = 1$ for $1 \le i \le k$.

In short: $\underline{u}_i \cdot \underline{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$ ← Kronecker delta

More Generalizations

- Not every vector space is finite dimensional. For example real valued functions defined over $[0,1]$ interval form a vector space with the following definitions of vector summation and scalar multiplication:

$$\underline{u} + \underline{v} \equiv u(x) + v(x)$$
$$\alpha \underline{u} \equiv \alpha u(x).$$

- Not every vector space has a dot or "inner" product and norm.

- If a vector space has a valid dot product, can define a norm based on this dot product as: $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$ ← called as "induced norm."

- A commonly used alternative for the dot product is:

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i w_i, \quad \text{where } w_i \ge 0, \ 1 \le i \le n.$$

This type of dot product is called as a weighted dot product

Extension to Complex Case

- Can use complex scalars rather than real scalars in matrices and vectors.
- Very useful in Electrical Engineering to investigate solutions to linear system of equations with complex valued coefficients.

- Care must be taken to define the proper dot products.

For $\underline{u} \in \mathbb{C}^n$, $\underline{v} \in \mathbb{C}^n$,

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i^*$$

Then $\|\underline{u}\|_2 = \sqrt{\sum_{i=1}^{n} u_i u_i^*} = \sqrt{\sum_{i=1}^{n} |u_i|^2}$

example

Find the solution set to following linear system of equations:

$$(1+j2)\,x_1 + (2-j)\,x_2 = 3+j8$$
$$(2-j)\,x_1 + (1+j)\,x_2 = 4+j3$$

Form the augmented coefficient matrix:

$$\begin{bmatrix} 1+j2 & 2-j & 3+j8 \\ 2-j & 1+j & 4+j3 \end{bmatrix} \xrightarrow{R_1/(1+j2) \to R_1} \begin{bmatrix} 1 & \frac{2-j}{1+j2} & \frac{3+j8}{1+j2} \\ 2-j & 1+j & 4+j3 \end{bmatrix}$$

$$\xrightarrow{R_2 - (2-j)R_1 \to R_2} \begin{bmatrix} 1 & \frac{2-j5}{5} & \frac{19+2j}{5} \\ 0 & \frac{6+j17}{5} & -4+6j \end{bmatrix}$$

$$\xrightarrow{\frac{R_2\,5}{6+j17} \to R_2} \begin{bmatrix} 1 & 2-j5 & \frac{19+j2}{5} \\ 0 & 1 & \frac{78+j104}{65} \end{bmatrix}$$

$$\xrightarrow{R_1 - (2-j5)R_2 \to R_1} \begin{bmatrix} 1 & 0 & \frac{676-j182}{65} \\ 0 & 1 & \frac{78+j104}{65} \end{bmatrix}$$

Thus, $x_1 = \dfrac{676-j182}{65}$

$x_2 = \dfrac{78+j104}{65}$

Note that, in general, manual operations with complex valued matrices requires considerably more work.

Span and Subspace (Section 9.7 in the textbook)

Def'n If $\underline{u}_1, \ldots, \underline{u}_k$ are vectors in vector space S, then the set of all vectors of the form:

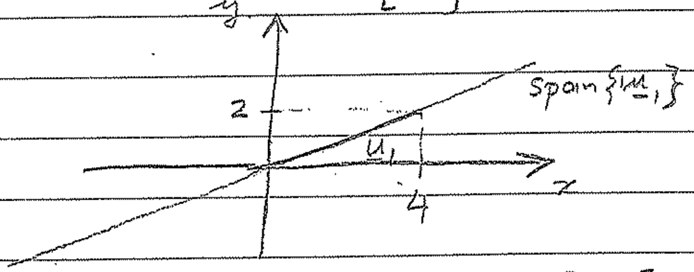
$$\underline{u} = \alpha_1 \underline{u}_1 + \cdots + \alpha_k \underline{u}_k, \quad \text{for } \alpha_i \in \mathbb{R},\ 1 \le i \le k$$

is called as the span of $\underline{u}_1, \ldots, \underline{u}_k$ and is denoted as:

$$\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$$

Def'n The set $\{\underline{u}_1, \ldots, \underline{u}_k\}$ is called as the generating set of $\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$.

ex's (i) $\underline{u}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ $\quad \text{span}\{\underline{u}_1\} = \left\{ \underline{u} : \underline{u} = \alpha \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$



(ii) $\text{span}\left\{ \underbrace{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} -8 \\ -4 \end{bmatrix}}_{\underline{u}_2} \right\} = \text{span}\left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$

because, $\underline{u}_2 = -2\underline{u}_1$. Hence, $\underline{u} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \alpha_1 \underline{u}_1 - 2\alpha_2 \underline{u}_1$

$$= (\alpha_1 - 2\alpha_2)\underline{u}_1$$

$$= \beta_1 \underline{u}_1.$$

Def'n If a subset T of a vector space S satisfies the following it is called as a subspace of S:

(i) $\forall$ $\underline{u}$ and $\underline{v}$ in T, $\underline{u} + \underline{v}$ is in T : closed under summat

(ii) $\forall \underline{u} \in T$, $\alpha \in \mathbb{R}$, $\alpha \underline{u} \in T$ as well.

since S is a vector space, to check if $T \subset S$ is a subspace or not it is sufficient to check $\Rightarrow$ T: a subspace is itself a vector space!

Theorem 9.7.1 If $\underline{u}_1, \ldots, \underline{u}_k$ are in S, vector space, then $\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$ is a subspace of S.

Proof

If $\underline{u} \in \text{Span}\{\underline{u}_1, \ldots, \underline{u}_k\}$, then there exists $\alpha_1, \ldots, \alpha_k$ such that $\underline{u} = \alpha_1 \underline{u}_1 + \cdots + \alpha_k \underline{u}_k$.

Similarly, if $\underline{v} \in \text{span}\{\underline{u}_1, \dots, \underline{u}_k\}$, then there exists $\beta_1, \dots, \beta_k$ such that $\underline{v} = \beta_1 \underline{u}_1 + \dots + \beta_k \underline{u}_k$.

Then $\underline{u} + \underline{v} = (\alpha_1 + \beta_1)\underline{u}_1 + \dots + (\alpha_k + \beta_k)\underline{u}_k \in \text{span}\{\underline{u}_1, \dots, \underline{u}_k\}$

Thus, the span satisfies the closeness under summation property.

Now consider $\underline{u} \in \text{span}\{\underline{u}_1, \dots, \underline{u}_k\} \Rightarrow \underline{u} = \alpha_1 \underline{u}_1 + \dots + \alpha_k \underline{u}_k$

Then $c \cdot \underline{u}$ for any $c \in \mathbb{R}$: $c\underline{u} = c\alpha_1 \underline{u}_1 + \dots + c\alpha_k \underline{u}_k \in \text{span}\{\underline{u}_1, \dots, \underline{u}_k\}$

Thus, $\text{span}\{\underline{u}_1, \dots, \underline{u}_k\}$ is a subspace of S. ∎

Linear Dependence

Def'n $\{\underline{u}_1, \dots, \underline{u}_k\}$ is said to be linearly dependent if at least one of them can be expressed as a linear combination of the others. If none can be expressed, the set is linearly independent

ex's (i) $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{u}_2}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{u}_3} \right\}$ is linearly dependent: $\underline{u}_1 = \frac{1}{2}\underline{u}_2 + \frac{1}{2}\underline{u}_3$

(ii) $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\underline{u}_2} \right\}$ is linearly independent

(iii) $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\underline{u}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{u}_3} \right\}$ is linearly dependent $\underline{u}_3 = 0 \cdot \underline{u}_1 + 0 \cdot \underline{u}_2$.

Theorem 9.8.1 Let $\underline{u}_1, \dots, \underline{u}_k$ be vectors in a vector space S. Then the set $\{\underline{u}_1, \dots, \underline{u}_k\}$ is LD if and only if there exists scalars α_i, not all zero, such that

$$\alpha_1 \underline{u}_1 + \dots + \alpha_k \underline{u}_k = \underline{0}.$$

Proof has two parts

(i) $\{\underline{u}_1, \dots, \underline{u}_k\}$ is LD $\Rightarrow \exists\, \alpha_1, \dots, \alpha_k$, $\alpha_1 \underline{u}_1 + \dots + \alpha_k \underline{u}_k = \underline{0}$ and not all α_i's are zero proof of (i) $\{\underline{u}_1, \dots, \underline{u}_k\}$ is LD implies that for some j

$$\underline{u}_j = \sum_{i \neq j} \alpha_i \underline{u}_i$$

Then, $\underline{0} = \sum_{i \neq j} \alpha_i \underline{u}_i - \underline{u}_j$

Thus, we have a sequence of coefficients α_i's such that $\alpha_j = -1$, hence not all zero, that provides $\underline{0} = \sum_{i=1}^{k} \alpha_i \underline{u}_i$. ∎

(ii) $\exists\, \alpha_1, \dots, \alpha_k$ not all zero
$\sum_{i=1}^{k} \alpha_i \underline{u}_i = \underline{0} \implies \{\underline{u}_1, \dots, \underline{u}_k\}$ is LD

proof of (ii) Since not all α_i's are zero, let α_j be nonzero.
Then $\sum_{i \neq j} \alpha_i \underline{u}_i + \alpha_j \underline{u}_j = \underline{0} \implies \underline{u}_j = \sum_{i \neq j} \left(-\frac{\alpha_i}{\alpha_j}\right) \underline{u}_i \implies$ the set is LD ∎

ex/ Consider $\left\{ \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\underline{u}_2}, \underbrace{\begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix}}_{\underline{u}_3} \right\}$.

Thm 9.8.1 provides us means to check LI easily. The $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is LI if $\underline{u}_1 \alpha_1 + \underline{u}_2 \alpha_2 + \underline{u}_3 \alpha_3 = \underline{0}$
has a unique solution for α_1, α_2 and α_3.

$$\implies \left.\begin{aligned} 2\alpha_1 \qquad + 2\alpha_3 &= 0 \\ \alpha_2 + 2\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + 3\alpha_3 &= 0 \\ -3\alpha_1 + \alpha_2 \qquad &= 0 \end{aligned}\right\}$$ This homogeneous linear system of equations must have a unique solution for α_1, α_2 and α_3.

Augmented Coefficient Matrix
$$\begin{array}{cccc} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 & \underline{0} \end{array}$$
$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

have 3 pivots and no free variables $\implies$ the solution is unique. $\implies \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ are LI.

see 24b

explain here
or Theorem 9.8.3

Theorem 9.8.4 If $\{\underline{u}_1, \ldots, \underline{u}_k\}$ is LI then

$$\alpha_1 \underline{u}_1 + \cdots + \alpha_k \underline{u}_k = \beta_1 \underline{u}_1 + \cdots + \beta_k \underline{u}_k \Rightarrow \alpha_i = \beta_i \quad 1 \le i \le k$$

Proof $\alpha_1 \underline{u}_1 + \cdots + \alpha_k \underline{u}_k = \beta_1 \underline{u}_1 + \cdots + \beta_k \underline{u}_k \Rightarrow (\alpha_1 - \beta_1)\underline{u}_1 + \cdots + (\alpha_k - \beta_k)\underline{u}_k = \underline{0}$
since $\{\underline{u}_1, \ldots, \underline{u}_k\}$ is LI $\Rightarrow \alpha_i - \beta_i = 0 \quad 1 \le i \le k \Rightarrow \alpha_i = \beta_i \quad 1 \le i \le k$ ∎

Theorem 9.8.5 Every finite set of non-zero orthogonal vectors is LI.

Proof Assume that there exists a non-zero orthogonal set of vectors $\{\underline{u}_1, \ldots, \underline{u}_k\}$ which is LD. This implies that for some j:

$$\underline{u}_j = \sum_{i \ne j} \alpha_i \underline{u}_i$$

By computing dot product of both sides with $\underline{u}_j$ we get:

$$\underline{u}_j \cdot \underline{u}_j = \left(\sum_{i \ne j} \alpha_i \underline{u}_i\right) \cdot \underline{u}_j = \sum_{i \ne j} \alpha_i \underbrace{(\underline{u}_i \cdot \underline{u}_j)}_{0} = 0$$

$\Rightarrow \underline{u}_j \cdot \underline{u}_j = 0 \Rightarrow \underline{u}_j = 0$, but this is a contradiction because we have assumed that $\underline{u}_i$'s were non-zero. ∎

Important Observation: If $\underline{0} \in U = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\}$ then the set U is linearly dependent because one can choose all the coef. except the coef. of $\underline{0}$ as zero to get the following linear combination
$0\underline{u}_1 + 0\underline{u}_2 + \cdots + 5\underline{0} + 0\underline{u}_i + \cdots + 0\underline{u}_k = 5\underline{0} = \underline{0}$
↳ non zero coefficient.

Bases (see 25b), Expansions, Dimension

+ Basis

Def'n A finite set of vectors $\{\underline{e}_1, \ldots, \underline{e}_k\}$ in a vector space S is a basis for S if each vector in S can be expressed uniquely as:

$$\underline{u} = \alpha_1 \underline{e}_1 + \cdots + \alpha_k \underline{e}_k = \sum_{i=1}^{k} \alpha_i \underline{e}_i$$

Theorem 9.9.1
$\{\underline{e}_1, \ldots, \underline{e}_k\}$ is a basis for S if $\text{span}\{\underline{e}_1, \ldots, \underline{e}_k\} = S$ and $\{\underline{e}_1, \ldots, \underline{e}_k\}$ is LI.

Proof
Let $\underline{u}$ be an arbitrary vector in S. If $\text{span}\{\underline{e}_1, \ldots, \underline{e}_k\} = S$, then $\underline{u} = \alpha_1 \underline{e}_1 + \cdots + \alpha_k \underline{e}_k$ for some α_i, $1 \le i \le k$.

To conclude that $\{\underline{e}_1, \dots, \underline{e}_k\}$ is a basis, we need to show that this expansion of $\underline{u}$ is unique, i.e., if

$$\underline{u} = \beta_1 \underline{e}_1 + \dots + \beta_k \underline{e}_k = \alpha_1 \underline{e}_1 + \dots + \alpha_k \underline{e}_k$$

then, $\beta_i = \alpha_i$. But this is immediate since $\{\underline{e}_1, \dots, \underline{e}_k\}$ is LI.

ex/ In $\mathbb{R}^2$, $\underline{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\underline{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ forms a basis.

For example if $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} \alpha_1 + \alpha_2 &= u_1 \\ 2\alpha_1 - \alpha_2 &= u_2 \end{aligned}$$

Augmented coefficient matrix is: $\left[\begin{array}{cc|c} 1 & 1 & u_1 \\ 2 & -1 & u_2 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss}} \left[\begin{array}{ccc} 1 & 1 & u_1 \\ 0 & -3 & u_2 - 2 \end{array}\right.$

$$\Rightarrow \alpha_2 = -\frac{1}{3} u_2 + \frac{2}{3} u_1$$

$$\alpha_1 = u_1 + \frac{1}{3} u_2 - \frac{2}{3} u_1 = \frac{1}{3} u_1 + \frac{1}{3} u_2.$$

Thus, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{(u_1 + u_2)}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{(2u_1 - u_2)}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Also note that the solution for α_1 and α_2 is unique since $\underline{e}_1$ and $\underline{e}_2$ are LI.

<u>Def'n</u> of dimension If the greatest number of LI vectors that can be found in a vector space S is k, $1 \le k < \infty$, then S is <u>k-dimensional</u>

$$\underline{\dim S = k}.$$

If $S = \{\underline{0}\}$ then $\dim S = 0$. If arbitrarily large number of LI vectors can be found in S, then S is infinite-dimensional.

<u>Theorem 9.9.2</u> If a vector space S admits a basis consisting of k vectors, then $\dim S = k$.

(because it is a basis)

<u>Proof</u> Let $\{\underline{e}_1, \dots, \underline{e}_k\}$ be the basis. Since it is LI,

$$\dim S \ge k.$$

Now assume that $\dim S > k$. This implies that, there exists a LI set of vectors $\{\underline{u}_1, \ldots, \underline{u}_{k+1}\}$.
Since $\{\underline{e}_1, \ldots, \underline{e}_k\}$ is a basis, we can uniquely represent each $\underline{u}_j$ as:

$$\underline{u}_j = \sum_{i=1}^{k} \alpha_{ji} \underline{e}_i, \quad 1 \le j \le k+1$$

Then, investigate linear combinations of $\underline{u}_j$ as:

$$\underline{0} = \beta_1 \underline{u}_1 + \cdots + \beta_{k+1} \underline{u}_{k+1} = \sum_{j=1}^{k+1} \beta_j \underline{u}_j = \sum_{j=1}^{k+1} \beta_j \sum_{i=1}^{k} \alpha_{ji} \underline{e}_i$$

$$= \sum_{i=1}^{k} \underline{e}_i \left[\sum_{j=1}^{k+1} \alpha_{ji} \beta_j \right]$$

$$\Rightarrow \sum_{j=1}^{k+1} \alpha_{ji} \beta_j = 0 \quad \text{for } 1 \le i \le k$$

in long form, this implies:

$$\left.\begin{aligned} \alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \cdots + \alpha_{k+1,1}\beta_{k+1} &= 0 \\ \alpha_{1,2}\beta_1 + \alpha_{2,2}\beta_2 + \cdots + \alpha_{k+1,2}\beta_{k+1} &= 0 \\ &\vdots \\ \alpha_{1,k}\beta_1 + \alpha_{2,k}\beta_2 + \cdots + \alpha_{k+1,k}\beta_{k+1} &= 0 \end{aligned}\right\} \begin{aligned} &k \text{ equations in} \\ &k+1 \text{ unknowns, } \beta_1, \ldots, \beta_{k+1} \end{aligned}$$

This homogeneous system admits nontrivial solutions where not all $\beta_j = 0$. Therefore, there exists linear combinations of the form:

$$\beta_1 \underline{u}_1 + \cdots + \beta_{k+1} \underline{u}_{k+1} = \underline{0} \quad \text{where not all } \beta_j = 0.$$

This is a contradiction since $\{\underline{u}_1, \ldots, \underline{u}_{k+1}\}$ was assumed to be LI. ∎

<u>Theorem 9.9.3</u> The dimension of $\mathbb{R}^n$ is n.

<u>Proof</u> Choose $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is LI.

Also, $\underline{u} \in \mathbb{R}^n$, then $\underline{u} = \sum_{i=1}^{n} \alpha_i \underline{e}_i$. Thus $\operatorname{span}\{\underline{e}_1, \ldots, \underline{e}_n\} = \mathbb{R}^n$.
Hence $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is a basis for $\mathbb{R}^n$. Therefore the $\dim \mathbb{R}^n = n$. ∎

(

Theorem 9.9.4 $\dim[\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}]$, where $\underline{u}_i$ not all zero is equal to the greatest number of linearly independent vectors within the ~~generating~~ generating set $\{\underline{u}_1, \ldots, \underline{u}_k\}$

Proof Let m be the greatest number of linearly independent vectors in $\{\underline{u}_1, \ldots, \underline{u}_k\}$. Let $\underline{u}_i$ be ordered such that the first m of them are LI. Then, the remaining vectors, $\underline{u}_{m+1}, \ldots, \underline{u}_k$ must be linearly dependent to $\underline{u}_1, \ldots, \underline{u}_m$. Therefore,

$$\underline{u}_i = \alpha_{i1} \underline{u}_1 + \cdots + \alpha_{im} \underline{u}_m, \quad m+1 \leq i \leq k.$$

Therefore if $\underline{v} \in \text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$, $\underline{v} = \sum_{i=1}^{k} \beta_i \underline{u}_i$

$$\Rightarrow \underline{v} = \sum_{i=1}^{m} \beta_i \underline{u}_i + \sum_{j=m+1}^{k} \beta_j \underline{u}_j$$

$$\underline{v} = \sum_{i=1}^{m} \beta_i \underline{u}_i + \sum_{j=m+1}^{k} \beta_j \left[\sum_{i=1}^{m} \alpha_{ji} \underline{u}_i \right]$$

$$\underline{v} = \sum_{i=1}^{m} \left[\beta_i + \sum_{j=m+1}^{k} \beta_j \alpha_{ji} \right] \underline{u}_i \in \text{span}\{\underline{u}_1, \ldots, \underline{u}_m\}$$

$$\Rightarrow \text{Span}\{\underline{u}_1, \ldots, \underline{u}_k\} = \text{Span}\{\underline{u}_1, \ldots, \underline{u}_m\}$$

Hence $\{\underline{u}_1, \ldots, \underline{u}_m\}$ is a basis for $\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$. Thus the dimension of $\text{span}\{\underline{u}_1, \ldots, \underline{u}_k\}$ is m. ∎

(see 28b) (optional)

Orthogonal Bases

For a given vector space of dimension 2 and above, there are infinitely many bases. In practice, we prefer orthogonal bases.

Let $\{\underline{e}_1, \ldots, \underline{e}_k\}$ be an orthogonal basis for S. Then, for any $\underline{u} \in$ $\underline{u} \in S$

$$\underline{u} = \alpha_1 \underline{e}_1 + \cdots + \alpha_k \underline{e}_k,$$

for a set of $\alpha_1, \ldots, \alpha_k$. When the basis is orthogonal, it is simple to find α_i's for a given $\underline{u}$.

$$\underline{u} \cdot \underline{e}_j = (\alpha_1 \underline{e}_1 + \cdots + \alpha_k \underline{e}_k) \cdot \underline{e}_j \overset{\text{orthogonality}}{=} \alpha_j \underline{e}_j \cdot \underline{e}_j$$

$$\Rightarrow \alpha_j = \frac{\underline{u} \cdot \underline{e}_j}{\underline{e}_j \cdot \underline{e}_j}. \Rightarrow \underline{u} = \sum_{i=1}^{k} \left(\frac{\underline{u} \cdot \underline{e}_i}{\underline{e}_i \cdot \underline{e}_i} \right) \underline{e}_i.$$

... product

Note that $\frac{\underline{e}_i}{\underline{e}_i \cdot \underline{e}_i} = \hat{\underline{e}}_i \Rightarrow \underline{u} = \sum_{i=1}^{N} (\underline{u} \cdot \underline{e}_i) \hat{\underline{e}}_i$

Furthermore, if the basis is orthonormal, then:

$$\alpha_i = (\underline{u}_i \cdot \underline{e}_i) / \underbrace{\underline{e}_i \cdot \underline{e}_i}_{1} = \underline{u}_i \cdot \underline{e}_i .$$

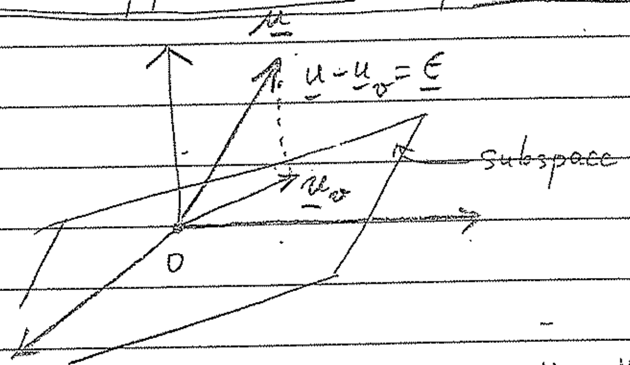
Best Approximation

Let S be a vector space and $\mathcal{V}$ be a subspace of S. Also let $\{\hat{\underline{e}}_1, \dots, \hat{\underline{e}}_N\}$ be an orthonormal basis for $\mathcal{V}$, i.e.,

$$\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = \delta_{ij} \quad , \quad 1 \le i, j \le N$$

and $\operatorname{span}\{\hat{\underline{e}}_1, \dots, \hat{\underline{e}}_N\} = \mathcal{V}$

In many engineering applications, we are faced with finding the best approximation of a given vector in a subspace.



