BALANCES OF POWER FROM STATIC EQUILIBRIA

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Abstract: In a world of n states, various notions of balance of power are examined with no special assumptions about states' behavior. Three definitions of static equilibrium yield three different concepts of balance: *balance, perfect balance*, and *threat-balance*, all legitimate in giving a meaning to balance of power within the realm of the simple model used. All three indicate the *capacity* of states, individually or collectively, to realize equal allocations against each other by so apportioning their resources. For each notion of balance, a characterization in terms of resource distribution is given and its relation with equilibrium allocations is rigorously analyzed. Most conditions for stability that have appeared in the balance of power literature can be understood, through these three notions of balance, without any specific reference to states' behavior.

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1 INTRODUCTION

This study is motivated by a desire to distinguish between those conditions for stability which result as consequences of behavioral assumptions and those that are somewhat more fundamental or structural. Towards this, we adopt a model which does not impose any behavior on the states. While this model may be a naive representation of reality, it is common to a number of more sophisticated models that appeared in the literature. The results obtained by studying this common denominator complement the aforementioned attempts towards identifying premises that best describe states in conflict.

The following are the defining features of the model. There is a world of n states having fixed nonnegative resources $r_1, ..., r_n$. The state i has full control over r_i and none over the resources of other states. Each state however knows all resource values. Each state allocates all of its resource against the remaining states so that each r_i is infinitely divisible and the allocated portions add up to r_i. The model itself is the set of all $n \times n$ matrices A with its ij-th entry $A(i, j)$ equal to the resource allocated by i against j. The i-th row sum of the matrix A is the resource r_i controlled by the state i.

Let us say that the world is "in equilibrium" if the resources are so apportioned that any two states have equal amounts allocated against each other, or equivalently, if the matrix A is symmetric. If an equilibrium exists, then let us say that the world is "balanced[1]". It is crucial to note the distinction between the equilibrium and the balance. The equilibrium has to do with allocated (apportionment, targeting) resources and the balance with resources controlled by states and with their relative magnitudes.[2]

Whether states are willing to achieve an equilibrium or not, whether they are cooperative or not, the following questions seem legitimate to ask on this simple model.

Q1. When is the world balanced? Equivalently, does there exist an equilibrium?

Q2. What are the allocations that achieve an equilibrium provided the world is balanced?

If $n = 2$, the answers are obvious: The world is balanced if and only if the two states have equal amounts of resources in which case an equilibrium is achieved with full allocations against each other. If $n \geq 3$, then Q1 and Q2 are answered by Theorem 1 below. The answer to Q1 turns out to be one of the most frequently encountered conditions in the literature, known as a "no predominance" or "no hegemony" condition. A world of n states is balanced precisely when no country controls more than half of the total world resource. Thus, the world is balanced in a continuum of situations ranging from equality of resources to the existence of a state controlling exactly one half of the total world resource. In Aumann and Machler (1964) and Niou and Ordeshook (1986), for $n = 3$, the condition characterizes, respectively, discrete bargaining set (with a different interpretation of r_i's!) and system stability. For larger n, the condition is necessary for system stability of Niou

[1]Both terms "equilibrium" and "balance" are loaded with a plethora of different meanings. In this paper, we do not intend to mean anything more than what is attributed by their formal definitions.

[2]We elaborate more on this point in Section 2.

and Ordeshook (1986) (but not sufficient). The main contention of Zinnes (1970) is that this no hegemony condition should be taken as the definition of balance of power. Theorem 1 clarifies exactly what type of balance this condition avails, namely, the capacity of states for equilibrium allocations.