Want to minimize $\|\underline{\epsilon}\|_2^2$ by choosing $\underline{u}_v \in \mathcal{V}$

Formally, $\min_{\underline{u}_v} \|\underline{\epsilon}\|_2^2 = \min_{\underline{u}_v} \|\underline{u} - \underline{u}_v\|_2^2$

Since $\underline{u}_v$ can be expressed in the given orthonormal basis as:

$$\underline{u}_v = \sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i$$

The minimization problem or best approximation problem can be written equivalently as:

$$\min_{\alpha_1, \dots, \alpha_N} \underbrace{\|\underline{u} - \sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i\|_2^2}_{J(\alpha_1, \dots, \alpha_N)}$$ cost function

$$J(\alpha_1, \dots, \alpha_N) = \left(\underline{u} - \sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i\right) \cdot \left(\underline{u} - \sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i\right)$$

$$= \underline{u} \cdot \underline{u} - 2 \sum_{i=1}^{N} \alpha_i \, \underline{u} \cdot \hat{\underline{e}}_i + \underbrace{\left(\sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i\right) \cdot \left(\sum_{i=1}^{N} \alpha_i \hat{\underline{e}}_i\right)}_{\sum^{N} \alpha_i^2}$$

$$\Rightarrow J(\alpha_1, \ldots, \alpha_N) = \overbrace{\underline{u}\cdot\underline{u} - 2\sum_{i=1}^{N} \alpha_i\, \underline{u}\cdot\hat{\underline{e}}_i + \sum_{i=1}^{N}\alpha_i^2}^{\text{quadratic function of } \alpha_i\text{'s}}$$

Optimal values of α_i can be found by finding: $\frac{\partial J}{\partial \alpha_i} = 0$

$$\frac{\partial J}{\partial \alpha_i} = -2\,\underline{u}\cdot\hat{\underline{e}}_i + 2\alpha_i = 0 \Rightarrow \underline{\alpha_i = \underline{u}\cdot\hat{\underline{e}}_i}$$

Therefore the best approximation of a given $\underline{u}$ in subspace $\mathcal{V}$ is

$$\underline{u}_{\mathcal{V}} = \sum_{i=1}^{N} (\underline{u}\cdot\hat{\underline{e}}_i)\,\hat{\underline{e}}_i$$

Note that, if $\underline{u} \in \mathcal{V}$, the best approximation of it will be itself.

Now, let us investigate the approximation error of the best approximation obtained above.

$$\underline{\epsilon} = \underline{u} - \underline{u}_{\mathcal{V}} = \underline{u} - \sum_{i=1}^{N}(\underline{u}\cdot\hat{\underline{e}}_i)\,\hat{\underline{e}}_i$$

<u>Claim</u> $\underline{\epsilon}$ is orthogonal to $\mathcal{V}$, i.e., $\underline{\epsilon}$ is orthogonal to every vector in $\mathcal{V}$.

<u>Proof</u> Any arbitrary vector in $\mathcal{V}$ can be expressed as

$$\underline{v} = \sum_{i=1}^{N} \beta_i \hat{\underline{e}}_i, \quad \text{since } \operatorname{span}\{\hat{\underline{e}}_1, \ldots, \hat{\underline{e}}_N\} = \mathcal{V}$$

Then

$$\underline{\epsilon}\cdot\underline{v} = \left(\underline{u} - \sum_{i=1}^{N}(\underline{u}\cdot\hat{\underline{e}}_i)\hat{\underline{e}}_i\right)\cdot\left(\sum_{i=1}^{N}\beta_i\hat{\underline{e}}_i\right)$$

$$= \sum_{i=1}^{N}\beta_i\, \underline{u}\cdot\hat{\underline{e}}_i - \left(\sum_{i=1}^{N}(\underline{u}\cdot\hat{\underline{e}}_i)\hat{\underline{e}}_i\right)\cdot\left(\sum_{i=1}^{N}\beta_i\hat{\underline{e}}_i\right)$$

$$= \sum_{i=1}^{N}\beta_i\, \underline{u}\cdot\hat{\underline{e}}_i - \sum_{i=1}^{N}(\underline{u}\cdot\hat{\underline{e}}_i)\beta_i = 0. \quad \blacksquare \qquad \left(\hat{\underline{e}}_i \cdot \sum_{i=1}^{N}\beta_i\hat{\underline{e}} = \beta_i\right)$$

Therefore, we call $\underline{u}_{\mathcal{V}}$ as the orthogonal projection of $\underline{u}$ on

$$\boxed{\operatorname{proj}_{\mathcal{V}} \underline{u} = \sum_{i=1}^{N}(\underline{u}\cdot\hat{\underline{e}}_i)\hat{\underline{e}}_i}$$

This form of the $\operatorname{proj}_{\mathcal{V}}\underline{u}$ requires availability of an orthonormal basis. What if we are given a non-orthonormal basis?

Answer Can use the Gram-Schmidt orthogonalization process!

The Gram-Schmidt Orthogonalization Process

Let $\{\underline{v}_1, \ldots, \underline{v}_N\}$ be a basis for subspace $\mathcal{V}$. Want to generate $\{\hat{\underline{e}}_1, \ldots, \hat{\underline{e}}_N\}$, an orthonormal basis for $\mathcal{V}$, from $\{\underline{v}_1, \ldots, \underline{v}_N\}$.

step1 > Choose $\hat{\underline{e}}_1 = \dfrac{\underline{v}_1}{\|\underline{v}_1\|}$; Note that $\hat{\underline{e}}_1 \cdot \hat{\underline{e}}_1 = 1$

step2 > (i) Then project $\underline{v}_2$ onto span$\{\hat{\underline{e}}_1\}$ to obtain $\underline{v}_{2p} = (\underline{v}_2 \cdot \hat{\underline{e}}_1)\hat{\underline{e}}_1$

(ii) Find the projection error $\underline{\epsilon}_2 = \underline{v}_2 - \underline{v}_{2p}$

↳ Then:

$$\hat{\underline{e}}_2 = \frac{\underline{\epsilon}_2}{\|\underline{\epsilon}_2\|}$$; Note that $\hat{\underline{e}}_2 \cdot \hat{\underline{e}}_1 = 0$ and $\|\hat{\underline{e}}_2\| = 1$.

step3 > Continue to determine the rest of $\hat{\underline{e}}_i$'s, for $2 < i \leq N$

- Project $\underline{v}_i$ onto the span$\{\hat{\underline{e}}_1, \ldots, \hat{\underline{e}}_{i-1}\}$ to obtain $\underline{v}_{ip}$

$$\underline{v}_{ip} = \sum_{j=1}^{i-1} (\underline{v}_i \cdot \hat{\underline{e}}_j)\,\hat{\underline{e}}_j$$

and the projection error $\underline{\epsilon}_i = \underline{v}_i - \underline{v}_{ip}$. Then set:

$$\hat{\underline{e}}_i = \frac{\underline{\epsilon}_i}{\|\underline{\epsilon}_i\|}.$$

Note that, $\hat{\underline{e}}_i$ is orthogonal to $\{\hat{\underline{e}}_1, \ldots, \hat{\underline{e}}_{i-1}\}$ because $\underline{\epsilon}_i$ is the projection error of $\underline{v}_i$ onto the span$\{\hat{\underline{e}}_1, \ldots, \hat{\underline{e}}_{i-1}\}$

Also, note that $\|\hat{\underline{e}}_i\| = 1$. Thus, G-S procedure generates an orthonormal basis from a given non-orthonormal basis

$$\Rightarrow \hat{\underline{e}}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}, \quad \hat{\underline{e}}_2 = \frac{\underline{v}_2 - (\underline{v}_2 \cdot \hat{\underline{e}}_1)\hat{\underline{e}}_1}{\|\underline{v}_2 - (\underline{v}_2 \cdot \hat{\underline{e}}_1)\hat{\underline{e}}_1\|}, \ldots,$$

$$\hat{\underline{e}}_i = \frac{\underline{v}_i - \sum_{j=1}^{i-1} (\underline{v}_i \cdot \hat{\underline{e}}_j)\hat{\underline{e}}_j}{\left\|\underline{v}_i - \sum_{j=1}^{i-1} (\underline{v}_i \cdot \hat{\underline{e}}_j)\hat{\underline{e}}_j\right\|}$$

Matrices and Linear Equations

(* Chapter 10 in the textbook)

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$, n columns, m rows. $a_{ij} \in \mathbb{R}$ or $a_{ij} \in \mathbb{C}$.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ -5 & 7 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 1+j & 2 \\ 3-j & 4+2j \end{bmatrix}$

Def'n $\underline{\underline{A}} = \{a_{ij}\}_{m\times n}$, $\underline{\underline{B}} = \{b_{ij}\}_{k\times l}$ are equal if $m=k$, $n=l$ and $a_{ij} = b_{ij}$ $1 \le i \le m$, $1 \le j \le n$.

Def'n $\underline{\underline{A}} = \{a_{ij}\}_{m\times n}$, $\underline{\underline{B}} = \{b_{ij}\}_{m\times n}$ then

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} = \{a_{ij} + b_{ij}\}_{m\times n}$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & -5 \\ -1 & 3 & 7 \end{bmatrix}$, $\underline{\underline{C}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} 2 & 4 & -2 \\ -3 & 8 & 13 \end{bmatrix}$ but $\underline{\underline{A}} + \underline{\underline{C}}$ and $\underline{\underline{B}} + \underline{\underline{C}}$ are not defi

Def'n If $\underline{\underline{A}} = \{a_{ij}\}_{m\times n}$, then $c\underline{\underline{A}} = \{c\,a_{ij}\}_{m\times n}$, $c \in \mathbb{R}$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $2\underline{\underline{A}} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$, $(-1)\cdot\underline{\underline{A}} = -\underline{\underline{A}} = \begin{bmatrix} -1 & -2 \\ -4 & -5 \end{bmatrix}$

Theorem 10.2.1 Properties of matrix addition and scalar multiplic

If $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are $m\times n$ matrices, $\underline{\underline{0}}$ is $m\times n$ zero matrix and α, β a any scalars, then:

(i) $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}$ (commutativity)

(ii) $(\underline{\underline{A}} + \underline{\underline{B}}) + \underline{\underline{C}} = \underline{\underline{A}} + (\underline{\underline{B}} + \underline{\underline{C}})$ (associativity)

(iii) $\underline{\underline{A}} + \underline{\underline{0}} = \underline{\underline{A}}$ (identity element of matrix addition

(iv) $\underline{\underline{A}} + (-\underline{\underline{A}}) = \underline{\underline{0}}$ (inverse element of matrix addition)
(v) $\alpha(\beta \underline{\underline{A}}) = (\alpha\beta)\underline{\underline{A}}$ (associativity of scalar multiplication)
(vi) $(\alpha+\beta)\underline{\underline{A}} = \alpha\underline{\underline{A}} + \beta\underline{\underline{A}}$ (distributivity)
(vii) $\alpha(\underline{\underline{A}}+\underline{\underline{B}}) = \alpha\underline{\underline{A}} + \alpha\underline{\underline{B}}$ (distributivity)
(viii) $1\cdot\underline{\underline{A}} = \underline{\underline{A}}$ (identity element of matrix multiplication)
(ix) $0\cdot\underline{\underline{A}} = \underline{\underline{0}}$
(x) $\alpha\cdot\underline{\underline{0}} = \underline{\underline{0}}$

Def'n Matrix Multiplication If $\underline{\underline{A}} = \{a_{ij}\}_{m\times n}$ and $\underline{\underline{B}} = \{b_{ij}\}_{n\times p}$
then $\underline{\underline{C}} = \underline{\underline{A}}\cdot\underline{\underline{B}} = \{c_{ij}\}_{m\times p}$ where:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad 1 \le i \le m, \quad 1 \le j \le p.$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 5 \\ 1 & 3 & 1 \end{bmatrix}$, $\underline{\underline{C}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 2 & 11 & 17 \\ 5 & 26 & 47 \end{bmatrix}$, $\underline{\underline{C}}\,\underline{\underline{A}} = \begin{bmatrix} 13 & 17 & 21 \\ 18 & 24 & 30 \end{bmatrix}$

but $\underline{\underline{A}}\,\underline{\underline{C}}$, $\underline{\underline{B}}\,\underline{\underline{C}}$, $\underline{\underline{B}}\,\underline{\underline{A}}$ and $\underline{\underline{C}}\,\underline{\underline{B}}$ are not defined.

Note that $\underline{\underline{A}}\,\underline{\underline{B}}$ is not necessarily equal to $\underline{\underline{B}}\,\underline{\underline{A}}$!

Def'n If $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ commute under multiplicat

This definition of matrix multiplication originates from linear system of equations:

$$\underbrace{\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}\right\}}_{\substack{\| \\ \underline{\underline{A}}\,\underline{x} = \underline{c}}} \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\underline{\underline{A}}_{m\times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\underline{x}_{n\times 1}} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}}_{\underline{c}_{m\times 1}}$$

Def'n If $A = \{a_{ij}\}_{n\times n}$, then $A^P = \underbrace{A \cdot A \cdots A}_{P \text{ factors}}$.

Def'n For $A = \{a_{ij}\}_{n\times n}$, If $a_{ij} = 0$ for $i \neq j$, then A is a diagonal matrix.

ex/ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$ not diagonal since A is not square

$B = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$ is a diagonal matrix

Def'n Diagonal matrix A is called as identity matrix if $a_{ii} = 1$.

ex/ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2x2 Identity matrix.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3x3 identity matrix.

Show that $A_{m\times n} \cdot I_n = A_{m\times n}$ and $I_m \times A_{m\times n} = A_{m\times n}$ for any A
(See extra note II)

Theorem 10.2.2 Exceptional Properties of matrix multiplication:

(i) $AB \neq BA$ in general

(ii) Even if $AB = AC$, B may not be equal to C

(iii) $AB = 0$ does not imply A or B is 0.

(iv) $A^2 = I$ does not imply $A = +I$ or $-I$

Theorem 10.2.3 Ordinary properties of matrix multiplication

(i) $(\alpha A)B = \alpha(AB) = A(\alpha B)$ (associativity)

(ii) $(AB)C = A(BC)$ (associativity)

(iii) $(A+B)C = AC + BC$ (distributivity)

(iv) $C(A+B) = CA + CB$ (distributivity)

(v) $A(\alpha B + \beta C) = \alpha AB + \beta AC$ (linearity)

Partioning in Matrix Operations

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{A}}_{11} & \underline{\underline{A}}_{12} & \cdots & \underline{\underline{A}}_{1n} \\ \vdots & & & \\ \underline{\underline{A}}_{m1} & \underline{\underline{A}}_{m2} & \cdots & \underline{\underline{A}}_{mn} \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} \underline{\underline{B}}_{11} & \cdots & \underline{\underline{B}}_{1q} \\ \vdots & & \\ \underline{\underline{B}}_{p1} & \cdots & \underline{\underline{B}}_{pq} \end{bmatrix}$$

if $m=p$, $n=q$ and each $\underline{\underline{A}}_{ij}$ is the same form as the correspond $\underline{\underline{B}}_{ij}$ block, then:

$$\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} \underline{\underline{A}}_{11} + \underline{\underline{B}}_{11} & \cdots & \underline{\underline{A}}_{1n} + \underline{\underline{B}}_{1n} \\ \vdots & & \vdots \\ \underline{\underline{A}}_{m1} + \underline{\underline{B}}_{m1} & & \underline{\underline{A}}_{mn} + \underline{\underline{B}}_{mn} \end{bmatrix}$$

if $n=p$, $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$

$$\underline{\underline{C}}_{ij} = \sum_{k=1}^{n} \underline{\underline{A}}_{ik} \underline{\underline{B}}_{kj}$$

provided that each $\underline{\underline{A}}_{ik} \underline{\underline{B}}_{kj}$ are defined.

ex/

$$\underline{\underline{A}} = \left[\begin{array}{cc|c} 2 & 4 & 1 \\ -1 & 3 & 0 \\ \hline 5 & 4 & 6 \end{array}\right] = \begin{bmatrix} \underline{\underline{A}}_{11} & \underline{\underline{A}}_{12} \\ \underline{\underline{A}}_{21} & \underline{\underline{A}}_{22} \end{bmatrix}$$

$$\underline{\underline{B}} = \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 2 & -4 & -1 \\ \hline 5 & 8 & 2 \end{array}\right] = \begin{bmatrix} \underline{\underline{B}}_{11} & \underline{\underline{B}}_{12} \\ \underline{\underline{B}}_{21} & \underline{\underline{B}}_{22} \end{bmatrix}$$

$$\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} \underline{\underline{A}}_{11} + \underline{\underline{B}}_{11} & \underline{\underline{A}}_{12} + \underline{\underline{B}}_{12} \\ \underline{\underline{A}}_{21} + \underline{\underline{B}}_{21} & \underline{\underline{A}}_{22} + \underline{\underline{B}}_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 5 & 4 \\ -1 & -1 & -1 \\ \hline 10 & 12 & 8 \end{array}\right]$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \left[\begin{array}{c|c} \underline{\underline{A}}_{11} \underline{\underline{B}}_{11} + \underline{\underline{A}}_{12} \underline{\underline{B}}_{21} & \underline{\underline{A}}_{11} \underline{\underline{B}}_{12} + \underline{\underline{A}}_{12} \underline{\underline{B}}_{22} \\ \hline \underline{\underline{A}}_{21} \underline{\underline{B}}_{11} + \underline{\underline{A}}_{22} \underline{\underline{B}}_{21} & \underline{\underline{A}}_{21} \underline{\underline{B}}_{12} + \underline{\underline{A}}_{22} \underline{\underline{B}}_{22} \end{array}\right] = \left[\begin{array}{cc|c} 13 & -6 & 4 \\ 6 & -13 & -6 \\ \hline 38 & 37 & 23 \end{array}\right]$$

Partioning helps for sparse matrices where most of the matrix entries are zero.

Def'n Transpose of matrix $\underline{\underline{A}} = \{a_{ij}\}_{m \times n}$ is defined as:

$$\underline{\underline{A}}^T = \{a^T_{ij}\}_{n \times m} \quad \text{where} \quad a^T_{ij} = a_{ji}.$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \underline{\underline{A}}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Theorem 10.3.1

(i) $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$

(ii) $(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$

(iii) $(\alpha \underline{\underline{A}})^T = \alpha \underline{\underline{A}}^T$

(iv) $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T \quad \overset{\text{extension}}{\Longrightarrow} \quad (\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}})^T = \underline{\underline{D}}^T \underline{\underline{C}}^T \underline{\underline{B}}^T \underline{\underline{A}}^T$

Proof for (iv) Let $\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}} \Longrightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$c^T_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} a^T_{kj} b^T_{ik} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj}$$

$\Rightarrow \underline{\underline{C}}^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$ ∎

Note that for vectors $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^n$

$$\underline{x}^T \underline{y} = [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i = \underline{x} \cdot \underline{y}$$

$$\boxed{\underline{x}^T \underline{y} = \underline{x} \cdot \underline{y}}$$

Def'n (i) If $\underline{\underline{A}}^T = \underline{\underline{A}}$, then $\underline{\underline{A}}$ is called as a symmetric matrix

(ii) If $\underline{\underline{A}}^T = -\underline{\underline{A}}$ then $\underline{\underline{A}}$ is called as a skew symmetric mat

Note that for any square matrix $\underline{\underline{A}}$:

$$\underline{\underline{A}} = \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T) + \frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^T)$$

where $\frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T)$ is a symmetric and $\frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^T)$ is a skew symmetric matrix (Show this)

HW Question: Determine all elements of the 3×3 skew-symmetric matrix $\underline{\underline{A}}$ with $a_{21} = 1$, $a_{31} = 3$, a_{2j}

Determinants

For a square matrix $\underline{\underline{A}} = \{a_{ij}\}_{n\times n}$:

$$\det \underline{\underline{A}} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ik} A_{ik}$$

where the cofactor A_{ik} is defined as:

$$A_{ik} = (-1)^{i+k} M_{ik}$$

where the minor M_{ik} is determinant of the $(n-1)\times(n-1)$ matrix that survives when the i^{th} row and k^{th} column of $\underline{\underline{A}}$ is removed.

ex/ (i) $\det [a_{11}] = |a_{11}| \equiv a_{11}$

(ii) $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}(-1)^{1+1}|a_{22}| + a_{12}(-1)^{1+2}|a_{21}|$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

(iii) $\det \begin{bmatrix} 4 & 7 & -2 \\ 0 & 3 & 2 \\ 1 & 5 & 6 \end{bmatrix} = 4\cdot(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 7(-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 1 & 6 \end{vmatrix} + (-2)\cdot(-1)^{1+3}\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix}$

$$= 4\cdot 8 - 7\cdot(-2) - 2\cdot(-3) = 32+14+6 = 52$$

Comments

(i) Since we can compute the cofactor expansion of a determinant along any row, it is preferable to expand the cofactors along the row with the most number of zero entries.

(ii) The cofactor expansion of large matrices are extremely time consuming. Number of computations in a cofactor expansion of an $n\times n$ matrix is $N(n) \sim e \cdot n!$

Properties of Determinants

know all of these

D1. If a multiple of a row is added to another, the determinant remains the same.

D2. If any two rows are interchanged, the polarity of the determinant changes.