Although the necessity of no hegemony condition for a balanced world is seen on a moment's reflection, the sufficiency which amounts to answering Q2 is more difficult. It turns out that while for $n = 3$ a unique allocation scheme achieves equilibrium, for $n \geq 4$, there are infinitely many schemes achieving equilibrium unless one state controls exactly one half of the total world resource. This "near predominance" or "almost hegemony" condition characterizes resource stability in the (static) cooperative game of Niou and Ordeshook (1986) and the perfect equilibrium in the noncooperative game of Wagner (1986). An interesting interpretation of the uniqueness result of Theorem 1 in terms of the cooperative n-person game for $n \geq 4$ of Niou and Ordeshook (1986) is that *the existence of unique and nonunique equilibrium allocations correspond, respectively, to the game being inessential and essential.* In fact, if the condition (5) holds with strict inequality, then there are infinitely many equilibrium allocations and the cooperative n-person game of Niou and Ordeshook (1986) has no core. If (5) holds with equality for some i, on the other hand, there is a unique choice of equilibrium allocations and the game of Niou and Ordeshook (1986) has a core.

If the resource values do not allow a balanced world, i.e., when there is a hegemon, then we may investigate what allocations yield as close an equilibrium as possible. This is asking for an optimal[3] equilibrium. In case $n = 2$, since the model allows no other possibility for the states but full allocations against each other, this scheme is optimal under any criterion of optimality. Theorem 2 shows that if "closeness" is quantified as minimizing the maximum disequilibrium among pairs of states, then the optimal allocations are as follows. If there is a hegemon, then it distributes its resource "evenly" among the weaker states while facing full allocations by every one of them. If there is no hegemon, then the optimal allocations are the equilibrium allocations of Theorem 1. The somewhat limited conclusion (since this is only one among infinitely many notions of optimality) one can draw from this result is that, the weak states take no action against each other for an optimal equilibrium in the presence of a hegemon.

The notion of balance we temporarily agreed upon is by no means the only one that can be allowed in our simple model. By preserving an equal treatment to every state, i.e., the symmetry among states, one can consider a stronger notion of equilibrium and balance. For want of a better term, let us say that the world is "in perfect equilibrium" if the total allocations by every pair of disjoint subsets of states, consisting of an equal number of members, against each other is the same. The world is called "perfectly balanced" if the resource values permit a perfect equilibrium. Here, the allocated resource by a subset of states to another subset of states is the sum of the allocations of the member states. After resolving some ambiguities this definition contains, we may ask the same questions as Q1 and Q2 for perfect balance. It turns out that (see Theorem 3) a perfectly balanced

[3] Although optimal means "the best", the entrenched usage does not carry any such association as there are many alternative criteria of optimality.

world is possible just in case the sum of the resources of the two weakest states is no less than the arithmetic average (the mean) of the resources of remaining states. If this condition holds, then there is a unique scheme of allocations, i.e., there is exactly one possible way of apportioning the resources, achieving the perfect equilibrium in any world of n states. The significance of the notion of perfect balance lies in this latter result. The nonuniqueness of the allocation schemes achieving equilibrium for $n \geq 4$ leaves space for considering a stronger notion of balance. Perfect balance is just strong enough a notion that narrows the allocation schemes down to a single one. At the time of the writing of this article, the relevance of perfect balance, or the condition for it, to more sophisticated game theoretic models is not clear to us.

The last definition of balance we examine is motivated by the concept of system stability of Niou and Ordeshook (1986). The definition requires incorporation of coalitions into our model. A coalition is simply a subset of states with the resource of a coalition taken to be the sum of the resources of member states. The world of n states is called "threat-balanced" if for every state there is a division of the remaining states into two coalitions such that the reduced world of three states is balanced. It turns out (see Theorem 4) that the world is threat-balanced if and only if it is system stable in the sense of Niou and Ordeshook. This also explains our reference to such a balance as threat-balance since in a system stable world each state faced by a threat has a viable counterthreat (Niou and Ordeshook, 1986).

The following are the main conclusions reached by our analysis:

1. Understanding the balance of power as the absence of a hegemon can be justified on the grounds that the lack of a hegemon is equivalent to the states being able to apportion (target) their resources to attain equal allocations in every one-against-one confrontation.

2. The absence of a hegemon, however, is not the only possible way of associating balance with equal allocations. Perfect balance which is the capacity of states to achieve a stronger equilibrium, *viz.*, equality of allocations between every pair of subsets of states with the same cardinality, is also a possibility.

3. The existence of an almost hegemon, which appears as a condition for resource stability of Niou and Ordeshook as well as subgame perfect equilibrium of Wagner, can be given a meaning devoid of any behavioral association in a world of more than three states, namely, it is a necessary and sufficient condition for the existence of a unique equilibrium allocation scheme.

4. The system stability of Niou and Ordeshook is another concept which has a structural meaning. It is a necessary and sufficient condition for a reduced world of three states consisting of the weakest state and two coalitions formed by the remaining states to be balanced.

5. Optimal allocation schemes towards achieving an equilibrium yield somewhat trivial and expected results in case a hegemon exists. If there is no hegemon, then

all straightforward definitions of optimality yield that optimal allocations are the equilibrium allocations.

6. Both perfect and threat-balance require viewing the world of n states as a reduced world of three effective states. Moreover, the two and three state worlds are distinguished by the uniqueness of equilibrium allocations whenever they exist. These indicate that conclusions reached by studying two and three country worlds can in many cases be easily extended to n country worlds. The recent concentration of efforts in Bueno de Mesquita and Lalman (1992), Wagner (1994), Fearon (1995), and Powell (1996) is thus on the right track.

2 A BALANCED WORLD

Given a set of states $\mathcal{N} = \{1, ..., n\}$, suppose that the i-th state has a resource $r_i \geq 0$ which it uses to allocate against the remaining states $\mathcal{N} - \{i\}$. Let a_{ij} denote the allocation of state i against the state j. The whole r_i is distributed among the remaining states so that a_{ij}'s satisfy

$$\sum_{j \in \mathcal{N} - \{i\}} a_{ij} = r_i \;\; \forall \;\; i \in \mathcal{N} \tag{1}$$

and

$$a_{ij} \geq 0 \;\; \forall \;\; \{i, j\} \subset \mathcal{N}. \tag{2}$$

We call a set of allocations $\{a_{ij} : \{i, j\} \subset \mathcal{N}\}$ a *b-equilibrium* ("b" for "bilateral") if (1), (2) hold and

$$a_{ij} = a_{ji} \;\; \forall \;\; \{i, j\} \subset \mathcal{N}. \tag{3}$$

Figure 1 illustrates a five-state-system at b-equilibrium. The lines emanating from a state denote its respective allocations and the sum of their values equals the total resource of that state. As two emanating lines are connected, this is interpreted as the equality of bilateral allocations. This section is mainly concerned with the question: *Given n states with resources $r_1, ..., r_n$, does there exist a b-equilibrium?* If the answer is in the affirmative, then such an n-state-world will be called *balanced.*

It is clear that the answer is not always in the affirmative. In fact, in the trivial case of $n = 2$, the admissible allocations are $a_{12} = r_1$ and $a_{21} = r_2$ due to the restrictions (1). Thus, (2) holds and the world is balanced if and only if $r_1 = r_2$. By way of emphasizing the distinction between b-equilibrium and balance, we note that in a balanced world of more than two states both equilibrium and disequilibrium allocations exist. In the world of three states with resources $r_1 = 8, r_2 = 6, r_3 = 4$, allocations $a_{12} = a_{21} = 5$,