D3. Determinant of triangular matrices is equal to the multiplication of diagonal entries

see extra notes on 38b before the examples.

example

$$\det \begin{bmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix} = - \begin{vmatrix} 2 & 0 & -4 \\ 4 & 3 & 5 \\ 0 & 2 & -1 \end{vmatrix}$$

← The first and third rows are interchanged.

$$\xrightarrow{-2r_1+r_2\to r_2} = - \begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 2 & -1 \end{vmatrix} \xrightarrow{-\frac{2}{3}r_2+r_3\to r_3} = - \begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 0 & -\frac{29}{3} \end{vmatrix} = -2\cdot 3\cdot\left(-\frac{29}{3}\right) = 58.$$

The triangularization process has a time complexity of

$$N_T(n) \approx \frac{2n^3}{3}$$

which is much less than $en!$ for n large.

Additional Properties of Determinants

know these as well! go together with 38c

D4. If $\underline{\underline{A}}$ has a column or row of zero, then $\det \underline{\underline{A}} = 0$.

D5. If any two rows or columns are proportional to each other than $\det \underline{\underline{A}} = 0$.

D6. If any row or column is linearly dependent to the rest of rows or columns, $\det \underline{\underline{A}} = 0$

D7. If all the elements in a row or column is multiplied by α, then the determinant is multiplied with the same α.

D8. $\det(\alpha \underline{\underline{A}}) = \alpha^n \det \underline{\underline{A}}$

D9. If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are the same except in a row or column then $\det(\underline{\underline{A}}+\underline{\underline{B}}) = \det(\underline{\underline{A}}) + \det(\underline{\underline{B}})$. ← gives the wrong impression.

D10. $\det(\underline{\underline{A}})^T = \det \underline{\underline{A}}$

D11. In general $\det(\underline{\underline{A}}+\underline{\underline{B}}) \neq \det \underline{\underline{A}} + \det \underline{\underline{B}}$

D12. $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = (\det \underline{\underline{A}})(\det \underline{\underline{B}})$

Row and Column Spaces

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \underline{r}_1^T \\ \underline{r}_2^T \\ \vdots \\ \underline{r}_m^T \end{bmatrix} \quad \text{where } \underline{r}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \in \mathbb{R}^n$$

$$\underline{\underline{A}} = \begin{bmatrix} \underline{a}_1 & \cdots & \underline{a}_n \end{bmatrix} \quad \text{where } \underline{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} \in \mathbb{R}^m$$

Def'n $\text{Row}(\underline{\underline{A}}) = \text{span}\{\underline{r}_1, \ldots, \underline{r}_m\}$, $\text{Col}(\underline{\underline{A}}) = \text{span}\{\underline{a}_1, \ldots, \underline{a}_n\}$

Def'n row rank $\equiv \dim(\text{Row}(\underline{\underline{A}}))$; column rank $= \dim(\text{Col}(\underline{\underline{A}}))$.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\underline{r}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \\ 3 \end{bmatrix}$, $\underline{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \\ 2 \end{bmatrix}$, $\underline{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 7 \end{bmatrix}$, $\underline{r}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{Row}(\underline{\underline{A}}) = \text{span}\{\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4\} = \text{span}\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$, row rank $= 3$.

$\text{Col}(\underline{\underline{A}}) = \text{span}\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5\} = \text{span}\{\underbrace{\underline{a}_1, \underline{a}_3, \underline{a}_4}_{\text{leading or pivot columns}}\}$, col. rank $= 3$.

Theorem If two matrices are row equivalent, they have the same row space and row rank.

Proof:

$\underline{\underline{A}}$ is row equivalent to $\underline{\underline{B}}$ implies that with a finite sequence of elementary row operations performed on $\underline{\underline{A}}$ we can obtain $\underline{\underline{B}}$. Since elementary row operations do not generate a row vector that is not linearly independent from the previous row vectors $\text{Row}(\underline{\underline{B}}) \subset \text{Row}(\underline{\underline{A}})$. But row equivalence of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ also implies that we can generate $\underline{\underline{A}}$ from $\underline{\underline{B}}$ by a finite sequence of elementary row operations as well. Therefore, $\text{Row}(\underline{\underline{A}}) \subset \text{Row}(\underline{\underline{B}})$. Thus

$$\text{Row}(\underline{\underline{A}}) = \text{Row}(\underline{\underline{B}}).$$

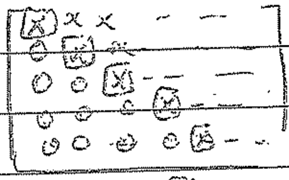
(*)

Theorem The non-zero row vectors of an echelon matrix are linearly independent and they form a basis for its row space.

Proof Let $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_k$ be the non zero row vectors of an echelon matrix $\underline{\underline{E}}$. Let $\underline{e}_i$'s first non-zero entry be positioned at p_i. Since $\underline{\underline{E}}$ is echelon, $p_1 < p_2 < \cdots < p_k$.

start contradiction: Now assume that, there exists an echelon matrix, which is non-zero, but its non-zero row vectors are linearly dependent. This implies that, there is a sequence of coefficients, $\alpha_1, \ldots$, which are not all zero, such that:

~~Draw~~ Draw the pattern



$\boxed{x}$ non-zero
0 : zero
x : can be anything.

$$\alpha_1 \underline{e}_1 + \cdots + \alpha_k \underline{e}_k = \underline{0}$$

But this implies that:

$$\alpha_1 e_{1p_1} + \alpha_2 e_{2p_1} + \cdots + \alpha_k e_{kp_1} = 0$$

Since $e_{2p_1} = e_{3p_1} = \cdots = e_{kp_1} = 0 \implies \alpha_1 = 0$.

Then, $\alpha_2 e_{2p_2} + \cdots + \alpha_k e_{kp_2} = 0 \implies \alpha_2 = 0$

Continuing like this, we see that $\alpha_i = 0$ for $1 \le i \le k$. Thus, we reached to a contradiction. Therefore, the assumption of having an echelon matrix whose non-zero rows are linearly dependent is wrong. ∎

Therefore, row space and row rank of a matrix can be found by reducing the matrix into an echelon form (by using elementary row operations) and identifying the non-zero row vectors of the obtained echelon matrix.

Now, focus on the Column space and column rank.

Theorem The column rank of ($\underline{\underline{A}}$ and its row echelon form $\underline{\underline{E}}$) is the same.

Proof

Consider solution to $\underline{\underline{A}}\,\underline{x} = \underline{0}$. By applying elementary row operations we obtain $\underline{\underline{E}}\,\underline{x} = \underline{0}$, such that solution spaces remains the same.

Therefore, $\begin{bmatrix} \underline{a}_1 \cdots \underline{a}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \underline{0} \iff \begin{bmatrix} \underline{e}_1 \cdots \underline{e}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \underline{0}$

same way as the ☆ proof of the Theorem

Now, observe that the leading or pivot columns span the column space of $\underline{\underline{E}}$ and they are linearly independent (Show this!). Therefore, if we denote those pivot columns as $\{\underline{e}_{p_1}, \ldots, \underline{e}_{p_r}\}$, the rest of the columns of $\underline{\underline{E}}$ are linearly dependent to these r pivot columns. i.e.,

$$\underline{e}_k = \sum_{i=1}^{r} c_{ki}\, \underline{e}_{p_i} \implies \underline{e}_k - \sum_{i=1}^{r} c_{ki}\, \underline{e}_{p_i} = \underline{0}, \quad 1 \le i \le r, \quad r < k \le n$$

Therefore, $\underline{\underline{E}} \cdot \left(\begin{array}{c} \\ k^{th}\text{ entry} \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underline{c}_k \right) = \underline{0}$. see (41b) an example.

Thus, also, $\underline{\underline{A}} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underline{c}_k \right) = \underline{0}$.

Therefore if $\underline{e}_k$ is a linear combination of columns of $\{\underline{e}_{p_1}, \ldots, \underline{e}_{p_r}\}$, $\underline{a}_k$ is a linear combination of columns of $\{\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}\}$.

Therefore $\text{span}\{\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}\} = \text{span}\{\underline{a}_1, \ldots, \underline{a}_n\}$

Furthermore, $\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}$ are linearly independent. This can be seen easily by assuming that they are linearly dependent, i.e., there is a non-zero set of coefficients that solves:

LI is proved by contradiction.

$$\begin{bmatrix} \underline{a}_{p_1} & \underline{a}_{p_2} & \cdots & \underline{a}_{p_r} \end{bmatrix} \cdot \begin{bmatrix} \beta_{p_1} \\ \vdots \\ \beta_{p_r} \end{bmatrix} = \underline{0}$$

β position n p_1: stands for position n

This means that $\underline{\underline{A}} \cdot \underline{z} = \underline{0}$ where $\begin{cases} z_i = 0 & \text{if } i \notin \{p_1, \ldots\} \\ z_i = \beta_i & \text{if } i \in \{p_1, \ldots\} \end{cases}$

Thus $\underline{\underline{E}} \cdot \underline{z} = \underline{0}$ (since $\underline{\underline{A}}\underline{x} = \underline{0}$ and $\underline{\underline{E}}\underline{x} = \underline{0}$ shares the same solution set)

$$\implies \begin{bmatrix} \underline{e}_{p_1} \cdots \underline{e}_{p_r} \end{bmatrix} \cdot \begin{bmatrix} \beta_{p_1} \\ \vdots \\ \beta_{p_r} \end{bmatrix} = \underline{0} \implies \{\underline{e}_{p_1}, \ldots, \underline{e}_{p_r}\} \text{ is LD}$$

But this is a contradiction since pivot columns are linearly independent. Thus $\{\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}\}$ is LI and $\text{span}\{\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}\} = \text{span}\{\underline{a}_1, \ldots, \underline{a}_n\} \implies \{\underline{a}_{p_1}, \ldots, \underline{a}_{p_r}\}$ is a basis for $\text{Col}(\underline{\underline{A}})$.

Hence, $\dim(\text{Col}(\underline{\underline{A}})) = r = \dim(\text{col}(\underline{\underline{E}}))$. ■

Theorem Column rank of a matrix $\underline{\underline{A}}$ is equal to its row rank.

Proof Have shown that

(i) $\text{Row}(\underline{\underline{A}}) = \text{row}(\underline{\underline{E}})$ where $\underline{\underline{E}}$ is an echelon form of $\underline{\underline{A}}$
row rank of $\underline{\underline{A}}$ is equal to number of pivots in $\underline{\underline{E}}$.

(ii) Column rank of $\underline{\underline{A}}$ is equal to the column rank of $\underline{\underline{E}}$ which is equal to number of pivots in $\underline{\underline{E}}$.

Thus, the column rank of $\underline{\underline{A}}$ is equal to its row rank.

Theorem 10.5.2 For any matrix $\underline{\underline{A}}$, the number of LI row vectors is equal to the number of LI column vectors.

Proof Number of LI row vectors is equal to the row rank. Number of LI column vectors is equal to the column rank. Since row rank is equal to the column rank, the theorem is true. ■

ex/ How many LI vectors are in $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix} \right\}$?

Form $\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \xrightarrow{R_1/2 \to R_1} \begin{bmatrix} 1 & 1/2 & -3/2 & 2 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \xrightarrow[R_4 - 2R_1 \to R_4]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 1/2 & -3/2 & \\ 0 & 3 & 1 & \\ 0 & 3 & 1 & \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{3}{2} & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 3-pivots $\Rightarrow$ 3 of the vectors are LI.

ex/ Consider the system $\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 2 \\ 1 & 0 & 3 & 3 & -1 & 6 \\ 2 & -1 & 2 & 1 & -1 & 7 \\ 1 & 0 & 5 & 8 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 9 \\ 1 \end{bmatrix}$

Reduce the augmented matrix to its echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 2 & 4 \\ 1 & 0 & 3 & 3 & -1 & 6 & 3 \\ 2 & -1 & 2 & 1 & -1 & 7 & 9 \\ 1 & 0 & 5 & 8 & -1 & 7 & 1 \end{bmatrix} \xrightarrow[\text{Row Operations}]{\text{Elementary}} \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 2 & 5 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \underbrace{x_1 \quad x_2 \quad x_3}_{\text{pivot variables}} \quad \underbrace{x_4 \quad x_5 \quad x_6}_{\text{free variables}} \end{array}$$

Observations (i) Rank of $\underline{\underline{A}}$ and Rank of $[\underline{\underline{A}} \vdots \underline{c}]$ are the same, hence, the system is consistent.

(ii) There are 3 free variables, therefore the solution family can be parametrized with 3 parameters.

Let $x_6 = \alpha$, $x_5 = \beta$, $x_4 = \gamma$, then can obtain the solution set by backsubstitution as:

$$x_3 = 1 - \frac{1}{2}\alpha - \frac{5}{2}\gamma$$
$$x_2 = 1 - 3\alpha + \beta + 5\gamma$$
$$x_1 = 6 - \frac{9}{2}\alpha + \beta + \frac{9}{2}\gamma$$

, which can be expressed as:

$$\underline{x} = \underbrace{\begin{bmatrix} 6 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{x}_0} + \alpha \underbrace{\begin{bmatrix} -9/2 \\ -3 \\ -1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\underline{x}_1} + \beta \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{x}_2} + \gamma \underbrace{\begin{bmatrix} \frac{9}{2} \\ 5 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{x}_3}$$

Observe that: (i) $\underline{x}_0$ is a particular solution $(\alpha = \beta = \gamma = 0)$.

(ii) $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ are homogeneous solutions since

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(\underline{x}_0 + \alpha \underline{x}_1 + \beta \underline{x}_2 + \gamma \underline{x}_3) = \underbrace{\underline{\underline{A}}\,\underline{x}_0}_{\underline{c}} + \alpha \underline{\underline{A}}\,\underline{x}_1 + \beta \underline{\underline{A}}\,\underline{x}_2 + \gamma \underline{\underline{A}}\,\underline{x}_3 =$$

$$\Rightarrow \forall \alpha, \beta, \gamma: \; \alpha \underline{\underline{A}}\,\underline{x}_1 + \beta \underline{\underline{A}}\,\underline{x}_2 + \gamma \underline{\underline{A}}\,\underline{x}_3 = \underline{0} \Rightarrow \underline{\underline{A}}\,\underline{x}_i = \underline{0} \text{ for } 1 \le i$$

(iii) $$[\underline{x}_1 \; \underline{x}_2 \; \underline{x}_3] = \begin{bmatrix} -9/2 & 1 & 9/2 \\ -3 & 1 & 5 \\ -1/2 & 0 & -5/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ has rank 3 $\Rightarrow \{\underline{x}_1, \underline{x}_2,$ are LI.

Generalization of this example is very important:

Consider $\underline{\underline{A}}\,\underline{x} = \underline{c}$; where $\underline{\underline{A}} = \{a_{ij}\}_{m \times n} \in \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$)

(i) If $r(\underline{\underline{A}}) < r([\underline{\underline{A}} \vdots \underline{c}])$ then there are no solutions. $\underline{\underline{A}}\underline{x} =$

(ii) If $r(\underline{\underline{A}}) = r([\underline{\underline{A}} \vdots \underline{c}])$ then there is at least one solution (cons

then If $n = r(\underline{\underline{A}})$, then the solution is unique.

then If $n > r(\underline{\underline{A}})$, then the solution is non-unique, it is a $(n - r$ dimensional parametric family.

(vii) If the system is consistent and $n > r(\underline{\underline{A}})$, can express any solution as:

$$\underline{x} = \underline{x}_p + \underline{x}_n$$

where $\underline{x}_p$ is a particular solution to $\underline{\underline{A}}\,\underline{x} = \underline{c}$ and $\underline{x}_n$ solves the homogeneous system: $\underline{\underline{A}}\,\underline{x}_n = \underline{0}$.

Def'n The solution set of $\underline{\underline{A}}\,\underline{x} = \underline{0}$ is called as the "null space of $\underline{\underline{A}}$".

Theorem $r(\underline{\underline{A}}) + \dim(\mathrm{Null}(\underline{\underline{A}})) = n$.

Proof Consider $\underline{\underline{A}}\,\underline{x} = \underline{0}$. Reduce $\underline{\underline{A}}$ to $\underline{\underline{E}}$, an echelon matrix, to identify the pivots and free variables. The $r(\underline{\underline{A}})$ is equal to the number of pivot variables. The dim Null$(\underline{\underline{A}})$ is equal to the number of free variables. Since,

$$\underbrace{\#\text{pivots}}_{r(\underline{\underline{A}})} + \underbrace{\#\text{free variables}}_{\dim(\mathrm{Null}(\underline{\underline{A}}))} = n \quad \blacksquare$$

Inverse Matrix; Cramer's Rule, Factorization

In solving scalar equations $ax = c$, we divide both sides by a to get $x = \frac{c}{a}$ if $a \neq 0$. This is equivalent to multiply both sides with a^{-1}:

$$\underbrace{a^{-1}a}_{1}\,x = a^{-1}c \Rightarrow x = a^{-1}c.$$

Want to generalize this idea to $\underline{\underline{A}}\,\underline{x} = \underline{c}$, in otherwords we would like to multiply both sides with a proper matrix $\underline{\underline{A}}^{-1}$ so that:

$$\underbrace{\underline{\underline{A}}^{-1}\underline{\underline{A}}}_{\text{Want this to be } \underline{\underline{I}}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$$

However, we have to careful. For example:

$$\begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{Since } \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Can think of multiplying both sides of given system by $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ to get:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

and propose that $x_1 = 3$, $x_2 = 4$ is a solution. But it is not! This is because of the fact that, the obtained system is not an equivalent one to the original.

Theorem In a linear system of equations $\underline{\underline{A}}\,\underline{x} = \underline{c}$, if $\underline{\underline{A}}$ is not a full rank square matrix, there exists no $\underline{\underline{A}}^{-1}$ such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ and $\underline{\underline{A}}\,\underline{x} = \underline{c}$ and $(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$ shares the same solution set for any $\underline{c}$.

Proof Assume that for $\underline{\underline{A}}$ which is not full rank square matrix, there exists $\underline{\underline{A}}^{-1}$ such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ and $(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$ and $\underline{\underline{A}}\,\underline{x} = \underline{c}$ shares the same solution set. We will show that this assumption leads to a contradiction, and hence, it is false.

Cases of interest (i) $\underline{\underline{A}}\,\underline{x} = \underline{c}$ is not consistent, thus the solution set is empty set. However $(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{x} = \underline{\underline{A}}^{-1}\underline{c} = \underline{x}$ has a unique solution $\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$. Therefore $\underline{\underline{A}}\,\underline{x} = \underline{c}$ must be consistent for all $\underline{c}$. This implies that $\underline{\underline{A}}$ must have full row rank, i.e. $r(\underline{\underline{A}}_{m\times n}) = m$.

$\underline{\underline{A}}_{m\times n} = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}_{m\times n}$

m: rows

n: columns

(ii) $\underline{\underline{A}}\,\underline{x} = \underline{c}$ has more than 1 solution for a certain $\underline{c}$. Then $(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{x} = \underline{\underline{A}}^{-1}\underline{c} = \underline{x}$ should have non-unique solutions as well. But it has the unique solution $\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$. Therefore $\underline{\underline{A}}\,\underline{x} = \underline{c}$ cannot have non-unique solutions for any $\underline{c}$. This implies that $\underline{\underline{A}}$ must have full column rank, i.e., $r(\underline{\underline{A}}_{m\times n}) = n$.

Combining (i) & (ii) we get $r(\underline{\underline{A}}_{m\times n}) = m = n \Rightarrow \underline{\underline{A}}$ is square and full rank. Hence, we reached to a contradiction. ■

Theorem If $\underline{\underline{A}}$ is a square matrix with full column rank, there exists an $\underline{\underline{A}}^{-1}$ such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$.

gives

Proof If $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}} \Rightarrow \underline{\underline{A}}^T(\underline{\underline{A}}^{-1})^T = \underline{\underline{I}}$.

In the following proof, we will show that the transposed form has a unique solution for $\underline{\underline{A}}$, full rank square matrix. Let $(\underline{\underline{A}}^{-1})^T = \underline{\underline{B}}$. Then, if we can find $\underline{\underline{B}}$, $\underline{\underline{B}}^T = \underline{\underline{A}}^{-1}$.

$\underline{\underline{A}}^T \underline{\underline{B}} = \underline{\underline{I}}$ can be re expressed as:

$$\underline{\underline{A}}^T \left[\underline{b}_1 \ \underline{b}_2 \cdots \underline{b}_n \right] = \left[\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_n \right], \text{ where } \underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ elem}$$

Thus, we can view the given matrix eqn as n linear system of equations:

$$\underline{\underline{A}}^T \underline{b}_i = \underline{e}_i \quad 1 \le i \le n.$$

Since $\underline{\underline{A}}$ is square and full rank so is $\underline{\underline{A}}^T$. Therefore, $\underline{\underline{A}}^T\underline{b}_i = \underline{e}_i$ is always consistent and it has unique solution. Thus, there exists a unique $\underline{\underline{B}}$, satisfying $\underline{\underline{A}}^T\underline{\underline{B}} = \underline{\underline{I}}$. Hence $\underline{\underline{A}}^{-1} = \underline{\underline{B}}^T$ do exists and it is unique.

Def'n If $\underline{\underline{A}}$ is a full rank square matrix, then it is called as invertible and its unique inverse is called as $\underline{\underline{A}}^{-1}$.

Theorem If $\underline{\underline{A}}$ is invertible, $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$.

Proof

Have shown that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$. Need to show that $\underline{\underline{A}}\,\underline{\underline{A}}^{-1}$ is also.

One way of showing this as follows:

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}} \Rightarrow \underline{\underline{A}}\,\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}} \Rightarrow (\underline{\underline{A}}\,\underline{\underline{A}}^{-1})\underline{\underline{A}} = \underline{\underline{A}}$$

$(AA^{-1})Ax = Ax$ for any x.

Since A is full rank, its range space is $\mathbb{R}^n$. Thus,

$(AA^{-1})z = z$ for any z in $\mathbb{R}^n$. Therefore $(AA^{-1} - I)z = 0$ for any z. This means that all the rows of $AA^{-1} - I$ is zero. Thus, $AA^{-1} = I$. ■

Def'n $A^{-n} \equiv (A^{-1})^n = \underbrace{(A^{-1}) \cdots (A^{-1})}_{n\ \text{times}}$

Theorem Algebra of inverse matrices:

(a) $(A^{-1})^{-1} = A$

(b) $(A^n)^{-1} = (A^{-1})^n$

(c) $(AB)^{-1} = B^{-1}A^{-1}$, if both A and B are invertible.

(d) A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T \Longrightarrow$ Proof $\underbrace{A^T(A^{-1})^T}_{(A^{-1}A)^T = I^T = I}$ and $\underbrace{(A^{-1})^T A^T}_{(AA^{-1})^T = I^T = I}$

Proof Left as an exercise.