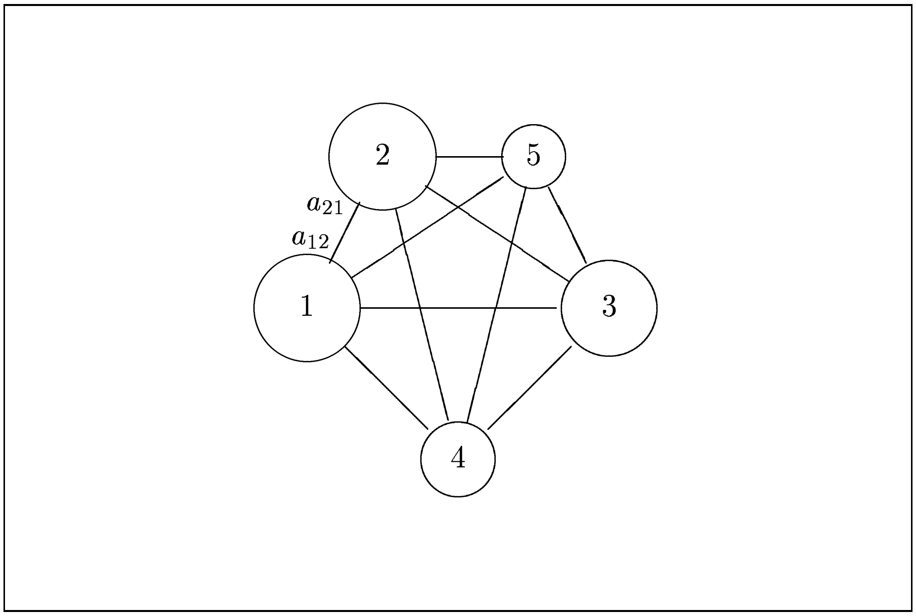

Figure 1: Bilateral-Equilibrium

$a_{13} = a_{31} = 3$, $a_{23} = a_{32} = 1$ yield a b-equilibrium. If the first state changes its allocations to $a_{12} = 4, a_{13} = 4$ the world is no longer in b-equilibrium. This world is balanced since there is at least one b-equilibrium. On the other hand, the world of three states of resources $r_1 = 10, r_2 = 6, r_3 = 3$ is not balanced since a b-equilibrium does not exist. [4] Note that if a state k has no resource, i.e., $r_k = 0$, then $a_{ki} = 0$ for all $i \neq k;\ i \in \mathcal{N}$ and for equilibrium the choices $a_{ik} = 0;\ i \neq k;\ i \in \mathcal{N}$ are enforced by (3). Consequently, the problem is reduced to the same problem with $n-1$ states and there is no loss of generality in assuming

$$r_i > 0 \ \ \forall \ \ i \in \mathcal{N}. \tag{4}$$

Theorem 1. *A world of n states with resources $r_1, ..., r_n$ satisfying (4) is balanced if and only if*

$$r_i \leq \sum_{j \in \mathcal{N} - \{i\}} r_j \ \ \forall \ \ i \in \mathcal{N}. \tag{5}$$

(i) If (5) holds and $n \leq 3$, then there is a unique b-equilibrium.

(ii) If (5) holds and $n > 3$, then there is a unique b-equilibrium if and only if equality holds in (5) for some $i \in \mathcal{N}$.

[4] Mathematically, the problem is to determine when the set of $n(n+1)/2$ equations (1), (3) in $n(n-1)$ unknowns have a solution satisfying the constraint (2). Although there are many algorithms to determine a solution to such problems (Chvátal, 1983), we are interested in determining a solvability condition in terms of the problem data $r_1, ..., r_n$.

(iii) If (5) holds, then there is always a b-equilibrium in which $n(n-3)$ of the a_{ij}'s are zero.