Theorem If A is invertible, the unique solution to $Ax = c$ can be obtained by $x = A^{-1}c$.

Proof Note that $A \cdot (A^{-1}c) = A \cdot A^{-1}c = c$, thus $A^{-1}c$ is a solution to the system. Also, the solution is unique because A has full column rank (since it is invertible). ■

↳ # of pivot elements $= n$ $(m=n)$

~~Determinants and Invertibility~~

→ The proof of each item consists of verifying that the appropriate matrix products yield the identity matrix.

(a) $A^{-1}A = I$ and $AA^{-1} = I$

(b) follows from (c) and use of definition

$A^n = \underbrace{A\,A \cdots A}_{n\text{-times}} \Rightarrow (A^n)^{-1} = [A\,A \cdots A]^{-1} = \underbrace{(A^{-1})(A^{-1}) \cdots A^{-1}}_{n\text{-times}} \overset{\text{definition}}{\equiv} A^{-n} = (A^{-1})^n$

(c) $\underbrace{AB(B^{-1}A^{-1})}_{A(BB^{-1})A^{-1}} = I$ and $\underbrace{(B^{-1}A^{-1})(AB)}_{B^{-1}\underbrace{A^{-1}A}_{I}B} = I$

$A(BB^{-1})A^{-1}$ $\qquad$ $B^{-1}\underbrace{A^{-1}A}_{I}B \Longrightarrow B^{-1}IB = B^{-1}B = I$

$A\,I\,A^{-1} = AA^{-1} = I$

Determinants and Invertibility

$\underline{\underline{A}}$ is invertible $\Longleftrightarrow$ $\det \underline{\underline{A}} \neq 0$.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \det \underline{\underline{A}} = 4 - 6 = -2 \neq 0 \Rightarrow \underline{\underline{A}}$ is invertible

Proof

$\underline{\underline{A}}$ is invertible $\Rightarrow$ $\underline{\underline{A}}$ is full rank.

Since $\det \underline{\underline{A}} = 0 \Rightarrow \det \underline{\underline{E}} = 0$ where $\underline{\underline{E}}$ is obtained from $\underline{\underline{A}}$ by Gauss elimination.

Hence (i) if $\underline{\underline{A}}$ is invertible $\Rightarrow$ $\underline{\underline{A}}$ is full rank

$\Rightarrow$ $\underline{\underline{E}}$ is full rank, i.e. # pivots $= \underbrace{n}_{\text{dimension of } \underline{\underline{A}}}$

$\Rightarrow \det(\underline{\underline{E}}) \neq 0$; since $\det(\underline{\underline{E}})$ is the multiplication of the pivots.

$\Rightarrow \det(\underline{\underline{A}}) \neq 0$.

(ii) if $\det \underline{\underline{A}} \neq 0 \Rightarrow \det(\underline{\underline{E}}) \neq 0$.

$\Rightarrow$ $\underline{\underline{E}}$ is full rank

$\Rightarrow$ $\underline{\underline{A}}$ is full rank

$\Rightarrow$ $\underline{\underline{A}}$ is invertible.

Theorem Cramer's Rule

Consider $n \times n$ linear system: $\underline{\underline{A}}\underline{x} = \underline{b}$, $\underline{\underline{A}} = [\underline{a}_1 \cdots \underline{a}_n]$.
If $\det \underline{\underline{A}} \neq 0$, then the system is uniquely solvable and the i^{th} entry of the solution is:

$$x_i = \frac{\det[\underline{a}_1 \cdots \underline{a}_{i-1} \; \underline{b} \; \underline{a}_{i+1} \cdots \underline{a}_n]}{\det \underline{\underline{A}}}.$$

Proof Since $\det \underline{\underline{A}} \neq 0 \Rightarrow \underline{\underline{A}}$ is invertible. Thus the system has a unique solution $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$. The Cramer's rule also provides us a closed form solution for $\underline{x}$ that is useful in analytical investigations.

$$\underline{\underline{A}}\,\underline{x} = [\underline{a}_1 \cdots \underline{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\underline{a}_1 x_1 + \cdots + \underline{a}_n x_n}_{\text{must be } \underline{b} \text{ when } \underline{x} \text{ is the unique solution}} = \sum_{k=1}^{n} \underline{a}_k x_k$$

$$\det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n] = \det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \sum_{k=1}^{n} \underline{a}_k x_k\ \underline{a}_{i+1} \cdots \underline{a}_n]$$

$$\xrightarrow{\text{succesive application of determinant property D7 and D9}} = \sum_{k=1}^{n} x_k \underbrace{\det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{a}_k\ \underline{a}_{i+1} \cdots \underline{a}_n]}_{= \begin{cases} 0 & \text{if } k \neq i \\ \det \underline{\underline{A}} & \text{if } k = i \end{cases}}$$

$$\Rightarrow \det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n] = x_i \det \underline{\underline{A}}$$

$$\Rightarrow x_i = \frac{\det[\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1} \cdots \underline{a}_n]}{\det \underline{\underline{A}}} \qquad \blacksquare$$

ex/ By using the Cramer's rule, solve: $x_1 + 4x_2 + 5x_3 = 2$

$4x_1 + 2x_2 + 5x_3 = 3$

$-3x_1 + 3x_2 - x_3 = 1.$

$$\det \underline{\underline{A}} = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{vmatrix} = \det \begin{vmatrix} 1 & 4 & 5 \\ 0 & -14 & -15 \\ 0 & 15 & 14 \end{vmatrix} = -14 \cdot 14 + 15 \cdot 15 = 29$$

$$x_1 = \frac{\det \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix}}{29} = \frac{33}{29} \ ; \quad x_2 = \frac{\det \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix}}{29} = \frac{35}{29}$$

$$x_3 = \frac{\det \begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix}}{29} = -\frac{23}{29}$$

Inverses and Adjoint Matrix

Can find inverse of $\underline{\underline{A}}$ (if it exists, i.e., $\underline{\underline{A}}$ is square full rank) by solving:

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}} \quad \text{or} \quad [\underline{a}_1 \cdots \underline{a}_n][\underline{x}_1 \cdots \underline{x}_n] = [\underline{e}_1 \cdots \underline{e}_n]$$

Hence we have to solve n linear systems for $1 \leq j \leq n$:

$$\underline{\underline{A}}\,\underline{x}_j = \underline{e}_j \quad , \ 1 \leq j \leq n.$$

By using the Cramer's Rule we get:

$$x_{ij} = \frac{|\,\underline{a}_1 \cdots \underline{a}_{i-1}\ \underline{e}_j\ \underline{a}_{i+1} \cdots \underline{a}_n\,|}{\det \underline{\underline{A}}} = \frac{A_{ji}}{\det \underline{\underline{A}}}$$

A_{ji}: ji^{th} cofactor of $\underline{\underline{A}}$

$$\Rightarrow \underline{\underline{X}} = \underline{\underline{A}}^{-1} = \frac{[\underline{\underline{A}}_{ij}]^T}{\det \underline{\underline{A}}} = \frac{\operatorname{Adj} \underline{\underline{A}}}{\det \underline{\underline{A}}} \qquad \blacksquare$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{bmatrix}$, $\det \underline{\underline{A}} = 29$

$$\underline{\underline{A}}\,[\underline{x}_1\ \underline{x}_2\ \underline{x}_3] = [\underline{e}_1\ \underline{e}_2\ \underline{e}_3]$$

$$\underline{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

$$x_{11} = \frac{\det \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix}}{\det \underline{\underline{A}}} = \frac{A_{11}}{\det \underline{\underline{A}}}$$

$$x_{21} = \frac{\det \begin{bmatrix} 1 & 1 & 5 \\ 4 & 0 & 5 \\ -3 & 0 & -1 \end{bmatrix}}{\det \underline{\underline{A}}} = \frac{A_{12}}{\det \underline{\underline{A}}}$$

$$x_{31} = \frac{\det \begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 0 \\ -3 & 3 & 0 \end{bmatrix}}{\det \underline{\underline{A}}} = \frac{A_{13}}{\det \underline{\underline{A}}}$$

$$\Rightarrow \underline{\underline{X}} = \underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{\det \underline{\underline{A}}} \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T}_{\operatorname{adj} \underline{\underline{A}}}$$

A more practical technique for manual computation of !

$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$ is equivalent to n simultaneous linear system of equations for $\underline{\underline{A}}_{n\times n}$

unknown columns of $\underline{\underline{A}}^{-1}$

Let $\underline{\underline{A}}^{-1} = [\underline{x}_1 \cdots \underline{x}_n]$.

standard basis vectors

Then $\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}[\underline{x}_1 \cdots \underline{x}_n] = \underline{\underline{I}} = [\underline{e}_1 \cdots \underline{e}_n]$

$\Rightarrow \quad \underline{\underline{A}}\,\underline{x}_i = \underline{e}_i \quad \Rightarrow$ $\underline{x}_i$ can be solved individually.
But they can be solved simultaneously as well!

Form the augmented system:

$$\left[\underline{\underline{A}} \; [\underline{e}_1 \cdots \underline{e}_n]\right] = [\underline{\underline{A}} \; \underline{\underline{I}}]$$

Then, by using elementary row operations, reduce it to

$$\left[\underline{\underline{I}} \; [\underline{x}_1 \cdots \underline{x}_n]\right] = [\underline{\underline{I}} \; \underline{\underline{A}}^{-1}].$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\quad [\underline{\underline{A}} \; \underline{\underline{I}}] = \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 9 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2/9} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/9 & 2/9 & 1/9 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/9 & 2/9 & 1/9 & 0 \\ 0 & 0 & 8/9 & -2/9 & -1/9 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 \, 9/8} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/9 & 2/9 & 1/9 & 0 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \frac{1}{9}R_3} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -3/8 & 3/8 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1/4 & -3/8 & 3/8 \\ 1/4 & 1/8 & -1/8 \\ -1/4 & -1/8 & 9/8 \end{matrix}}_{\underline{\underline{A}}^{-1}}$$

LU-Factorization

LU-Factorization is another way for solving $\underline{\underline{A}}\,\underline{x} = \underline{c}$ for a square matrix $\underline{\underline{A}}$.

Any square matrix $\underline{\underline{A}}$ can be factorized as:

$$\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$$

upper triangular.

lower triangular

Note that the factorization is not unique!

ex/ For $\underset{=}{A}_{3\times 3}$ it has 9 entries a_{ij}, $1\le i\le 3, 1\le j\le 3$

$$\underset{=}{A} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{6 \text{ parameters}} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{6 \text{ parameters}} \quad (\text{for } n\times n:\ n^2+n)$$

Total 12 unknowns.

In Doolittle's Method set $l_{11} = l_{22} = l_{33} = 1$, then solve for remaining 9 parameters. For $n\times n$ case $l_{11} = \cdots = l_{nn} = 1$ and solve for remaining n^2 parameters.

ex/

$$\underbrace{\begin{bmatrix} 2 & -3 & 3 \\ 6 & -8 & 7 \\ -2 & 6 & -1 \end{bmatrix}}_{\underset{=}{A}} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$2 = u_{11}$, $-3 = u_{12}$, $3 = u_{13}$

$6 = l_{21}u_{11} \Rightarrow l_{21} = 6/2 = 3$; $-8 = l_{21}u_{12} + u_{22}$

$\Rightarrow u_{22} = -8 - 3\cdot(-3) = 1$

$7 = l_{21}u_{13} + u_{23} \Rightarrow u_{23} = 7 - 3\cdot 3 = -2$

$-2 = l_{31}u_{11} \Rightarrow l_{31} = -1$; $6 = l_{31}u_{12} + l_{32}u_{22}$

$\Rightarrow l_{32} = 3$

$-1 = l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = -1 - (-1)\cdot 3 - 3\cdot(-2) = 8$

$$\Rightarrow \underset{=}{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 8 \end{bmatrix}$$

Once $\underset{=}{A} = \underset{=}{L}\,\underset{=}{U}$ is obtained then $\underset{=}{A}\underline{x} = \underline{c}$ can be solved as:

$\underset{=}{L}\underbrace{\underset{=}{U}\underline{x}}_{\underline{y}} = \underline{c} \Rightarrow \underset{=}{L}\underline{y} = \underline{c}$ solve for $\underline{y}$ by forward substitut

then $\underset{=}{U}\underline{x} = \underline{y}$ solve for $\underline{x}$ by backward substituti

Note that this technique is faster than Gauss-Jordan by a factor of

The Eigenvalue Problem

(Chapter 11 in the textbook, Lecture notes)

For a square matrix $\underline{\underline{A}}$, we want to identify vectors $\underline{x}$ such that:

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \text{for a scalar } \lambda.$$

Note that $\underline{\underline{A}}\,\underline{v}$ can be viewed as a transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$. Therefore $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$ can be viewed as the identification of those $\underline{x}$'s whose transform is a scaled version of itself.

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \implies \underline{\underline{A}}\,\underline{x} - \lambda \underline{x} = \underline{0} \implies \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{x} = \underline{0}$$

Since $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$ is a homogeneous equation in terms of $\underline{x}$ for a given λ, it is always consistent and $\underline{x} = \underline{0}$ is always a solution. But in the eigenvalue problem we are interested in non-zero solutions to $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$.

Non-zero solutions exists iff $\underbrace{\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})}_{\text{a polynomial in } \lambda!} = 0$

Thus, non-trivial solutions to $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$ exists only for a set of λ that solves $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

Def'n The pairs $(\lambda, \underline{x})$, $\underline{x} \neq \underline{0}$ that solves $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$ are eigenpairs, λ is an eigenvalue and $\underline{x}$ is a corresponding eigenvector.

Def'n $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ is called as the characteristic polynomial of matrix $\underline{\underline{A}}$.

Hence, roots of $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ are the eigenvalues of matrix $\underline{\underline{A}}$.

ex/ Want to determine the eigenvalues, λ and their corresponding eigenvectors or eigenspaces for:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

$$= (\lambda-5)(\lambda-1)^2$$

So, $\lambda_1 = 5$, $\lambda_2 = 1$ is a repeated eigenvalue with multiplicity 2.
Now, want to find eigenspaces corresponding to these eigenvalues:

For $\lambda_1 = 5$: $(\underline{\underline{A}} - 5\underline{\underline{I}})\underline{x} = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \underline{x} = \underline{0}$

Gauss elimination results in:

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$ x_1 & x_2 are pivots, x_3 is a free variable.

Thus: $x_3 = \alpha \Rightarrow x_2 = \alpha \Rightarrow x_1 = \alpha \Rightarrow \underline{x} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence, the eigenspace of $\lambda_1 = 5$ is span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 1$: $(\underline{\underline{A}} - \underline{\underline{I}})\underline{x} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \underline{x} = \underline{0}$

Gauss elimination yields:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \implies$$ x_1 is a pivot and x_2 & x_3 are free variables.

$x_3 = \alpha$, $x_2 = \beta \Rightarrow x_1 = -2\beta - \alpha$. $\underline{x} = \begin{bmatrix} -2\beta-\alpha \\ \beta \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Hence, the eigenspace associated with $\lambda_2 = 1$ is span$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Notes (i) If $\underline{x}$ is an eigenvector $c\underline{x}$, $c \neq 0$ is also an eigenvector.
(ii) If λ is an eigenvalue with multiplicity k, its eigenspace may be lower than k-dimensional.

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$

$\Rightarrow \lambda_1 = 1$ is the only eigenvalue with multiplicity 3.
Its eigenspace is the set of solutions to:

$(\underline{\underline{A}} - \underline{\underline{I}})\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ x_1 & x_3 are pivots and x_2 is free varia

$\Rightarrow x_2 = \alpha \Rightarrow x_1 = 0$ and $x_3 = 0 \Rightarrow \underline{x} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Hence the eigenspace of $\lambda_1 = 1$ is Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ 1-dimensiona

A special Case: Symmetric Matrices

Theorem 11.3.1 If $\underline{\underline{A}}$ is symmetric, all its eigenvalues are real.

Proof Let $(\lambda, \underline{x})$ be an eigenpair of $\underline{\underline{A}}$. Then:

$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \Rightarrow \underline{x}^H(\underline{\underline{A}}\,\underline{x}) = \lambda \underline{x}^H \underline{x}$ ($\underline{x}^H \equiv$ complex conjugate transpose of)

$= (\underline{\underline{A}}\,\underline{x})^H \underline{x} = (\lambda \underline{x})^H \underline{x}$

$= \lambda^* \underline{x}^H \underline{x}$

$\underline{\underline{A}}$ is symmetric and real.

$\Rightarrow \lambda \underline{x}^H \underline{x} = \lambda^* \underline{x}^H \underline{x} \Rightarrow (\lambda - \lambda^*)\underline{x}^H \underline{x} = 0$. $\underline{x}^H \underline{x} > 0$

$\Rightarrow \lambda = \lambda^*$ ∎

Theorem 11.3.2 If λ is an eigenvalue of real symmetric $\underline{\underline{A}}$ with multiplicity k, then its corresponding eigenspace is k-dimension

Proof
Will provide after Theorem 11.4.4.

Theorem 11.3.3 If $\underline{\underline{A}}$ is symmetrix, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof Let $(\lambda_1, \underline{x}_1)$ and $(\lambda_2, \underline{x}_2)$ be two eigenpairs of $\underline{\underline{A}}$ with $\lambda_1 \neq \lambda_2$. Then:

$$\underline{\underline{A}}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \Rightarrow \underline{x}_2^H (\underline{\underline{A}}\,\underline{x}_1) = \lambda_1 \underline{x}_2^H \underline{x}_1$$

λ_2 is real.

$\underline{\underline{A}}$ is real symmetric

$$= (\underline{\underline{A}}\,\underline{x}_2)^H \underline{x}_1 = \lambda_2^* \underline{x}_2^H \underline{x}_1 = \lambda_2 \underline{x}_2^H \underline{x}_1$$

$$\Rightarrow \lambda_1 \underline{x}_2^H \underline{x}_1 = \lambda_2 \underline{x}_2^H \underline{x}_1 \Rightarrow (\lambda_1 - \lambda_2)\underline{x}_2^H \underline{x}_1 = 0$$

$$\lambda_1 - \lambda_2 \neq 0 \Rightarrow \underline{x}_2^H \underline{x}_1 = 0 \quad \blacksquare$$

Theorem 11.3.4 If $\underline{\underline{A}}_{n\times n}$ is symmetric, then its eigenvectors provide an orthogonal basis for $\mathbb{R}^n$.

Proof (i) If $\underline{\underline{A}}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ then their corresponding eigenvectors form the basis since they are orthogonal (Thm. 11.3.3)

(ii) If $\underline{\underline{A}}$ has repeated eigenvalues $\lambda_1, \ldots, \lambda_m$ where λ_i is an eigenvalue with multiplicity k_i then:

- Each eigenspace is k_i dimensional (Thm. 11.3. Therefore there exists an orthogonal basis for each eigenspace such as: $\{\underline{u}_{i1}, \ldots, \underline{u}_{ik_i}\}$
- Since eigenspaces corresponding to distinct eigenval are orthogonal (Thm. 11.3.3) the following set is an orthogonal basis for $\mathbb{R}^n$:

$$\bigcup_{i=1}^{m} \{\underline{u}_{i1}, \ldots, \underline{u}_{ik_i}\}$$

Note that $\sum_{i=1}^{m} k_i = n$.

Diagonalization

Given a square matrix $\underline{\underline{A}}$, if we can find a $\underline{\underline{Q}}$ such that:

$$\underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{D}} \leftarrow \text{diagonal}$$

then we say $\underline{\underline{A}}$ is diagonalizable, and $\underline{\underline{Q}}$ diagonalizes $\underline{\underline{A}}$.

Theorem 11.4.1 Let $\underline{\underline{A}}$ be $n\times n$ matrix.

(i) $\underline{\underline{A}}$ is diagonalizable $\Longleftrightarrow$ $\underline{\underline{A}}$ has n LI eigenvectors.

(ii) If $\underline{\underline{A}}$ has n LI eigenvectors $\underline{e}_1, \ldots, \underline{e}_n$, $\underline{\underline{Q}} = [\underline{e}_1 \cdots \underline{e}_n]$ then $\underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{D}}$ is diagonal, $d_i = \lambda_i$.

Proof

(i) We will show first $(\Longrightarrow)$ and then $(\Longleftarrow)$ parts.

$(\Longrightarrow)$: $\underline{\underline{A}}$ is diagonalizable $\Rightarrow$ there exists a $\underline{\underline{Q}} = [\underline{e}_1 \cdots$ such that:

$$\underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{D}} \Longrightarrow \underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{D}} = [\underline{e}_1 \cdots \underline{e}_n]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$\Longrightarrow \underline{\underline{A}}\,[\underline{e}_1 \cdots \underline{e}_n] = [d_1\underline{e}_1 \; d_2\underline{e}_2 \cdots d_n\underline{e}_n]$$

$\Longrightarrow \underline{\underline{A}}\,\underline{e}_i = d_i\,\underline{e}_i$, since $\underline{e}_i \neq \underline{0}$ (otherwise $\underline{\underline{Q}}^{-1}$ does not exist) $\underline{e}_i$ is an eigenvector associated with eigenvalue d_i. Since $\underline{\underline{Q}}^{-1}$ exists $\underline{e}_i$ must be LI. ∎

$(\Longleftarrow)$ $\underline{\underline{A}}$ has n linearly independent eigenvectors $\{\underline{e}_1, \ldots, \underline{e}_n\}$. Then:

$$\underline{\underline{A}}\,\underline{e}_i = \lambda_i\,\underline{e}_i \quad \text{or} \quad \underline{\underline{A}}\,[\underline{e}_1 \cdots \underline{e}_n] = \underbrace{[\underline{e}_1 \cdots \underline{e}_n]}_{\underline{\underline{Q}}}\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\underline{\underline{\Lambda}}}$$

Since columns of $\underline{\underline{Q}}$ are LI, $\underline{\underline{Q}}$ is invertible

$$\Longrightarrow \underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{\Lambda}} \leftarrow \text{diagonal}$$

$\Longrightarrow$ $\underline{\underline{A}}$ is diagonalizable ∎

<u>Theorem 11.4.2</u> $\underset{=}{A}_{n\times n}$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ then their corresponding eigenvalues are linearly independent

<u>Proof</u>

$\{\underline{e}_1, \ldots, \underline{e}_n\}$ is LI $\Rightarrow$ $\alpha_1 \underline{e}_1 + \cdots + \alpha_n \underline{e}_n = \underline{0}$ only if $\alpha_1 = \cdots = \alpha_n = 1$

$$\Rightarrow \begin{bmatrix} \alpha_1 \underline{e}_1 & \cdots & \alpha_n \underline{e}_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{0}$$

Since
$$\underset{=}{A}(\alpha_1 \underline{e}_1 + \cdots + \alpha_n \underline{e}_n) = \lambda_1 \alpha_1 \underline{e}_1 + \cdots + \lambda_n \alpha_n \underline{e}_n$$
$$\vdots$$
$$\underset{=}{A}^{n-1}(\alpha_1 \underline{e}_1 + \cdots + \alpha_n \underline{e}_n) = \lambda_1^{n-1} \alpha_1 \underline{e}_1 + \cdots + \lambda_n^{n-1} \alpha_n \underline{e}_n$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \underline{e}_1 & \cdots & \alpha_n \underline{e}_n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vandermonde matrix with non-zero determinant if } \lambda_i \neq \lambda_j \text{ (Exercise 17 in section 10.4)}} = \begin{bmatrix} \underline{0} & \underline{0} & \cdots & \underline{0} \end{bmatrix} = \underset{=}{0}_{n\times}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \underline{e}_1 & \cdots & \alpha_n \underline{e}_n \end{bmatrix} = \underset{=}{0}$$

since $\underline{e}_i \neq \underline{0}$

$\Rightarrow \alpha_1 = \cdots = \alpha_n = 0. \Rightarrow \{\underline{e}_1, \ldots, \underline{e}_n\}$ is LI.

<u>Theorem 11.4.3</u>

If $\underset{=}{A}_{n\times n}$ has n distinct eigenvalues, it is diagonalizable.

<u>Proof</u> $\underset{=}{A}_{n\times n}$ has n distinct eigenvalues $\xRightarrow{\text{Thm. 11.4.2}}$ it has n LI eigenvectors

$\underset{=}{A}_{n\times n}$ has n LI eigenvectors $\xRightarrow{\text{Thm. 11.4.1}}$ It is diagonalizable

<u>Theorem 11.4.4</u>

Every symmetric matrix $\underset{=}{A}$ is diagonalizable.

<u>Proof</u>

We will provide a proof that does not assume the validity of Theorem 11.3.2.

The proof will be based on method of induction.
For a symmetric matrix of dimension 1×1:

$\underline{\underline{A}} = [a_{11}]$ is alread in diagonal form therefore it is certainly diagonalizable.

Assume that every symmetric matrix of $(n-1)\times(n-1)$ dimensions is diagonalizable in the following form:

$$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{D}} \leftarrow \text{diagonal} \quad \text{where} \quad \underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}} \text{ or } \underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T.$$

(Note that for $\underline{\underline{A}} = [a_{11}]$, $\underline{\underline{Q}} = 1$.)

Now; we will show that for $\underline{\underline{A}}_{n\times n}$, there exists a $\underline{\underline{Q}}_{n\times n}$ that diagonalizes $\underline{\underline{A}}_{n\times n}$.
Let $(\lambda, \underline{x})$ be an eigenpair of $\underline{\underline{A}}_{n\times n}$, where λ is real (Thm. 11.3) and $\underline{x}^T\underline{x} = 1$; unit-norm.
Let $\{\underline{y}_1, \dots, \underline{y}_{n-1}, \underline{x}\}$ be an orthonormal basis for $\mathbb{R}^n$.

Then form $\underline{\underline{Y}}_{n\times n-1} = [\underline{y}_1 \cdots \underline{y}_{n-1}]$.

Note that: (i) $\underline{\underline{Y}}^T \underline{\underline{Y}} = \underline{\underline{I}}_{n-1}$ since $\underline{y}_i^T \underline{y}_j = \delta_{ij}$.

(ii) $\underline{x}^T \underline{\underline{Y}} = \underline{0}^T$ since $\underline{x}^T \underline{y}_i = 0$.

(iii) $\underline{\underline{Y}}^T \underline{\underline{A}}\, \underline{\underline{Y}}$ is symmetric for symmetric $\underline{\underline{A}}$.

$\rightarrow$ Since $\underline{\underline{Y}}^T \underline{\underline{A}}\, \underline{\underline{Y}}$ is an $(n-1)\times(n-1)$ dimensional symmetric matrix, by the inductive hypothesis, there exists a $\underline{\underline{Q}}_{(n-1)\times(n-1)}$ matrix that diagonalizes it:

$$\underline{\underline{Q}}^T_{(n-1)\times(n-1)} \underline{\underline{Y}}^T \underline{\underline{A}}\, \underline{\underline{Y}}\, \underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{\underline{D}}, \quad (\star 1)$$

Now consider $\underline{\underline{Q}}_{n\times n} = [\underline{x} \;\; \underline{\underline{Y}}\, \underline{\underline{Q}}_{(n-1)\times(n-1)}]$.

Observations: (i) $\underline{\underline{Q}}_{n\times n}$ is orthogonal:

$$\underline{\underline{Q}}^T_{n\times n}\,\underline{\underline{Q}}_{n\times n} = \left[\underline{x}\;\; \underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right]^T \left[\underline{x}\;\; \underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right]$$

$$= \begin{bmatrix} \underline{x}^T \\ \underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underline{\underline{Y}}^T \end{bmatrix} \left[\underline{x}\;\; \underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right]$$

$$= \begin{bmatrix} \underline{x}^T\underline{x} & \underline{x}^T\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} \\ \underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underline{\underline{Y}}^T\underline{x} & \underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underline{\underline{Y}}^T\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} \end{bmatrix}$$

- $\underline{x}^T\underline{x} = 1$
- $\underline{x}^T\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{0}^T$ since $\underline{x}^T\underline{\underline{Y}} = \underline{0}^T$
- $\underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underline{\underline{Y}}^T\underline{x} = \underline{0}$ since $\underline{\underline{Y}}^T\underline{x} = \underline{0}$.
- $\underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underline{\underline{Y}}^T\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{\underline{Q}}^T_{(n-1)\times(n-1)}\underbrace{\left(\underline{\underline{Y}}^T\underline{\underline{Y}}\right)}_{\underline{\underline{I}}_{(n-1)}}\underline{\underline{Q}}_{(n-1)\times(n-1)}$

$$= \underline{\underline{Q}}^T_{(n-1)\times(n-1)}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{\underline{I}}_{(n}$$

$$\Longrightarrow \underline{\underline{Q}}^T_{n\times n}\,\underline{\underline{Q}}_{n\times n} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \underline{\underline{I}}_{n-1} \end{bmatrix} = \underline{\underline{I}}_n.$$

(ii) $$\underline{\underline{Q}}^T_{n\times n}\,\underline{\underline{A}}_{n\times n}\,\underline{\underline{Q}}_{n\times n} = \left[\underline{x}\;\; \underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right]^T \underline{\underline{A}}_{n\times n} \left[\underline{x}\;\; \underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right]$$

$$= \begin{bmatrix} \underline{x}^T\underline{\underline{A}}\,\underline{x} & \underline{x}^T\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} \\ \left(\underline{x}^T\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right)^T & \left(\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)}\right)^T\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} \end{bmatrix}$$

- $\underline{x}^T\underline{\underline{A}}\,\underline{x} = \underline{x}^T\lambda\underline{x} = \lambda$
- $\underbrace{\underline{x}^T\underline{\underline{A}}}_{\lambda\underline{x}^T}\underline{\underline{Y}}\,\underline{\underline{Q}}_{(n-1)\times(n-1)} = \lambda\underbrace{\underline{x}^T\underline{\underline{Y}}}_{\underline{0}^T}\underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{0}^T$
- $\underline{\underline{Q}}^T_{(n-1)\times(n-1)}\left(\underline{\underline{Y}}^T\underline{\underline{A}}\,\underline{\underline{Y}}\right)\underline{\underline{Q}}_{(n-1)\times(n-1)} = \underline{\underline{D}}$ (See Eqn. ★1 above

$$\Longrightarrow \underline{\underline{Q}}^T_{n\times n}\,\underline{\underline{A}}_{n\times n}\,\underline{\underline{Q}}_{n\times n} = \begin{bmatrix} \lambda & \underline{0}^T \\ \underline{0} & \underline{\underline{D}} \end{bmatrix}$$ is diagonal, $\underline{\underline{Q}}^T_{n\times n}\underline{\underline{Q}}_{n\times n} =$

Thus $\underline{\underline{A}}_{n\times n}$ is also diagonalizable. ■

Revisit Theorem 11.3.2: If λ is an eigenvalue of a real symmetric $\underline{\underline{A}}$ with multiplicity k, then its corresponding eigenspace k dimensional.

Proof: • Theorem 11.4.4 shows that for $\underline{\underline{A}}$ is real symmetric $\underline{\underline{A}}$ is diagonalizable: $\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{Q}}^{-1} \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{D}}$.

• Theorem 11.4.1 shows that if $\underline{\underline{A}}_{n\times n}$ is diagonalizable, it must have n LI eigenvectors.

Now assume that there exists a real symmetric matrix $\underline{\underline{A}}$ which has an eigenvalue λ with multiplicity k but its corresponding eigenspace is less than k dimensional. Then the LI set of eigenvectors will have less than n vectors in it. But since $\underline{\underline{A}}$ is diagonalizable it must have n LI eigenvectors. This is a contradiction. Thus the assumption of an eigenvalue with multiplicity k and eigenspace with less than k-dimensions is wrong. ∎

Closing Note

Even if $\underline{\underline{A}}$ is not diagonalizable, it can be triangularized:

$$\underbrace{\underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}} = \underline{\underline{J}}}_{\text{Jordan form for } \underline{\underline{A}}.} \leftarrow \text{upper triangular.}$$

See exercise 11.4.10 on page 582 in the textbook.

Cayley–Hamilton Theorem

Let $\underline{\underline{A}}_{n\times n}$ has a characteristic equation:

$$\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0.$$

Then:

$$\underline{\underline{A}}^n + \alpha_1 \underline{\underline{A}}^{n-1} + \cdots + \alpha_{n-1}\underline{\underline{A}} + \alpha_n \underline{\underline{I}} = \underline{\underline{0}}.$$

Proof In the following proof, we will assume that $\underline{\underline{A}}$ has a full set of eigenvectors. Then:

$$\underline{\underline{A}}\,[\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_n] = \underbrace{[\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_n]}_{\underline{\underline{V}}} \overbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}^{\underline{\underline{\Lambda}}}$$

Note that some of the λ_i's may be the same, but columns of $\underline{\underline{V}}$ are linearly independent. Therefore $\underline{\underline{V}}$ is invertible. Hence:

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}.$$

Note that: $(\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}})^2 = \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underbrace{\underline{\underline{V}}\,\underline{\underline{V}}^{-1}}_{\underline{\underline{I}}}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}^{-1}\underline{\underline{A}}^2\underline{\underline{V}} = \underline{\underline{\Lambda}}^2$

and

$$(\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}})^k = \underline{\underline{V}}^{-1}\underline{\underline{A}}^k\underline{\underline{V}} = \underline{\underline{\Lambda}}^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

Since each λ_i satisfies the characteristic equation:

$$\begin{bmatrix} \lambda_1^n + \alpha_1\lambda_1^{n-1} + \cdots + \alpha_{n-1}\lambda_1 + \alpha_n & & \\ & \ddots & \\ & & \lambda_n^n + \alpha_1\lambda_n^{n-1} + \cdots + \alpha_{n-1}\lambda_n + \alpha_n \end{bmatrix} = \underline{\underline{0}}$$

$$= \underline{\underline{\Lambda}}^n + \alpha_1\underline{\underline{\Lambda}}^{n-1} + \cdots + \alpha_{n-1}\underline{\underline{\Lambda}} + \alpha_n\underline{\underline{I}} = \underline{\underline{0}}$$

$$\Rightarrow \underline{\underline{V}}^{-1}\underline{\underline{A}}^n\underline{\underline{V}} + \alpha_1\underline{\underline{V}}^{-1}\underline{\underline{A}}^{n-1}\underline{\underline{V}} + \cdots + \alpha_{n-1}\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} + \alpha_n\underline{\underline{V}}^{-1}\underline{\underline{I}}\,\underline{\underline{V}} = \underline{\underline{0}}$$

$$\Rightarrow \underline{\underline{V}}^{-1}(\underline{\underline{A}}^n + \alpha_1\underline{\underline{A}}^{n-1} + \cdots + \alpha_{n-1}\underline{\underline{A}} + \alpha_n\underline{\underline{I}})\underline{\underline{V}} = \underline{\underline{0}}$$

Can multiply both sides from the left with $\underline{\underline{V}}$ and right with $\underline{\underline{V}}^{-1}$ to get:

$$\underline{\underline{A}}^n + \alpha_1\underline{\underline{A}}^{n-1} + \cdots + \alpha_{n-1}\underline{\underline{A}} + \alpha_n\underline{\underline{I}} = \underline{\underline{0}}. \blacksquare$$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, Characteristic eqn: $\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2+3\lambda+2=0$

$\lambda_1=-1,\ \lambda_2=-2$

$\underline{\underline{A}}^2 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$, $\underline{\underline{A}}^2+3\underline{\underline{A}}+2\underline{\underline{I}} = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -6 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \underline{\underline{0}}$

Note that, this implies: $\underline{\underline{A}}^2 = -3\underline{\underline{A}} - 2\underline{\underline{I}}$.

Hence, any power of $\underline{\underline{A}}$ can be reduced to:

$$\underline{\underline{A}}^k = \beta_1 \underline{\underline{A}} + \beta_0 \underline{\underline{I}} \quad \text{where } \alpha_1 \text{ and } \alpha_0 \text{ depends on } k.$$

In general $\underline{\underline{A}}^k = \sum_{i=0}^{n-1} \beta_i \underline{\underline{A}}^i$

If λ_j is an eigenvalue of $\underline{\underline{A}}$ and $\underline{v}_j$ is an eigenvector associated with λ_j:

$$\underbrace{\underline{\underline{A}}^k \underline{v}_j}_{} = \left(\sum_{i=0}^{n-1} \beta_i \underline{\underline{A}}^i\right) \underline{v}_j$$

$$\lambda_j^k \underline{v}_j = \left(\sum_{i=0}^{n-1} \beta_i \lambda_j^i\right) \underline{v}_j \Rightarrow \boxed{\lambda_j^k = \sum_{i=0}^{n-1} \beta_i \lambda_j^i}$$

Thus, we can determine β_i based on these conditions.

ex/ For the above example, compute $\underline{\underline{A}}^5$.

$$\underline{\underline{A}}^5 = \beta_1 \underline{\underline{A}} + \beta_0 \underline{\underline{I}} \ ; \quad \lambda_1 = -1,\ \lambda_2 = -2$$

Hence β_1 and β_0 satisfies the following equations:

$$\left.\begin{aligned} \lambda_1^5 &= \beta_1 \lambda_1 + \beta_0 \\ \lambda_2^5 &= \beta_1 \lambda_2 + \beta_0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \lambda_1^5 \\ \lambda_2^5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -32 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = -\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -32 \end{bmatrix} = \begin{bmatrix} 30 \\ 31 \end{bmatrix}$$

Hence: $\underline{\underline{A}}^5 = 31\underline{\underline{A}} + 30\underline{\underline{I}} = \begin{bmatrix} 0 & 31 \\ -62 & -93 \end{bmatrix} + \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 30 & 3 \\ -62 & -63 \end{bmatrix}$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $\lambda_1 = \lambda_2 = -1$, find $\underline{\underline{A}}^{10}$

$$\underline{\underline{A}}^{10} = \beta_1 \underline{\underline{A}} + \beta_0 \underline{\underline{I}}$$

This implies that: $\lambda_1^{10} = \beta_1 \lambda_1 + \beta_0 \Rightarrow 1 = -\beta_1 + \beta_0$

Since $\lambda_2 = \lambda_1$, we cannot get another independent equation for β_1 and β_0

However, since λ_1 is a double root we get another equation by computing the derivative of: $\lambda^{10} = \beta_1 \lambda + \beta_0 \Big|_{\lambda=\lambda_1}$

$$\Rightarrow 10\lambda_1^9 = \beta_1 \Rightarrow -10 = \beta_1$$

Thus: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \end{bmatrix} \Rightarrow \beta_1 = -10,\ \beta_0 = -9$

Hence: $\underline{\underline{A}}^{10} = -10\underline{\underline{A}} - 9\underline{\underline{I}} = \begin{bmatrix} 0 & -10 \\ 10 & -20 \end{bmatrix} - \begin{bmatrix} +9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 10 & -29 \end{bmatrix}$

ex/ $\underline{\underline{A}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, find $e^{\underline{\underline{A}}t}$:

Since $\underline{\underline{A}}$ is 2x2, and $\lambda_1 = 1$, $\lambda_2 = 1$

$$e^{\underline{\underline{A}}t} = \beta_1 \underline{\underline{A}} + \beta_0 \underline{\underline{I}}$$

$$\left.\begin{aligned} e^{\lambda_1 t} = \beta_1 \lambda + \beta_0 &\Rightarrow e^t = \beta_1 + \beta_0 \\ \frac{d}{d\lambda} e^{\lambda t}\Big|_{\lambda=1} = \frac{d}{d\lambda}(\beta_1 \lambda + \beta_0)\Big|_{\lambda=1} &\Rightarrow te^t = \beta_1 \end{aligned}\right\} \Rightarrow \begin{aligned} \beta_1 &= te^t \\ \beta_0 &= (1-t)e^t \end{aligned}$$

Thus: $e^{\underline{\underline{A}}t} = te^t \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + (1-t)e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} e^t & -te^t \\ 0 & e^t \end{bmatrix}$$

eigenvalue in the diagonal

Alternatively: $f(\underline{\underline{A}}) = \sum_{r=0}^{p} \beta_r \underline{\underline{A}}^r$ if $\underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{\Lambda}}$

$$\underline{\underline{Q}}^{-1} f(\underline{\underline{A}}) \underline{\underline{Q}} = \sum_{r=0}^{p} \beta_r \underline{\underline{Q}}^{-1} \underline{\underline{A}}^r \underline{\underline{Q}}$$

$$\Rightarrow \underline{\underline{Q}}^{-1} f(\underline{\underline{A}}) \underline{\underline{Q}} = \sum_{r=0}^{p} \beta_r \underline{\underline{\Lambda}}^r \quad , \text{ since } \underbrace{(\underline{\underline{Q}}^{-1} \underline{\underline{A}} \underline{\underline{Q}})^r}_{\underline{\underline{\Lambda}}^r} = \underline{\underline{Q}}^{-1} \underline{\underline{A}}^r \underline{\underline{Q}}$$

$$\Rightarrow \underline{\underline{Q}}^{-1} f(\underline{\underline{A}}) \underline{\underline{Q}} = \begin{bmatrix} \sum_{r=0}^{p} \beta_r \lambda_1^r & & \\ & \ddots & \\ & & \sum_{r=0}^{p} \beta_r \lambda_n^r \end{bmatrix}$$

$$= \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

$$\Rightarrow f(\underline{\underline{A}}) = \underline{\underline{Q}} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} \underline{\underline{Q}}^{-1} .$$

ex/ Calculate $e^{\underline{\underline{A}}t}$ for $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\lambda_1 = -1$, $\lambda_2 = -2$

Now, need to compute eigenvectors of $\underline{\underline{A}}$ as well.

$$\left.\begin{array}{l} (\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}) \underline{e}_1 = \underline{0} \Rightarrow \underline{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ (\underline{\underline{A}} - \lambda_2 \underline{\underline{I}}) \underline{e}_2 = \underline{0} \Rightarrow \underline{e}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{array}\right\} \quad \begin{array}{l} \underline{\underline{Q}} = [\underline{e}_1 \ \underline{e}_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \\ \underline{\underline{Q}}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \end{array}$$

Then:

$$e^{\underline{\underline{A}}t} = \underline{\underline{Q}} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \underline{\underline{Q}}^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Quadratic Forms

$f(x_1, x_2) = a_{11} x_1^2 + a_{22} x_2^2 + 2a_{12} x_1 x_2$ is called a quadratic form in the variables x_1 and x_2.

An n-variable quadratic form can be represented as:

$$f(x_1, \ldots, x_n) = \underline{x}^T \underline{\underline{A}} \underline{x}$$

where $\underline{\underline{A}}$ can be chosen to be symmetric.

<u>Def'n</u> $f(x_1, \ldots, x_n)$ is called as a canonical quadratic form if:

$$f(x_1, \ldots, x_n) = \underline{x}^T \underline{\underline{D}} \underline{x} , \ \underline{\underline{D}} \text{ is diagonal.}$$

Theorem 11.6.1 Every quadratic form can be transformed into a canonical form.

Proof

Let $f(x_1, \dots, x_n) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$

Since $\underline{\underline{A}}$ is symmetric it can be diagonalized as:

$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{\Lambda}}$ where $\underline{\underline{Q}}$ has unit-norm eigenvectors as its columns.

$$\text{Then: } \underbrace{f(x_1, \dots, x_n)}_{f(\underline{x})} = \underline{x}^T \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T \underline{x}$$

$$= (\underline{\underline{Q}}^T \underline{x})^T \underline{\underline{\Lambda}} (\underline{\underline{Q}}^T \underline{x})$$

$$= \tilde{\underline{x}}^T \underline{\underline{\Lambda}}\, \tilde{\underline{x}}, \quad \tilde{\underline{x}} = \underline{\underline{Q}}^T \underline{x}$$

$$\text{or } \underline{x} = \underline{\underline{Q}}\, \tilde{\underline{x}}$$

Thus $f(\underline{\underline{Q}}\, \tilde{\underline{x}}) = \tilde{\underline{x}}^T \underline{\underline{\Lambda}}\, \tilde{\underline{x}}$ is in canonical form.

Def'n A quadratic form $\underline{x}^T \underline{\underline{A}}\, \underline{x}$ is called as positive definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \ \forall \underline{x} \neq \underline{0}$.

Negative definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0 \ \forall \underline{x} \neq \underline{0}$.

Likewise, $\underline{\underline{A}}$ is called positive (negative) definite if its quadratic form is positive (negative) definite.

Theorem 11.6.2 For $\underline{\underline{A}}$ symmetric, $\underline{\underline{A}}$ is positive (negative) definite if every eigenvalue of $\underline{\underline{A}}$ is positive (negative).