A proof of Theorem 1 is given in the Appendix. Here, we give a set of a_{ij}'s achieving a b-equilibrium provided (5) holds. Let us number the states in such a way that

$$r_1 \geq r_2 \geq ... \geq r_n. \tag{6}$$

Let k be the smallest integer such that the following inequalities are satisfied:

$$\sum_{i=1}^{k} r_{2i-1} \leq \sum_{j=1}^{n} r_j - \sum_{i=1}^{k} r_{2i-1}, \tag{7}$$

$$\sum_{i=1}^{k+1} r_{2i-1} \geq \sum_{j=1}^{n} r_j - \sum_{i=1}^{k+1} r_{2i-1}. \tag{8}$$

Thus k is such that the sum of the resources of the odd-numbered states $1, 3, ..., 2k-1$ is not more than the sum of the resources of the remaining states but when the resource of the state $2k+1$ is added, the situation is reversed. By (5), such a $k \geq 1$ always exists and by (6) satisfies $2k+1 \leq n$ for $n \geq 3$. Let

$$c = \sum_{j=1}^{n} r_j - 2\sum_{i=1}^{k} r_{2i-1},$$
$$d = 2\sum_{i=1}^{k+1} r_{2i-1} - \sum_{j=1}^{n} r_j.$$

A b-equilibrium is then given by

$$\begin{aligned}
&a_{1(2k+1)} = a_{(2k+1)1} = d/2, \\
&a_{1(2k)} = a_{(2k)1} = d/2 + r_{2k} - r_{2k+1}, \\
&a_{(2k)(2k+1)} = a_{(2k+1)(2k)} = c/2, \\
&a_{(2l)1} = a_{1(2l)} = r_{2l} - r_{2l+1}, \\
&a_{(2l)(2l+1)} = a_{(2l+1)(2l)} = r_{2l+1}, \text{ for } l = 1, ..., k-1 \text{ if } k \geq 2, \\
&a_{1j} = a_{j1} = r_j, \text{ for } j = 2k+2, ..., n \text{ if } 2k+2 \leq n, \text{ and} \\
&a_{ij} = 0 \text{ if it does not appear above.}
\end{aligned} \tag{9}$$

2.1 On the Condition for Balance

The simple condition (5) reads: *the strongest state's resource is not more than the sum of the resources of remaining states*, or alternatively, *no state controls more than half of* *the total world resource.* The condition (5) is a "no hegemony" or "no predominance" condition in that no state has a resource exceeding the sum of resources of all other states.

The condition may very well fail to hold for a given n and given resources, a state may have hegemony. For each $n \geq 2$, there are infinitely many resource values satisfying (5) and not satisfying (5). The following is true: *Given n, suppose $r_1, ..., r_n$ do not satisfy (5). There exists r_{n+1} such that $r_1, ..., r_{n+1}$ satisfy (5).* In fact, with (6), if r_1 is greater than the sum of resources of remaining states, let $r_{n+1} := r_1 - r_2 - ... - r_n$. Then, although state-1 is still the strongest, it no longer has hegemony and in the new world of $n+1$ states an equilibrium can be achieved. An alternative and obvious way of achieving a balanced world without increasing the number of states is of course a reduction, by way of self-consumption, in the resource of state-1 by the amount $r_1 - r_2 - ... - r_n$. This amount can be viewed as the maximum the hegemonic state can internally consume while a b-equilibrium prevails.

2.2 On the Choice of Allocations

If (5) holds and a balanced world is possible, the sets of all a_{ij}'s achieving a b-equilibrium can be obtained through the intersection of a finite number of polyhedra in a suitable space due to the nonnegativity constraint (2). In (9) we gave expressions for one b-equilibrium. The above choice of a_{ij}'s are such that $n(n-3)$ of them are always zero which means that $\frac{n(n-3)}{2}$ *pairs of states* need not allocate any resources against each other.