Proof $\underline{x}^T \underline{\underline{A}}\, \underline{x} = \underline{x}^T (\underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T) \underline{x}$, where $\underline{\underline{Q}}$ has unit-norm eigenvectors as its column

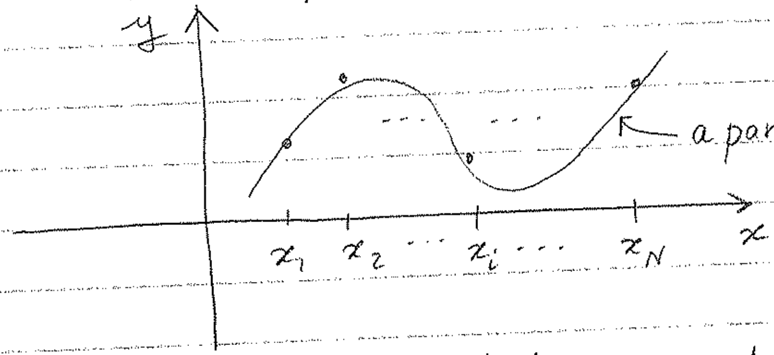
$$= (\underline{\underline{Q}}^T \underline{x})^T \underline{\underline{\Lambda}} (\underline{\underline{Q}}^T \underline{x})$$

$$= \tilde{\underline{x}}^T \underline{\underline{\Lambda}}\, \tilde{\underline{x}}$$

$$= \sum_{i=1}^{n} \lambda_i \tilde{x}_i^2 \quad \begin{matrix} > 0 & \text{if } \lambda_i > 0, & 1 \le i \le n \\ < 0 & \text{if } \lambda_i < 0, & 1 \le i \le n \end{matrix}$$

Curve Fitting

In many applications, we are given a set of measurements such as: (x_i, y_i), $1 \le i \le N$, where x_i may represent the time of the measurement or position of the measurement.



The relationship between x and y is typically assumed to be a parametric function: $y = f(x; \underline{\theta})$ ← the set of parameters in vector form

Then, we can choose the set of parameters that provides the best fit to the given measurements. For example, we can choose $\underline{\theta}$ to minimize:

$$\sum_{i=1}^{N} \left(y_i - f(x_i; \underline{\theta})\right)^2, \text{ or } \quad \underline{\theta}^* = \arg\min_{\underline{\theta}} \sum_{i=1}^{N} \left(y_i - f(x_i; \underline{\theta})\right)^2$$

This problem can be solved by using optimization tools, like the ones that are available in MATLAB. But, for certain cases, for instance when $f(x; \underline{\theta})$ is a polynomial in x:

$$f(x; \underline{\theta}) = \theta_n x^n + \theta_{n-1} x^{n-1} + \cdots + \theta_1 x + \theta_0$$

the optimization problem can be reduced to a system of linear equations. Let e_i be the fit error at x_i:

$$e_i = \left(y_i - \left(\theta_n x_i^n + \theta_{n-1} x_i^{n-1} + \cdots + \theta_1 x_i + \theta_0\right)\right)$$

Then:

$$\underbrace{\begin{bmatrix} e_1 \\ \vdots \\ \vdots \\ e_N \end{bmatrix}}_{\underline{e}} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_N \end{bmatrix}}_{\underline{y}} - \underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \\ \vdots & & & \\ 1 & x_N & \cdots & x_N^n \end{bmatrix}}_{\underline{\underline{X}}} \underbrace{\begin{bmatrix} \theta_0 \\ \vdots \\ \vdots \\ \theta_n \end{bmatrix}}_{\underline{\theta}}$$

Hence, the optimization problem becomes:

$$\min_{\underline{\theta}} \|\underline{e}\|^2 = \min_{\underline{\theta}} \|\underline{y} - \underline{\underline{X}}\,\underline{\theta}\|^2$$

Thus, optimal $\underline{\theta}$ is chosen to minimize $\underline{y} - \underline{\underline{X}}\,\underline{\theta}$.
This problem can easily be solved by invoking the following orthogonality condition: $\underline{e}$ at optimal $\underline{\theta}$ must be orthogonal to the column space of $\underline{\underline{X}}$:

$$\underline{\underline{X}}^T \underline{e} = \underline{0} \Rightarrow \underline{\underline{X}}^T(\underline{y} - \underline{\underline{X}}\,\underline{\theta}) = 0 \Rightarrow \underbrace{\underline{\underline{X}}^T\underline{\underline{X}}\,\underline{\theta} = \underline{\underline{X}}^T\underline{y}}_{\text{called as the "normal equation"}}$$

This last equation is always consistent (show this). Hence can be solved to get the optimal values for the parameters $\underline{\theta}$.

ex/ Let $\underline{y} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$; $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$

If we want to fit a second order polynomial to the given set of data, we have 3-parameters θ_0, θ_1 and θ_2. The normal equations becomes:

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T}_{\underline{\underline{X}}^T} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{\underline{\underline{X}}}} \underbrace{\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix}}_{\underline{\theta}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T}_{\underline{\underline{X}}^T} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}}_{\underline{y}}$$

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 29 \end{bmatrix}$$

Can solve this system to obtain $\underline{\theta}$ as:

$$\underline{\theta} = \begin{bmatrix} 9/20 \\ -1/20 \\ 1/4 \end{bmatrix}.$$

Linear Constant Coefficient Difference Equations

$$\underbrace{x[n] = \overbrace{a_1 x[n-1] + \cdots + a_k}^{\text{coefficients}} x[n-k]}_{k^{th}\text{ order difference equation}}$$

Occurs very frequently discrete-time systems or discretized models of continuous time systems. Can investigate the solution based on the following vector-matrix relationship:

$$\underline{x}_n = \begin{bmatrix} x[n] \\ \vdots \\ x[n-k+1] \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} a_1 & \cdots & a_k \\ 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{bmatrix} = \begin{bmatrix} [a_1 \cdots a_k] \\ \underline{\underline{I}} \quad \underline{0} \end{bmatrix}$$

$$\Rightarrow \underline{x}_n = \underline{\underline{A}}\, \underline{x}_{n-1}.$$

Since $\underline{x}_{n-1} = \underline{\underline{A}}\, \underline{x}_{n-2}$ we can write:

$$\underline{x}_n = \underline{\underline{A}} \left(\underline{\underline{A}} \left(\cdots \left(\underline{\underline{A}}\, \underline{x}_0 \right) \cdots \right) \right) = \underline{\underline{A}}^n \underline{x}_0$$

where $\underline{x}_0 = \begin{bmatrix} x[0] \\ \vdots \\ x[-k+1] \end{bmatrix}$ } initial conditions

Therefore, given $\underline{x}_0$, can find $\underline{x}_n$ for any n by multiplying $\underline{x}_0$ n-times with $\underline{\underline{A}}$. But this is not the most efficient technique, and also would not provide us a closed form solution. We can get more insight as follows:

Theorem DE.1

If $(\lambda, \underline{x}_0)$ is an eigenpair of $\underline{\underline{A}}$, then $\underline{x}_n = \lambda^n \underline{x}_0$ is a solution to $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0$.

Proof $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0 = \lambda^n \underline{x}_0$.

<u>Theorem DE.2</u> If $\underline{\underline{A}}$ has a full set of eigenvectors that span $\mathbb{R}^k$, then $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0$ can be solved by:

$$\underline{x}_n = \sum_{i=1}^{k} \alpha_i \lambda_i^n \underline{v}_i \quad \text{where} \quad \underline{x}_0 = \sum_{i=1}^{k} \alpha_i \underline{v}_i$$

<u>Proof</u> Simply multiply $\underline{x}_0$ by $\underline{\underline{A}}^n$ to get:

$$\underline{\underline{A}}^n \underline{x}_0 = \underline{\underline{A}}^n \left(\sum_{i=1}^{k} \alpha_i \underline{v}_i \right) = \sum_{i=1}^{k} \alpha_i \underline{\underline{A}}^n \underline{v}_i = \underbrace{\sum_{i=1}^{n} \alpha_i \lambda_i^n \underline{v}_i}_{\underline{x}_n} \quad \blacksquare$$

<u>example</u> $x[n] = 2x[n-1] + x[n-2]$, where initial conditions $x[0]=3, x[-1]$

$$\underline{x}_n = \begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 2 & +1 \\ 1 & 0 \end{bmatrix} \Rightarrow \overbrace{\lambda_1 = 1+\sqrt{2}, \; \lambda_2 = 1-\sqrt{2}}^{\text{eigenvalues}}$$

To get $\underline{v}_1$:

$$(\underline{\underline{A}} - (1+\sqrt{2})\underline{\underline{I}})\, \underline{v} = \underline{0} \Rightarrow \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

To get $\underline{v}_2$:

$$(\underline{\underline{A}} - (1-\sqrt{2})\underline{\underline{I}})\, \underline{v} = \underline{0} \Rightarrow \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} -1+\sqrt{2} \\ -1 \end{bmatrix}$$

$$\underline{x}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 = \begin{bmatrix} 1+\sqrt{2} & -1+\sqrt{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\Rightarrow \alpha_2 = \frac{2-\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}-1}{2} = -\frac{1-\sqrt{2}}{2}$$

$$\alpha_1 = 1 + \alpha_2 = \frac{\sqrt{2}+1}{2}$$

Thus:

$$\underline{x}_n = \alpha_1 \lambda_1^n \underline{v}_1 + \alpha_2 \lambda_2^n \underline{v}_2 = \frac{(\sqrt{2}+1)}{2} (1+\sqrt{2})^n \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} - \frac{(1-\sqrt{2})}{2} (1-\sqrt{2})^n \begin{bmatrix} \sqrt{2}-1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{x}_n = \begin{bmatrix} \dfrac{(1+\sqrt{2})^{n+2}}{2} + \dfrac{(1-\sqrt{2})^{n+2}}{2} \\ \dfrac{(1+\sqrt{2})^{n+1}}{2} + \dfrac{(1-\sqrt{2})^{n+1}}{2} \end{bmatrix}$$

$$\Rightarrow x[n] = \frac{1}{\ } \cdot \left[(1+\sqrt{2})^{n+2} + (1-\sqrt{2})^{n+2} \right], \; n \geq 0$$

Another alternative approach is based on the following observation:

Given difference equation:

$x[n] = a_1 x[n-1] + a_2 x[n-2] + \cdots + a_k x[n-k]$

a sequence of the form $x[n] = r^n$ would solve the DE if:

$$r^n = a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_k r^{n-k}$$

or: $r^n - a_1 r^{n-1} - a_2 r^{n-2} - \cdots - a_k r^{n-k} = 0 \quad \forall n.$

$$\Rightarrow r^{n-k} \underbrace{\left(r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k\right)}_{\text{characteristic eqn of the DE}} = 0$$

characteristic eqn of the DE $= 0$ at r.

$\Rightarrow$ r should be a root of the characteristic eqn.

☆ Show that this is equivalent to r being an eigenvalue of $\underline{\underline{A}} = \begin{bmatrix} a_1 \; \cdots & a_k \\ \underline{\underline{I}}_{(k-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix}$.

Hence, there can be at most k distinct r values. If they are all distinct, then a solution to the original problem be formulated easily by:

$$x[n] = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n = \sum_{i=1}^{k} \alpha_i r_i^k \quad , \quad n \geq 0$$

Where if we are given a set of initial conditions such as $\begin{bmatrix} x[0] \\ \vdots \\ x[-k+1] \end{bmatrix}$ or $\begin{bmatrix} x[-1] \\ \vdots \\ x[-k] \end{bmatrix}$

we can find α_i's as the solution to:

$$\underbrace{\begin{bmatrix} r_1^{-1} & r_2^{-1} & \cdots & r_k^{-1} \\ r_1^{-2} & r_2^{-2} & \cdots & r_k^{-2} \\ \vdots & & & \\ r_1^{-k} & r_2^{-k} & \cdots & r_k^{-k} \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} x[-1] \\ \vdots \\ \vdots \\ x[-k] \end{bmatrix}$$

ex/ Let us apply this approach to the previous example:

$x[n] = 2x[n-1] + x[n-2]$

Characteristic eqn. is: $r^2 - 2r - 1 = 0 \Rightarrow r_1 = 1+\sqrt{2}, r_2 = 1-\sqrt{2}$

Thus $x[n] = \alpha_1 r_1^n + \alpha_2 r_2^n$, $x[0] = 3$, $x[-1] = 1$

$$\Rightarrow \left.\begin{aligned} 3 &= \alpha_1 + \alpha_2 \\ 1 &= \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} \end{aligned}\right\} \Rightarrow \alpha_1 = \frac{r_1 r_2 - 3r_1}{r_2 - r_1} = \frac{4+3\sqrt{2}}{2\sqrt{2}}$$

$$r_1 r_2 = r_2\alpha_1 + r_1\alpha_2 \qquad \alpha_2 = 3 - \alpha_1 = \frac{3\sqrt{2}-4}{2\sqrt{2}}$$

Hence: $x[n] = \left(\frac{4+3\sqrt{2}}{2\sqrt{2}}\right)\cdot\left(1+\sqrt{2}\right)^n + \left(\frac{3\sqrt{2}-4}{2\sqrt{2}}\right)\left(1-\sqrt{2}\right)^n$, $n \geq$

Show that this solution is the same with the previously obtained one.

ex/ This second approach can be modified easily if the characteristic eqn. has repeated roots as well.

$x[n] = 2x[n-1] - x[n-2]$, $x[-1] = 3$, $x[-2] = 4$

the characteristic equation is:

$r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$.

Then, the solution $x[n]$ can be written as:

$x[n] = \alpha_1 r_1^n + \alpha_2 n r_1^n$

(We will provide a proof for a similar situation for differential equations

Then:

$x[-1] = 3 = \alpha_1 r_1^{-1} - \alpha_2 r_1^{-1} = (\alpha_1 - \alpha_2) r_1^{-1}$

$x[-2] = 4 = \alpha_2 r_1^{-2} - 2\alpha_2 r_1^{-2} = (\alpha_1 - 2\alpha_2) r_1^{-2}$

$$\Rightarrow \left.\begin{aligned} \alpha_1 - \alpha_2 &= 3r_1 = 3 \\ \alpha_1 - 2\alpha_2 &= 4r_1^2 = 4 \end{aligned}\right\} \underline{\alpha_2 = -1, \alpha_1 = 2}$$

$\Rightarrow x[n] = (2-n)$, $n \geq 0$.

The z-Transform

A very useful transformation that is commonly used to Investigate and design of discrete-time systems. It is also useful in the solution to DE's.

Def'n Given a sequence $x[n]$, its z-transform $X(z)$ is defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

when the summation converges.

Facts ① For finite length sequences z-transform converges for any z, except maybe $z=0$.

② If the z-transform of a sequence converges, it converges for all $|z| > r_0$, where r_0 depends on the sequence.

Def'n Those z values for which $X(z)$ is defined are said to be in the Region of Convergence, or ROC.

Properties:

(i) Z-transform is linear: $\forall\, x_1[n], x_2[n], \alpha_1$ and α_2:

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad ROC_1$$
$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad ROC_2$$

Then

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longleftrightarrow \alpha_1 X_1(z) + \alpha_2 X_2(z), \quad ROC \supset ROC_1 \cap ROC_2$$

Proof

$$\sum_{n=0}^{\infty} (\alpha_1 x_1[n] + \alpha_2 x_2[n]) z^{-n} = \sum_{n=0}^{\infty} (\alpha_1 x_1[n] z^{-n} + \alpha_2 x_2[n] z^{-n})$$

$$= \alpha_1 \underbrace{\sum_{n=0}^{\infty} x_1[n] z^{-n}}_{X_1(z)} + \alpha_2 \underbrace{\sum_{n=0}^{\infty} x_2[n] z^{-n}}_{X_2(z)} = \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

(ii) $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$

$x[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}X(z) + x[-1]$.

Proof $\displaystyle\sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}]z^{-n} = \sum_{n'=-1}^{\infty} x[n']z^{-(n'+1)}$

$$= \sum_{n'=0}^{\infty} x[n']z^{-1}z^{-n'} + x[-1]$$

$$= z^{-1}\underbrace{\sum_{n'=0}^{\infty} x[n']z^{-n'}}_{X(z)} + x[-1]$$

$$= z^{-1}X(z) + x[-1] \quad \blacksquare$$

(iii) $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$

$x[n-k] \longleftrightarrow z^{-k}X(z) + z^{-k+1}x[-1] + z^{-k+2}x[-2] + \cdots + x[-k]$

$k \geq 1$

Proof $\displaystyle\sum_{n=0}^{\infty} x[\underbrace{n-k}_{n'}]z^{-n} = \sum_{n'=-k}^{\infty} x[n']z^{-(n'+k)} = z^{-k}\sum_{n'=0}^{\infty} x[n']z^{-n'}$

$$+ z^{-k}\left(x[-1]z + x[-2]z^2 + \cdots + x[-k]z^k\right)$$

$$= z^{-k}X(z) + z^{-k+1}x[-1] + \cdots + x[-k]$$

(iv) $x[n] = a^n, \ n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1 \Rightarrow \underbrace{|a| < |z|}_{\text{ROC}}.$$

(v) $x[n] = na^n, \ n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} na^n z^{-n} = \left[\sum_{n=0}^{\infty} na^n z^{-(n+1)}\right] \cdot z = \left(\frac{d}{dz}\sum_{n=0}^{\infty} a^n z^{-n}\right) z$$

$$= -\left[\frac{d}{dz}\frac{1}{(1-az^{-1})}\right] z$$

$$X(z) = +\frac{az^{-1}}{(1-az^{-1})^2} \quad \blacksquare$$

ex/ $x[n] = \frac{5}{6} x[n-1] - \frac{1}{6} x[n-2]$, $x[-1] = -4$, $x[-2] = -14$.

To find the solution, transform both sides of the equation to the z-domain:

$$X(z) = \frac{5}{6}\left(z^{-1}X(z) + x[-1]\right) - \frac{1}{6}\left(z^{-2}X(z) + z^{-1}x[-1] + x[-2]\right)$$

$$X(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = -\frac{20}{6} + \frac{4}{6}z^{-1} + \frac{14}{6}$$

$$= \frac{4}{6}z^{-1} - 1$$

partial fraction expansion

$$X(z) = \frac{\frac{4}{6}z^{-1} - 1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} = \frac{\frac{4}{6}z^{-1} - 1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{A}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{B}{\left(1 - \frac{1}{3}z^{-1}\right)}$$

$$A = \left(1 - \frac{1}{2}z^{-1}\right) \cdot X(z)\Big|_{z=\frac{1}{2}} = \frac{\left(4/6 \cdot 2 - 1\right)}{\left(1 - \frac{2}{3}\right)} = \frac{1/3}{1/3} = 1$$

$$B = \left(1 - \frac{1}{3}z^{-1}\right) X(z)\Big|_{z=\frac{1}{3}} = \frac{\left(4/6 \cdot 3 - 1\right)}{\left(1 - \frac{1}{2} \cdot 3\right)} = \frac{1}{-1/2} = -2.$$

$$\Rightarrow X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} - \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)} \xleftrightarrow{z^{-1}} x[n] = \left(\frac{1}{2}\right)^n - 2 \cdot \left(\frac{1}{3}\right)^n, \; n \geq$$

ex/ $x[n] = x[n-1] - \frac{1}{4} x[n-2]$, $x[-1] = 1$, $x[-2] = -1$.

Transform both sides of the difference equation to z-domain

$$X(z) = \left(z^{-1}X(z) + x[-1]\right) - \frac{1}{4}\left(z^{-2}X(z) + z^{-1}x[-1] + x[-2]\right)$$

$$= z^{-1}X(z) + 1 - \frac{1}{4}z^{-2}X(z) - \frac{1}{4}z^{-1} + \frac{1}{4}$$

$$\Rightarrow X(z)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right) = \frac{5}{4} - \frac{1}{4}z^{-1}$$

$$\Rightarrow X(z) = \frac{5/4 - 1/4\, z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)} = \frac{5/4 - 1/4\, z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} = \frac{A}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{B}{\left(1 - \frac{1}{2}z^{-1}\right.}$$

$$\Rightarrow \frac{5}{4} - \frac{1}{4}z^{-1} = A\left(1 - \frac{1}{2}z^{-1}\right) + B = A + B - \frac{A}{2}z^{-1}$$

$$\Rightarrow A + B = 5/4$$

$$\frac{A}{2} = \frac{1}{4} \Rightarrow A = \frac{1}{2} \Rightarrow B = \frac{5}{4} - A = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}.$$

Hence, $X(z) = \frac{1}{2}\frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{3}{4}\frac{1}{(1-\frac{1}{2}z^{-1})^2}$

Since $\frac{1}{(1-az^{-1})} \xleftrightarrow{Z^{-1}} a^n, \quad n \geq 0$

$\frac{1}{(1-az^{-1})^2} \xleftrightarrow{Z^{-1}} (n+1)a^n, \quad n \geq 0$ (Show this!)

Therefore:

$$X(z) \xleftrightarrow{Z^{-1}} \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}(n+1)\left(\frac{1}{2}\right)^n, \quad n \geq 0$$

$$\Rightarrow x[n] = \frac{5}{4}\left(\frac{1}{2}\right)^n + \frac{3}{4}n\left(\frac{1}{2}\right)^n, \quad n \geq 0$$

//

Linear Constant Coefficient Differential Equations

(Chapter 3, sections 4 and 5 in the textbook)

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = 0$$ $\left\{\right.$ Homogeneous n^{th}-order Linear Constant Coefficient DE.

ex/ $\frac{dy}{dx} + a_1 y = 0$, consider a solution in the form of $y(x) = e^{\lambda x}$

Then: $\frac{dy}{dx} = \lambda e^{\lambda x}$ hence: $\lambda e^{\lambda x} + a_1 e^{\lambda x} = 0 \Rightarrow \underline{a_1 = -\lambda}$

Hence, the general solution is of the form $Ce^{-a_1 x}$.
To determine C, we need an initial condition such as $y(0)$. Then: $y(0) = Ce^{-a_1 0} = C \Rightarrow y(x) = y(0)e^{-a_1 x}$

ex/ $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 = y'' + a_1 y' + a_2 y$

Also consider solutions in the form: $y(x) = e^{\lambda x}$

$\Rightarrow \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_2 e^{\lambda x} = 0$

$\Rightarrow (\lambda^2 + a_1\lambda + a_2) e^{\lambda x} = 0 \Rightarrow \underbrace{\lambda^2 + a_1\lambda + a_2 = 0}_{\text{characteristic eqn.}}$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a}}{2}$$

for real a_1 and a_2, we may have two real roots or a pair of complex conjugate roots.

The general solution is of the form:

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

If we are given initial conditions such as $y(0)$ and $y'(0)$ can uniquely determine C_1 and C_2.

ex/ $y'' - 9y = 0 \Rightarrow \underbrace{\lambda^2 - 9 = 0}_{\text{characteristic eqn.}} \Rightarrow \lambda_1 = 3, \lambda_2 = -3.$

The general solution is: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

Assuming we are given $y(0) = 4$, $y'(0) = -5$:

$$\Rightarrow \left.\begin{aligned} 4 &= C_1 + C_2 \\ -5 &= 3C_1 - 3C_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &6C_1 = 7 \Rightarrow C_1 = \tfrac{7}{6} \\ &C_2 = 4 - C_1 = 4 - \tfrac{7}{6} = \tfrac{17}{6} \end{aligned}$$

$$\Rightarrow y(x) = \frac{7}{6} e^{3x} + \frac{17}{6} e^{-3x}.$$

ex/ $y'' + 9y = 0 \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda_1 = j3, \lambda_2 = -j3.$

$$\Rightarrow y(x) = C_1 e^{j3x} + C_2 e^{-j3x} = \left(\frac{C_1 + C_2}{2} + \frac{C_1 - C_2}{2}\right) e^{j3x} + \left(\frac{C_2 + C_1}{2} + \frac{C_2 - C_1}{2}\right) e^{-j3x}$$

$$\Rightarrow y(x) = \left(\frac{C_1 + C_2}{2}\right)\left(e^{j3x} + e^{-j3x}\right) + j\left(\frac{C_1 - C_2}{2j}\right)\left(e^{j3x} - e^{-j3x}\right)$$

$$= (C_1 + C_2)\cos 3x + j(C_1 - C_2)\sin 3x.$$

$$= A\cos 3x + B\sin 3x$$ (A and B can be real or complex)

ex/ $y'' + 4y' + 7y = 0$, Characteristic eqn is: $\lambda^2 + 4\lambda + 7 = 0$

$$\Rightarrow \lambda_1 = \frac{-4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm j\sqrt{3}$$

$$\Rightarrow y(x) = C_1 e^{(-2+j\sqrt{3})x} + C_2 e^{(-2-j\sqrt{3})x}$$
$$= e^{-2x}\left(C_1 e^{j\sqrt{3}x} + C_2 e^{-j\sqrt{3}x}\right)$$
$$= e^{-2x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

Given $y(0) = 1$, $y'(0) = 0$

$$\Rightarrow 1 = A, \quad y'(0) = -2e^{-2x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + e^{-2x}\left(-\sqrt{3}A\sin\sqrt{3}x + \sqrt{3}B\cos\sqrt{3}x\right)\Big|_{x=0}$$
$$= -2A + \sqrt{3}B = 0 \Rightarrow B = \frac{2}{\sqrt{3}}$$

$$\Rightarrow y(x) = e^{-2x}\left(\cos\sqrt{3}x + \frac{2}{\sqrt{3}}\sin\sqrt{3}x\right).$$

Theorem 3.4.1

If the characteristic eqn. of an n^{th}-order linear constant coefficient DE has n-distinct roots, $\lambda_1, \ldots, \lambda_n$, the set $\{e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\}$ is LI in any interval $I \subset \mathbb{R}$.

Proof

Assume that they are LD. Then, there should be a set of coefficients not all zero such that

$$\alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \cdots + \alpha_n e^{\lambda_n x} = 0, \quad \forall x \in I.$$

By taking successive derivatives of both sides we get:

$$\alpha_1 \lambda_1 e^{\lambda_1 x} + \alpha_2 \lambda_2 e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n e^{\lambda_n x} = 0$$
$$\vdots$$
$$\alpha_1 \lambda_1^{(n-1)} e^{\lambda_1 x} + \alpha_2 \lambda_2^{(n-1)} e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n^{(n-1)} e^{\lambda_n x} = 0.$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{This is full rank Vandermonde matrix for } \lambda_i \neq \lambda_j} \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \alpha_2 e^{\lambda_2 x} \\ \vdots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \vdots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \underline{0} \Rightarrow \alpha_i e^{\lambda_i x} = 0$$
$$\Rightarrow \alpha_i = 0$$
$$\Rightarrow\Leftarrow$$

Theorem 3.4.2 Repeated Roots Case

If λ_1 is a root of order k of the characteristic equation, then $e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^{k-1}e^{\lambda_1 x}$ are LI solutions of the DE.

Proof Let $L[y]=0$ be the operator form of:

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y \Rightarrow L = \frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}\frac{d}{dx} + a_n$$

Then: $L[e^{\lambda x}] = \left(\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_0\right)e^{\lambda x}$

$= (\lambda-\lambda_1)^k p(\lambda) e^{\lambda x}$ (Since λ_1 is a root with multiplicity k and order of $p(\lambda)$ is $(n-k)$.)

$\Rightarrow L[e^{\lambda_1 x}] = 0 \Rightarrow e^{\lambda_1 x}$ is a solution.

Now, want to show that $xe^{\lambda_1 x}$ is also a solution.

$$\frac{d}{d\lambda}L[e^{\lambda x}] = \frac{d}{d\lambda}\left[(\lambda-\lambda_1)^k p(\lambda)e^{\lambda x}\right] = k(\lambda-\lambda_1)^{k-1}p(\lambda)e^{\lambda x} + (\lambda-\lambda_1)^k\frac{d}{d\lambda}\left(p(\lambda)e^{\lambda x}\right)$$

Note that: $\frac{d}{d\lambda}L[e^{\lambda x}] = L\left[\frac{d}{d\lambda}e^{\lambda x}\right] = L[xe^{\lambda x}]$.

$$L[xe^{\lambda x}]\Big|_{\lambda=\lambda_1} = \underbrace{k(\lambda-\lambda_1)^{k-1}p(\lambda)e^{\lambda x} + (\lambda-\lambda_1)^k\frac{d}{d\lambda}\left(p(\lambda)e^{\lambda x}\right)\Big|_{\lambda=\lambda_1}}_{0}$$

$\Rightarrow L[xe^{\lambda_1 x}] = 0 \Rightarrow xe^{\lambda_1 x}$ is also a solution.

Can apply this same procedure to prove that $x^2e^{\lambda_1 x}, \ldots, x^{(k-1)}e^{\lambda_1 x}$ are also solutions.

__example__ $y^{(iv)} - 8y''' + 26y'' - 40y' + 25y = 0$

characteristic eqn is: $\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25 = 0$

has repeated complex roots: $\lambda_1 = 2+j$ with multiplicity 2

$\lambda_2 = 2-j$ with multiplicity 2

Then the general solution can be written as:

$$y(x) = (C_1 + C_2 x)e^{(2+j)x} + (C_3 + C_4 x)e^{(2-j)x}$$
$$= e^{2x}\left(C_1 e^{jx} + C_3 e^{-jx} + x(C_2 e^{jx} + C_4 e^{-jx})\right)$$
$$= e^{2x}\left((A\cos x + B\sin x) + x(C\cos x + D\sin x)\right)$$

Def'n Stability: $\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_n y = 0$ $\quad$ (DE_n)

has a general solution: $y(t) = C_1 y_1(t) + \cdots + C_n y_n(t)$
is stable if all of its solutions are bounded for $t > 0$, i.e.,
$|y_i(t)| < M_i < \infty$, $1 \le i \le n$, $t \ge 0$.

Theorem 3.4.3 (DE_n) is stable if and only if the characteristi[c] equation have no roots to the right of the imaginary axis in the complex plane and that any roots on the imaginary axis be non-repeated.

Proof

(i) If $\lambda = a + jb$ is a non-repeated root of the characteristic eqn. then it can contribute a solution of the form $e^{\lambda t} = e^{at} e^{jbt}$
Hence, if $a \le 0$, $|e^{\lambda t}| = |e^{at} e^{jbt}| = |e^{at}| \underbrace{|e^{jbt}|}_{1} = e^{at} \le 1$, t...

(ii) If $\lambda = a + jb$ is a repeated root of order k, then it can contribute k solutions of the form $\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}\}$.
For $a < 0$, we want to show that all of these solutions are stable.
For $t > 0$: $|t^i e^{\lambda t}| = |t^i| \cdot |e^{at} \cdot e^{jbt}| = t^i e^{at}$.
Since $\lim_{t \to \infty} t^i e^{at} = 0$ for $a < 0$, and $t^i e^{at}$ is always finite for any interval $0 < t < T$, $t^i e^{at}$ is bounded for $t > 0$.

(iii) Note that if for simple root $\lambda = a + jb$, $a > 0$ or for repeated root $\lambda = a + jb$, $a \ge 0$, the solutions contributec[d] by this root will be unstable.

Theorem 3.4.4 Coefficients of mixed sign

If the coefficients of the characteristic equation are real and of mixed sign (there is at least one positive and one negative coefficient), then there is at least one root λ with $Re\{\lambda\} > 0$ so the system is unstable.

** Note that the Thm does not imply that if a system is unstable, its characteristic polynomial will have mixed signs.
→ We will prove that a stable system cannot have mixed sign.

Proof Let us use method of induction. Since roots can be real or complex conjugate pairs, at each step we should consider addition of a real or complex conjugate pairs to the existing roots.

For $n = 1$ % meaning that 1 real root or a pair of complex conjugate roots

The characteristic eqn is of the form:

(a) $\lambda - \lambda_1$ for a real root. Then if $\lambda_1 > 0$ we have mixed coefficients and an unstable solution in the form $e^{\lambda_1 t}$

(b) $(\lambda - (a+jb))(\lambda - (a-jb)) = \lambda^2 - 2a\lambda + (a^2+b^2)$

If $a > 0$ we have mixed signs and unstable solution

If $a \leq 0$ we do not have mixed signs and have a stable solution

Assume that we have an $(n-1)$ order characteristic equation which has stable roots and no mixed signed coefficients. We want to show that an addition of a real root or a complex conjugate pair as a root won't change the signs of the coefficients and won't create unstability if the roots introduce are stable.

(a) The newly introduced root is a real stable one:

Then the characteristic equation becomes: $p(\lambda) = q(\lambda) \cdot (\lambda$

Since the newly introduced root is stable $\lambda_r < 0$.

Thus the coefficients of $p(\lambda)$ can be found as:

$$p_i = q_{i-1} - \lambda_r q_i$$

Since all $q_i \geq 0$ and $\lambda_r < 0$, $p_i \geq 0$ for all i.

Thus the new root won't create mixed signs.

(b) The newly introduced root is a real unstable one:

Then the characteristic polynomial coefficients become:

$$p_i = q_{i-1} - \lambda_r q_i \Rightarrow p_0 = q_0 = 1 > 0$$

$$p_n = -\lambda_r q_n < 0 \text{ since } q_n >$$

Thus, the $p_n(\lambda)$ polynomial has mixed signs due to the newly introduced unstable root.

(c) The newly introduced roots are stable complex conjugate pairs. Then

$$p(\lambda) = q(\lambda)\cdot(\lambda-(a+jb))(\lambda-(a-jb))$$
$$= q(\lambda)\cdot(\lambda^2-2a\lambda+(a^2+b^2))$$

Since original coefficients in $q(\lambda)$ are all non-negative and $a<0$, $p(\lambda)$ is a multiplication of two non-mixed coefficient polynomials, hence the result will be a non-mixed coefficient polynomial as well.

(d) The newly introduced roots are unstable complex conjugate pairs. Then:

$$p(\lambda) = q(\lambda)\,(\lambda^2-2a\lambda+(a^2+b^2)) \quad ; \quad a>0$$

is multiplication of a non-mixed coefficient polynomial with a mixed coefficient polynomial. This may result a mixed coefficient polynomial. Then, we will be able to say that the system is unstable.

Note that even if the introduced roots are unstable, $p(\lambda)$ may still have no coefficient sign mix, e.g.,

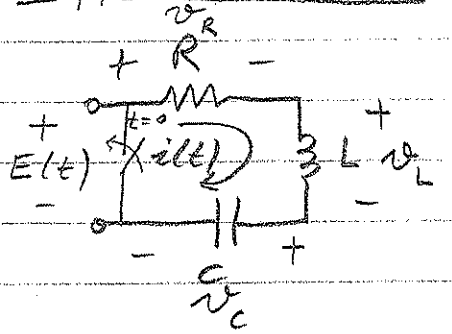
$$q(\lambda) = (\lambda+1), \quad \lambda^2-2a\lambda+(a^2+b^2) = \lambda^2-\epsilon\lambda+1, \ \epsilon>0$$

Then: $p(\lambda) = q(\lambda)\cdot(\lambda^2-2a\lambda+(a^2+b^2)) = (\lambda+1)(\lambda^2-\epsilon\lambda+1)$

$$= \lambda^3+(1-\epsilon)\lambda^2+(1-\epsilon)\lambda$$

if $0<\epsilon<1$, $p(\lambda)$ won't have mixed signed coefficients.

Application Example: Oscillator Circuit



$$E(t) = v_R(t)+v_L(t)+v_C(t)$$
$$v_R(t) = R\,i(t)$$
$$v_L(t) = L\frac{di(t)}{dt}$$
$$i(t) = C\frac{dv_C(t)}{dt} \Rightarrow v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$$

$$\Rightarrow E(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$$

$$\Rightarrow E'(t) = L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t)$$

For the free oscillator case: $E(t) = 0$:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Characteristic equation: $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$

$$\Rightarrow \lambda_1 = \frac{-R/L + \sqrt{R^2/L^2 - 4/LC}}{2}, \quad \lambda_2 = \frac{-R/L - \sqrt{R^2/L^2 - 4/LC}}{2}$$

(i) if $\frac{R^2}{L^2} - \frac{4}{LC} > 0 \Rightarrow R^2C - 4L > 0 \Rightarrow \boxed{R > 2\sqrt{\frac{L}{C}}}$.

then we have two real roots both smaller than 0.
Thus:

$$i(t) = C_1 e^{+\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad \lambda_1 < 0, \lambda_2 < 0$$

No oscillatory components: overdamped case

(ii) If $R = 2\sqrt{\frac{L}{C}}$, then have $\lambda_1 = \lambda_2 = -\frac{R}{2L}$

$$i(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$

No oscillatory component: critically damped case

(iii) If $R < 2\sqrt{\frac{L}{C}}$, then have complex conjugate pairs

$$\lambda_1 = \frac{-R/L + \sqrt{R^2/L^2 - 4/LC}}{2} = \frac{-R/L + j\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}}{2} = \lambda_2^*$$

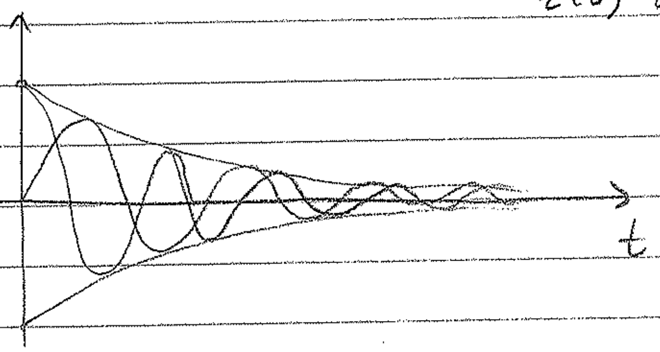
Then:

$$i(t) = e^{-\frac{R}{2L}t}\left(A\cos(\omega t) + B\sin(\omega t)\right), \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

time-constant

natural frequency

Underdamped oscillations. A and B can be determined once $i(0)$ and $i'(0)$ are given.



Laplace Transform*
(*Chapter 5 in the textbook)

General Linear integral transform is of the following form:

$$F(s) = \int_a^b K(t,s) f(t)\,dt\,, \quad K(t,s) : \text{Transformation kernel}$$

The Laplace transform: $\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$

* s: complex valued transform domain variable.

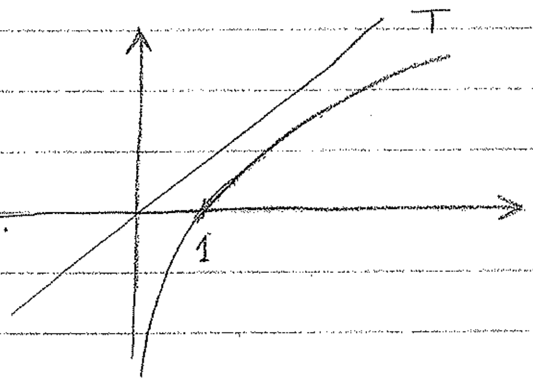
Similar to z-transform, which reduces difference equations to linear algebraic equations, the Laplace transform reduces the linear constant coefficient differential equations to linear algebraic equations.

Calculation of the transform:

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$, convergence is assured if $|f(t)| \le Ke^{c}$ for $t > T$.

ex/ $f(t) = \sin(wt)$ is exponential order because:
$$|f(t)| \le e^{0t} = 1, \quad t > 0.$$

ex/ $f(t) = t^2$, $\quad t^2 \le e^t$ for $t > T$ where $T^2 \le e^T$
$\Rightarrow 2 \ln T \le T$
$\Rightarrow \underline{T > 0}$.



Also, can check:

$$\lim_{t\to\infty} \frac{t^2}{e^t} = \lim_{t\to\infty} \frac{2t}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} =$$

l'Hopital's rule.

ex/ $f(t) = e^{t^2}$ is not of exponential order:

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{ct}} = \lim_{t\to\infty} e^{(t^2-ct)} = \infty, \text{ for any } c.$$

Theorem 5.2.1 For $f(t)$ satisfying:
(i) $f(t)$ is piecewise continuous on $0 \le t \le A$, with finite number of discontinuities,
(ii) $f(t)$ is of exponential order, $|f(t)| \le Ke^{ct}$, $t \ge T$,
the Laplace transform $F(s)$ exists for $Re\{s\} > c$.

Proof

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \underbrace{\int_0^T f(t)e^{-st}dt}_{\substack{\text{exists for all } s \\ \text{if (i) is valid.}}} + \int_T^\infty f(t)e^{-st}dt$$

$$\left|\int_T^\infty f(t)e^{-st}dt\right| \le \int_T^\infty |f(t)e^{-st}|\,dt \le \int_T^\infty |f(t)||e^{-st}|\,dt$$

$$\le \int_T^\infty Ke^{ct}e^{-Re\{s\}t}dt = K\int_T^\infty e^{-(Re\{s\}-c)t}dt = K\frac{e^{-(Re\{s\}-c)T}}{(Re\{s\}-c)} < \infty$$

for $Re\{s\} > c$.

ex/ $f(t) = e^{at}$: $F(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt$

$$= \frac{1}{s-a}, \quad Re\{s\} > a.$$

ex/ $f(t) = \sin at$: $F(s) = \int_0^\infty \sin at\, e^{-st}dt = \int_0^\infty \left(\frac{e^{jat}-e^{-jat}}{2j}\right)e^{-st}dt$

$$= \frac{1}{2j}\int_0^\infty e^{-(s-ja)t}dt + \frac{1}{2j}\int_0^\infty e^{-(s+ja)t}dt$$

$$= \frac{1}{2j}\frac{1}{(s-ja)} - \frac{1}{2j}\frac{1}{(s+ja)} = \frac{1}{2j}\frac{(s+ja)-(s-ja)}{(s^2+a^2)} = \frac{a}{s^2+a^2}$$

$Re\{s\} > 0.$

Notes

(i) The inverse Laplace transform is also an integral transform but requires techniques that will be introduced in EEE-242. Therefore, we will use the inspection technique like we did with the inversion of the z-transform.

(ii) The inverse of the Laplace transform is unique, i.e.,

$$\text{if} \quad f_1(t) \xleftrightarrow{\mathcal{L}} F_1(s)$$
$$f_2(t) \xleftrightarrow{\mathcal{L}} F_2(s)$$
$$\text{and} \quad \int_0^\infty |f_1(t) - f_2(t)|\, dt > 0, \text{ then } F_1(s) \neq F_2(s).$$

Properties of the Laplace Transform

(i) $L\{\alpha u(t) + \beta v(t)\} = \alpha L\{u(t)\} + \beta L\{v(t)\}$
is satisfied for $\forall \alpha, \beta, u(t)$ and $v(t)$ which are of exponential order.

(ii) $L^{-1}\{\alpha U(s) + \beta V(s)\} = \alpha L^{-1}\{U(s)\} + \beta L^{-1}\{V(s)\}$
is satisfied for $\forall \alpha, \beta, U(s)$ and $V(s)$.

(iii) $L\{f'(t)\} = s L\{f(t)\} - f(0)$

proof $$L\{f'(t)\} = \int_0^\infty f'(t) e^{-st} dt = f(t) e^{-st}\Big|_0^\infty + s\int_0^\infty f(t) e^{-st} dt$$
$$= -f(0) + sF(s).$$

(Note that this property is valid even if $f'(t)$ is piecewise cont.