Let us now examine (9) more closely for some small n. For $n = 3$, the inequalities (7) and (8) give with $k = 1$,

$$r_1 \leq r_2 + r_3, \; r_1 + r_3 \geq r_2$$

which are satisfied by (5). The unique b-equilibrium is

$$\begin{aligned} a_{13} &= a_{31} = \tfrac{1}{2}(r_1 + r_3 - r_2), \\ a_{12} &= a_{21} = \tfrac{1}{2}(r_1 + r_2 - r_3), \\ a_{23} &= a_{32} = \tfrac{1}{2}(r_2 + r_3 - r_1). \end{aligned}$$

For $n = 4$, the inequalities (7) and (8) with $k = 1$ give

$$r_1 \leq r_2 + r_3 + r_4, \; r_1 + r_3 \geq r_2 + r_4,$$

which are satisfied by (5) and by (6). A b-equilibrium is obtained from (9) as

$$\begin{aligned} a_{13} &= a_{31} = \tfrac{1}{2}(r_3 + r_1 - r_4 - r_2), \\ a_{21} &= a_{12} = \tfrac{1}{2}(r_2 + r_1 - r_4 - r_3), \\ a_{32} &= a_{23} = \tfrac{1}{2}(r_4 + r_3 + r_2 - r_1), \\ a_{14} &= a_{41} = r_4, \\ a_{42} &= a_{24} = a_{43} = a_{34} = 0. \end{aligned}$$

For $n = 5$, the inequalities (7) and (8) give

$$\begin{aligned} k = 1: &\quad r_1 \le r_2 + r_3 + r_4 + r_5, \; r_1 + r_3 \ge r_5 + r_4 + r_2, \\ k = 2: &\quad r_1 + r_3 \le r_5 + r_2 + r_4, \; r_1 + r_3 + r_5 \ge r_2 + r_4. \end{aligned}$$

By (5) and by (6), one of the two sets of inequalities is satisfied. An admissible choice of allocations is obtained from (9) as

$$\begin{aligned} a_{31} &= a_{13} = \tfrac{1}{2}(r_3 + r_1 - r_5 - r_2 - r_4), \\ a_{21} &= a_{12} = \tfrac{1}{2}(r_2 + r_1 - r_5 - r_4 - r_3), \\ a_{32} &= a_{23} = \tfrac{1}{2}(r_5 + r_2 + r_3 + r_4 - r_1), \\ a_{41} &= a_{14} = r_4, \\ a_{15} &= a_{51} = r_5, \\ a_{54} &= a_{45} = a_{53} = a_{35} = a_{52} = a_{25} = a_{34} = a_{43} = a_{24} = a_{42} = 0, \end{aligned}$$

if $k = 1$ and as

$$\begin{aligned} a_{15} &= a_{51} = \tfrac{1}{2}(r_1 + r_3 + r_5 - r_2 - r_4), \\ a_{41} &= a_{14} = \tfrac{1}{2}(r_4 + r_3 + r_1 - r_5 - r_2), \\ a_{54} &= a_{45} = \tfrac{1}{2}(r_5 + r_2 + r_4 - r_3 - r_1), \\ a_{21} &= a_{12} = r_2 - r_3, \\ a_{23} &= a_{32} = r_3, \\ a_{34} &= a_{43} = a_{53} = a_{35} = a_{52} = a_{25} = a_{31} = a_{13} = a_{24} = a_{42} = 0, \end{aligned}$$

if $k = 2$.

One can observe the following general features of the allocations achieving equilibrium:

i) State-i needs to know the resource of every other state in order to determine its allocations towards achieving an equilibrium. The actual attainment of an equilibrium is instantaneous in case there is a unique equilibrium. It may however be a long and tedious process in case of nonunique equilibria.

ii) In the bordering case of state-1 having a resource equal to the sum of the resources of the remaining states, the unique equilibrium allocations are $a_{1j} = a_{j1} = r_j;\ j \in \mathcal{N}$ and $a_{ij} = 0;\ i \neq 1, j \neq 1$. Each state allocates its whole resource against the *almost hegemon* and none against each other.

iii) Note that the value of k is the largest possible in a world in which all states have the same amount of resource. For small k relative to $\frac{n-1}{2}$, the gap between the stronger and the weaker states is larger; for such k, the weaker states tend to allocate as little resource as possible towards each other and as large as possible against the stronger states. As k gets larger, the world is closer to an equilibrium to start with and discrimination becomes less pronounced.

iv) From items (ii) and (iii) emerges the *ad hoc* principle that the equilibrium seeking states are pushed (perhaps against their will) to coalitions by their dicriminating allocations against the weaker and the stronger states. Note that in order to make this *endogenous formation of coalitions* more precise, one needs to examine all possible equilibrium allocation schemes.