(iv) $$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$
$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$\Rightarrow L\{f^{(k)}(t)\} = s^k L\{f(t)\} - \sum_{i=1}^{k} s^{(k-i)} f^{(i-1)}(0).$$

(useful in the solution to Differential Equations with given initial conditions)

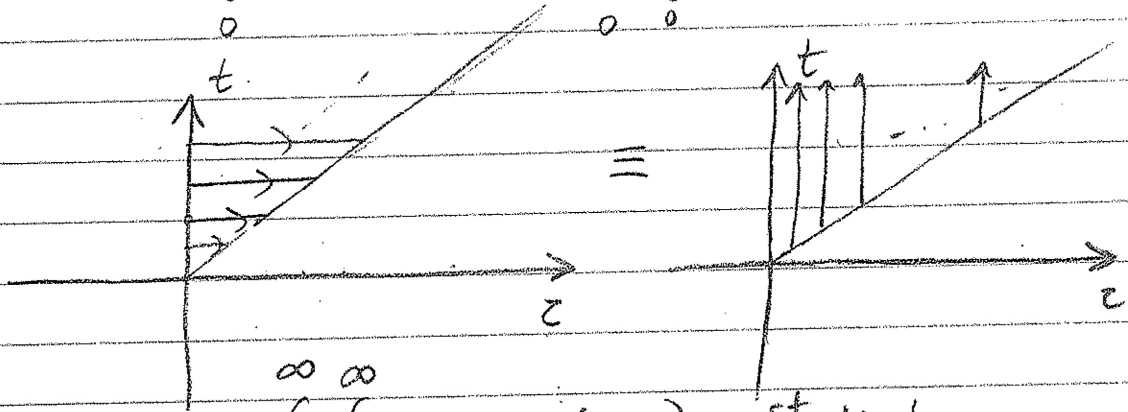
(v) Convolution property:

For $f(t)$ and $g(t)$ which are zero for $t<0$, their convolution is defined as:

$$h(t) = f(t) * g(t) = (f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

The Laplace transform of $h(t)$ is:

$$H(s) = \int_0^\infty h(t)\, e^{-st} dt = \int_0^\infty \int_0^t f(\tau)\, g(t-\tau)\, d\tau\, e^{-st} dt$$



$$\Rightarrow H(s) = \int_0^\infty \int_\tau^\infty f(\tau)\, g(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_0^\infty f(\tau) \left[\int_\tau^\infty g\,(\underbrace{t-\tau}_{\mu})\, e^{-st}\, dt \right] d\tau$$

$$= \int_0^\infty f(\tau) \left[\int_0^\infty g(\mu)\, e^{-s(\tau+\mu)}\, d\mu \right] d\tau$$

$$= \int_0^\infty f(\tau) \left[\int_0^\infty g(\mu) e^{-s\mu}\, d\mu \right] e^{-s\tau} d\tau$$

$$= \left[\int_0^\infty f(\tau) e^{-s\tau} d\tau \right] \left[\int_0^\infty g(\mu) e^{-s\mu} d\mu \right]$$

$$\boxed{H(s) = F(s) \cdot G(s)}$$

ex/ $F(s) = \dfrac{3}{s^2+3s-10} = \dfrac{3}{(s+5)(s-2)}$

(i) $\dfrac{3}{(s+5)(s-2)} = \dfrac{3}{7}\left[\dfrac{1}{s-2} - \dfrac{1}{s+5}\right]$

$$\Rightarrow f(t) = \frac{3}{7}\left(e^{2t} - e^{-5t}\right), \quad t \geq 0.$$

(ii) $\dfrac{3}{(s+5)(s-2)} = 3 \cdot \dfrac{1}{(s+5)} \cdot \dfrac{1}{(s-2)}$

$$\Rightarrow f(t) = 3 \cdot \left(e^{-5t} * e^{2t}\right)$$

$$= 3 \cdot \int_0^t e^{2\tau} e^{-5(t-\tau)} d\tau$$

$$= 3 \cdot e^{-5t} \int_0^t e^{7\tau} d\tau = 3e^{-5t} \frac{1}{7}\left(e^{7\tau}\Big|_0^t\right)$$

$$= \frac{3}{7} e^{-5t}\left(e^{7t} - 1\right)$$

$$= \frac{3}{7}\left(e^{2t} - e^{-5t}\right), \quad t \geq 0.$$

Solution to Linear Constant Coefficient Differential Equations by Laplace Transform

Ex $\quad x'' + ax' + bx = f(t)$

$$\left(s^2 X(s) - sx(0) - x'(0)\right) + a\left(sX(s) - x(0)\right) + bX(s) = F(s)$$

$$\Rightarrow (s^2+as+b)\,X(s) = s\,x(0) + a\,x(0) + x'(0) + F(s)$$

$$\Rightarrow X(s) = \underbrace{\frac{(s+a)\,x(0) + x'(0)}{(s^2+as+b)}}_{\text{invert by partial fraction expansion}} + \underbrace{\frac{F(s)}{(s^2+as+b)}}_{\text{invert by using either partial fraction expansion or convolution property.}}$$

ex/ $x' + px = q(t)$, $x(0) = x_0$

In Laplace Transform domain:

$$sX(s) - x(0) + pX(s) = Q(s)$$

$$\Rightarrow \quad X(s) = \frac{x_0}{s+p} + \frac{Q(s)}{s+p}$$

$$\Rightarrow \quad x(t) = x_0 e^{-pt} + \int_0^t q(\tau) e^{-p(t-\tau)} d\tau$$

$$x(t) = x_0 e^{-pt} + e^{-pt} \int_0^t q(\tau) e^{-p\tau} d\tau$$

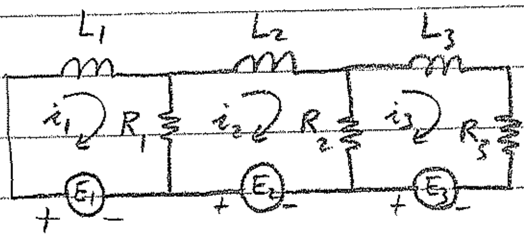
$$x(t) = \left[x_0 + \int_0^t q(\tau) e^{-p\tau} d\tau \right] e^{-pt}$$

· Systems of Linear Differential Equations

$$\left.\begin{array}{l} a_{11}(t)x_1' + \cdots + a_{1n}(t)x_n' + b_{11}(t)x_1 + \cdots + b_{1n}(t)x_n = f_1(t) \\ \vdots \\ a_{n1}(t)x_1' + \cdots + a_{nn}(t)x_n' + b_{n1}(t)x_1 + \cdots + b_{nn}(t)x_n = f_n(t) \end{array}\right\} \text{forcing functio}$$

$a_{ij}(t)$ and $b_{ij}(t)$ are known coefficients.

ex/



$$\left.\begin{array}{l} L_1 i_1' + R_1(i_1 - i_2) = E_1(t) \\ L_2 i_2' + R_2(i_2 - i_3) + R_1(i_2 - i_1) = E_2(t) \\ L_3 i_3' + R_3 i_3 + R_2(i_3 - i_2) = E_3(t) \end{array}\right\} \begin{array}{l} L_1 i_1' + R_1 i_1 - R_1 i_2 = E_1(\\ L_2 i_2' - R_1 i_1 + (R_2+R_1) i_2 - R_2 i_3 = E_2(\\ L_3 i_3' - R_2 i_2 + (R_2+R_3) i_3 = E_3(t \end{array}$$

Theorem 3.9.1 Let $a_{ij}(t)$, $1 \le i, j \le n$ and $f_i(t)$, $1 \le i \le n$ be continuous on a closed interval I. Also, let $x_i(t_0) = b_i$, $1 \le i$ for $t_0 \in I$. Then the system:

$$x_1' = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t)$$

$$\vdots$$

$$x_n' = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t)$$

has a unique solution on the entire interval I.

Note that (i) The left hand side has individual derivatives. This form can be obtained by using Gauss-Jordan elimination on the original system.

(ii) The system can be written as:

$$\underline{x}' = \underline{\underline{A}}(t)\,\underline{x} + \underline{f}$$

where $\underline{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\underline{\underline{A}}(t) = [a_{ij}(t)]_{n\times n}$, $\underline{f} = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

ex/

$$\left.\begin{aligned} x_1' &= x_1 + x_2 + 3t \\ x_1' + x_2' &= 5x_1 + 2x_2 + 5 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1' &= x_1 + x_2 + 3t \\ x_2' &= 4x_1 + x_2 + 5 - 3t \end{aligned}$$

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3t \\ 5-3t \end{bmatrix}}_{\text{In Laplace transform domain}}$$

$$(s\underline{X} - \underline{x}_0) = \underline{\underline{A}}\,\underline{X} + \underline{F}, \quad \underline{X} = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

$$\underline{F} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

where $F_1(s) = \int_0^\infty 3t e^{-st}\,dt$

$$= \underbrace{-\frac{3te^{-st}}{s}\Big|_0^\infty}_{0} + \frac{3}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{-\frac{1}{s}e^{-st}\big|_0^\infty} = \frac{3}{s^2}$$

$$F_2(s) = \int_0^\infty (5-3t)e^{-st}\,dt = \frac{5}{s} - \frac{3}{s^2}$$

$$\Rightarrow (s\underline{\underline{I}} - \underline{\underline{A}})\,\underline{X} = \underline{x}_0 + \underline{F}$$

$$\underline{X} = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}(\underline{x}_0 + \underline{F})$$

where $(s\underline{\underline{I}} - \underline{\underline{A}}) = \begin{bmatrix} s-1 & -1 \\ -4 & s-1 \end{bmatrix}$

$$(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} = \frac{1}{(s^2 - 2s - 3)}\begin{bmatrix} s-1 & 1 \\ 4 & s-1 \end{bmatrix}$$

$$\Rightarrow \quad \underline{X} = \frac{1}{(s-3)(s+1)} \begin{bmatrix} s-1 & 1 \\ 4 & s-1 \end{bmatrix} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3/s^2 \\ \frac{5}{s} - \frac{3}{s^2} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \dfrac{(s-1)x_1(0) + x_2(0)}{(s-3)(s+1)} \\[2ex] \dfrac{4x_1(0) + (s-1)x_2(0)}{(s-3)(s+1)} \end{bmatrix} + \begin{bmatrix} \dfrac{(s-1)3}{s^2(s-3)(s+1)} + \dfrac{(5s-3)}{s^2(s-3)(s+1)} \\[2ex] \dfrac{12}{s^2(s-3)(s+1)} + \dfrac{(5s-3)(s-1)}{s^2(s-3)(s+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2x_1(0)+x_2(0)}{4(s-3)} + \dfrac{2x_1(0)-x_2(0)}{4(s+1)} \\[2ex] \dfrac{x_2(0)+2x_1(0)}{2(s-3)} + \dfrac{x_2(0)-2x_1(0)}{2(s+1)} \end{bmatrix} + \begin{bmatrix} -\dfrac{4}{s} + \dfrac{2}{s^2} + \dfrac{1/2}{(s-3)} + \dfrac{7/2}{(s+1)} \\[2ex] \dfrac{10/3}{s} - \dfrac{1}{s^2} + \dfrac{2/3}{(s-3)} - \dfrac{4}{(s+1)} \end{bmatrix}$$

Thus;
$$\underline{x}(t) = \begin{bmatrix} \left(\dfrac{2x_1(0)+x_2(0)}{4}\right)e^{+3t} + \left(\dfrac{2x_1(0)-x_2(0)}{4}\right)e^{-t} \\[2ex] \dfrac{(x_2(0)+2x_1(0))}{2}e^{3t} + \dfrac{(x_2(0)-2x_1(0))}{2}e^{-t} \end{bmatrix}$$
} Given $x_1(0)$ and $x_2(0)$ this part is determined

$$+ \begin{bmatrix} -4 + 2t + \frac{1}{2}e^{3t} + \frac{7}{2}e^{-t} \\[2ex] \frac{10}{3} - t + \frac{2}{3}e^{3t} - 4e^{-t} \end{bmatrix}$$
} Particular solution corresponding to zero initial conditions

There is another alternative approach based on state-transition matrices.

Want to solve:

$$\underline{x}' = \underline{\underline{A}}(t)\,\underline{x} + \underline{f}(t) \quad \text{with initial values } \underline{x}(0) = \underline{x}_0.$$

Observations

(i) The set of solutions to the homogeneous system:

$$\underline{x}' = \underline{A}(t)\,\underline{x}$$

is an n-dimensional subspace of all continuous functions.

(ii) Can show this easily by showing that if $\underline{x}_1(t)$ is a solution so is $\alpha x_1(t)$, and if $x_2(t)$ is a solution so is $x_1(t) + x_2(t)$ as well

(ii) Given n linearly independent solutions of $\underline{x}' = \underline{\underline{A}}(t)\underline{x}$ $\{\phi_1(t), \ldots, \phi_n(t)\}$, then:

$\underline{x}(t) = \alpha_1 \underline{\phi}_1(t) + \cdots + \alpha_n \underline{\phi}_n(t)$ for arbitrary α_i generates all the solutions. In matrix form:

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{\alpha}$$

where $\underline{\underline{\Phi}}(t) = \begin{bmatrix} \underline{\phi}_1(t) & \cdots & \underline{\phi}_n(t) \end{bmatrix}$, $\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

(iii) Notation: $\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0) = \left[\underline{\phi}(t; t_0, \underline{x}_1) \cdots \underline{\phi}(t; t_0, \underline{x}_n) \right]$

where $\underline{\phi}(t; t_0, \underline{x}_i)$ is the solution to:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}, \text{ where } \underline{x}(t_0) = \underline{x}_i$$

(iv) A special case of fundamental matrix $\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)$ is for $\underline{\underline{X}}_0 = \underline{\underline{I}}$ is called as the <u>state-transition matrix</u>

$$\underline{\underline{\Phi}}(t; t_0) = \left[\underline{\phi}(t; t_0, \underline{e}_1) \cdots \underline{\phi}(t; t_0, \underline{e}_n) \right]$$

(v) The unique solution to $\underline{x}' = \underline{\underline{A}}(t)\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$ can be found as:

$$\underline{\phi}(t; t_0, \underline{x}_0) = \sum_{i=1}^{n} \underline{\phi}(t; t_0, \underline{x}_i)\alpha_i = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\underline{\alpha}$$

where $\underline{x}_0 = \alpha_1 \underline{x}_1 + \cdots + \alpha_n \underline{x}_n = \underbrace{[\underline{x}_1 \cdots \underline{x}_n]}_{\underline{\underline{X}}_0} \underline{\alpha} \Rightarrow \underline{\alpha} = \underline{\underline{X}}_0^{-1} \underline{x}_0$.

$$\Rightarrow \underline{\phi}(t; t_0, \underline{x}_0) = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\underline{\underline{X}}_0^{-1}\underline{x}_0$$

$$\Rightarrow \underline{\phi}(t; t_0, \underline{x}_0) = \underbrace{\underline{\underline{\Phi}}(t; t_0)}_{\text{state-transition matrix}} \underline{x}_0$$

$$\Rightarrow \boxed{\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0) = \underline{\underline{\Phi}}(t; t_0)\underline{\underline{X}}_0}$$

Note that

(i) $\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)$ is non-singular for all t and t_0 in I, the solution interval

proof Assume that it is singular for some t_1 and t_0 in I.

then: $\underline{\underline{\Phi}}(t_1; t_0, \underline{\underline{X}}_0)\, \underline{c} = \underline{0}$ for $\underline{c} \neq \underline{0}$.

Now let $\underline{\psi}(t) = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\, \underline{c}$

Then $\underline{\psi}(t)$ is the unique solution to $\underline{x}' = \underline{\underline{A}}(t)\underline{x}$, $\underline{x}(t_1) = \underline{0}$

Thus $\underline{\psi}(t) = \underline{0}$ ← zero function

$\Rightarrow \underline{\psi}(t_0) = \underline{\underline{X}}_0\, \underline{c} = \underline{0} \Rightarrow \underline{\underline{X}}_0$ is singular $\Rightarrow\!\Leftarrow$.

(ii) $\underline{\underline{\Phi}}(t; t_1)\, \underline{\underline{\Phi}}(t_1; t_0) = \underline{\underline{\Phi}}(t; t_0)$ for all t, t_1, t_0 in I.

proof Let $\underline{\underline{\Psi}}(t) = \underline{\underline{\Phi}}(t; t_1)\, \underline{\underline{\Phi}}(t_1; t_0)$, then

$$\underline{\underline{\Psi}}' = \underline{\underline{\Phi}}'(t; t_1)\, \underline{\underline{\Phi}}(t_1; t_0) = \underline{\underline{A}}(t)\, \underbrace{\underline{\underline{\Phi}}(t; t_1)\, \underline{\underline{\Phi}}(t_1; t_0)}_{\underline{\underline{\Psi}}(t)}.$$

$$\Rightarrow \underline{\underline{\Psi}}' = \underline{\underline{A}}(t)\, \underline{\underline{\Psi}}(t)$$

and

$$\underline{\underline{\Psi}}(t_1) = \underbrace{\underline{\underline{\Phi}}(t_1; t_1)}_{\underline{\underline{I}}}\, \underline{\underline{\Phi}}(t_1; t_0) = \underline{\underline{\Phi}}(t_1; t_0).$$

Thus, $\underline{\underline{\Psi}}(t)$ is the unique solution to $\underline{\underline{X}}' = \underline{\underline{A}}(t)\underline{\underline{X}}$, $\underline{\underline{X}}(t_1) = \underline{\underline{\Phi}}(t_1; t_0)$

but, $\underline{\underline{\Phi}}(t; t_0)$ is also a solution to this problem.

Hence: $\underline{\underline{\Psi}}(t) = \underline{\underline{\Phi}}(t; t_0) \Rightarrow \underline{\underline{\Phi}}(t; t_0) = \underline{\underline{\Phi}}(t; t_1)\, \underline{\underline{\Phi}}(t_1; t_0)$

(iii) Since, $\underline{\underline{\Phi}}(t_0, t_0) = \underline{\underline{I}} = \underline{\underline{\Phi}}(t_0; t_1)\, \underline{\underline{\Phi}}(t_1; t_0)$ $\forall\, t_0, t_1 \in I$

$$\Rightarrow \underline{\underline{\Phi}}^{-1}(t_1; t_0) = \underline{\underline{\Phi}}(t_0; t_1), \text{ or } \boxed{\underline{\underline{\Phi}}^{-1}(t; t_0) = \underline{\underline{\Phi}}(t_0; t)}$$

The non-homogeneous Case

$$\underline{x}' = \underline{\underline{A}}(t)\,\underline{x} + \underline{f}(t).$$

Let $\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)$ be any fundamental matrix corresponding to the homogeneous case, then:

$\underline{x} = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}$ solves the homogeneous problem.

Idea: To obtain solution to the non-homogeneous case use the variation of parameters technique and postulate a solution as:

$$\underline{x} = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}(t)$$

Then, substitute $\underline{x}$ back in the non-homogeneous problem:

$$\underline{x}' = \underline{\underline{\Phi}}'(t; t_0, \underline{\underline{X}}_0)\,\underline{v}(t) + \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}'(t)$$

$$= \underline{\underline{A}}(t)\,\underbrace{\underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}(t)}_{\underline{x}} + \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}'(t)$$

$$= \underline{\underline{A}}(t)\,\underline{x} + \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}'(t)$$

$$\Rightarrow \quad \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,\underline{v}'(t) = \underline{f}(t)$$

$$\Rightarrow \quad \underline{v}'(t) = \underline{\underline{\Phi}}^{-1}(t; t_0, \underline{\underline{X}}_0)\,\underline{f}(t)$$

$$\Rightarrow \quad \underline{v}(t) = \underbrace{\int \underline{\underline{\Phi}}^{-1}(t; t_0, \underline{\underline{X}}_0)\,\underline{f}(t)\,dt}_{\underline{V}(t)} + \underline{c} = \underline{V}(t) + \underline{c}$$

Hence: $\underline{x} = \underline{\underline{\Phi}}(t; t_0, \underline{\underline{X}}_0)\,(\underline{V}(t) + \underline{c})$

$$= \underline{\phi}_p(t) + \underline{\phi}_c(t)$$

particular solution   complementary solution

If we are given initial condition $\underline{x}(t_0)=\underline{x}_0$, then choose $\underline{\underline{\Phi}}(t;t_0,\underline{X}_0)$ as the state transition matrix to obtain:

$$\underline{V}(t)=\int_{t_0}^{t}\underline{\underline{\Phi}}^{-1}(\tau,t_0)f(\tau)\,d\tau=\int_{t_0}^{t}\underline{\underline{\Phi}}(t_0;\tau)f(\tau)\,d\tau$$

Then $\underline{V}(t_0)=\underline{0}$, hence $\underline{c}$ must be $\underline{x}_0$. Thus:

$$\underline{x}=\underbrace{\underline{\underline{\Phi}}(t;t_0)\,\underline{x}_0}_{\underline{\underline{\Phi}}_0(t)}+\underbrace{\int_{t_0}^{t}\underline{\underline{\Phi}}(t;\tau)\cdot f(\tau)\,d\tau}_{\underline{\underline{\Phi}}_f(t)}$$

due to initial condition $\qquad$ due to input

(Note that: $\underline{\underline{\Phi}}(t;t_0)\cdot\underline{\underline{\Phi}}(t_0;\tau)=\underline{\underline{\Phi}}(t;\tau)$)

ex/ Find the solution to the following initial value problem:

$$\underline{x}'=\begin{bmatrix}-1 & 1\\ -2 & 1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\ 2\end{bmatrix}u(t),\quad \underline{x}(0)=\underline{0},\quad u(t)=\begin{cases}1 & t>0\\ 0 & t\le 0\end{cases}$$

Since $\underline{\phi}_1=\begin{bmatrix}\cos t\\ \cos t-\sin t\end{bmatrix}$, $\underline{\phi}_2=\begin{bmatrix}\sin t\\ \cos t+\sin t\end{bmatrix}$ are two linearly independent solutions to the homogeneous problem a fundamental matrix can be obtained as:

$$\underline{\underline{\Phi}}(t)=\left[\underline{\phi}_1(t)\ \ \underline{\phi}_2(t)\right]=\begin{bmatrix}\cos t & \sin t\\ \cos t-\sin t & \cos t+\sin t\end{bmatrix}$$

Then, the state transition matrix can be obtained as:

$$\underline{\underline{\Phi}}(t;t_0)=\underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}^{-1}(t_0)=\begin{bmatrix}\cos t & \sin t\\ \cos t-\sin t & \cos t+\sin t\end{bmatrix}\begin{bmatrix}\cos t_0+\sin t_0 & -\sin t_0\\ -\cos t_0+\sin t_0 & \cos t_0\end{bmatrix}$$

$$=\begin{bmatrix}\cos(t-t_0)-\sin(t-t_0) & \sin(t-t_0)\\ -2\sin(t-t_0) & \cos(t-t_0)+\sin(t-t_0)\end{bmatrix}$$

For $t>0$: $u(t)=1$.

$$\underline{x} = \underline{\underline{\Phi}}(t,0)\,\underline{v}(t)$$

$$\underline{v}'(t) = \underline{\underline{\Phi}}^{-1}(t;0)\,\underline{f}(t) = \begin{bmatrix} \cos t+\sin t & -\sin t \\ -\cos t+\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t>0$$

$$= \begin{bmatrix} \cos t - \sin t \\ \cos t + \sin t \end{bmatrix}$$

$$\Rightarrow \underline{v}(t) = \begin{bmatrix} \sin t + \cos t + c_1 \\ \sin t - \cos t + c_2 \end{bmatrix}$$

Hence: $\underline{x} = \underline{\underline{\Phi}}(t;0)\,\underline{v}(t) = \begin{bmatrix} 1 + c_1 \cos t + c_2 \sin t \\ (c_1+c_2)\cos t + (c_2-c_1)\sin t \end{bmatrix}$

Impose the initial conditions to solve c_1 & c_2:

$$\underline{x}(0)=\underline{0} \Rightarrow \begin{bmatrix} 1+c_1 \\ c_1+c_2 \end{bmatrix} = \underline{0} \Rightarrow c_1=-1,\ c_2=1.$$

Hence, the solution becomes:

$$\underline{x}(t) = \begin{bmatrix} 1-\cos t+\sin t \\ 2\sin t \end{bmatrix}, \quad t\geq 0.$$

Alternatively

$$x(t) = \int_0^t \underline{\underline{\Phi}}(t;\tau)\,\underline{f}(\tau)\,d\tau = \int_0^t \begin{bmatrix} \cos(t-\tau)-\sin(t-\tau) & \sin(t-\tau) \\ -2\sin(t-\tau) & \cos(t-\tau)+\sin(t-\tau) \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} \cos(t-\tau)+\sin(t-\tau) \\ 2\cos(t-\tau) \end{bmatrix} d\tau = \begin{bmatrix} 1-\cos t+\sin t \\ 2\sin t \end{bmatrix}, \text{ Same as above.}$$