2.3 A Minimal Disequilibrium

It is tempting to rephrase the question at the beginning of this section in an optimality setting: *Given n states with resources $r_1, ..., r_n$, what are the allocations satisfying (1) and (2) such that the world is as close to an equilibrium as possible?*

There are many ways of quantifying the closeness to equilibrium. Here, we follow the most obvious. We call a world of n states to be in an *L_∞-optimal* [5] *equilibrium* if the allocations minimize

$$\max_{\substack{i,j \\ i \neq j}} |a_{ij} - a_{ji}|, \tag{10}$$

where $|a|$ denotes the absolute value of a number a. Note that if the resources satisfy (5), then the minimum (10) is equal to zero and the allocations achieving this minimum are the equilibrium allocations. If (5) fails, then by Theorem 1, it is known that the minimum is a positive number. The following result yields the minimum value and the allocations achieving this minimum.

Theorem 2. *In a world of n states with resources $r_1, ..., r_n$ satisfying (4) and (6), suppose that (5) fails. Such a world is in L_∞-optimal equilibrium if and only if*

$$\begin{aligned} a_{1j} &= \tfrac{1}{(n-1)}\left(r_1 + (n-1)r_j - \sum_{t=2}^{n} r_t\right), \\ a_{j1} &= r_j, \\ a_{ij} &= 0,\ i,j = 2, ..., n. \end{aligned} \tag{11}$$

The allocations (11) achieve the value

$$\min \max_{\substack{i,j \\ i \neq j}} |a_{ij} - a_{ji}| = \frac{1}{n-1}\left(r_1 - \sum_{t=2}^{n} r_t\right). \tag{12}$$

A proof of Theorem 2 is given in the Appendix. We observe from this result that an L_∞-optimal allocation scheme always exists, it is unique, and is rather simple: the hegemon,

[5] See e.g., Chvátal, 1983.

state-1, distributes its resource "evenly" among all the other states, i.e., in such a way that its excess resource against other states are uniformly the same. The other states take full action against the hegemon and none against each other. In the bordering case of the state-1 being an almost hegemon, (11) yield the unique equilibrium allocations described in remark (ii) of the previous subsection.

We note that if other well known definitions of optimality such as L_2 or L_1 optimality are adopted, then while the optimal allocation scheme will no longer be unique, two features will still be present: *(i)* If (5) holds, then optimal allocations are the equilibrium allocations. *(ii)* If (5) fails and a hegemon exists, then optimal allocations are such that all the other states take full actions against the hegemon and none against each other.

2.4 Allowing internal consumption

Let us now suppose that each state i uses a portion $\alpha_i r_i$, $\alpha_i \in [0,1]$, of its resource for allocations against the other states. The remaining portion $(1-\alpha_i)r_i$ is used for internal consumption or, as in the next section, for some other purpose. It follows by Theorem 1, that given k states $\{1,...,k\} = \mathcal{K}$ with total resource $r(\mathcal{K}) = \sum_{i=1}^k r_i$ and total allocation $\sum_{i=1}^k \alpha_i r_i =: I \in [0, r(\mathcal{K})]$, a b-equilibrium in $\mathcal{K}$ exists if and only if there exist $\alpha_i \in [0,1]$, $i \in \mathcal{K}$ satisfying

$$\sum_{i\in\mathcal{K}} \alpha_i r_i = I, \quad 0 \le \alpha_i r_i \le \min\{r_i, I/2\}. \tag{13}$$

The following result characterizes the internal equilibrium and the possible choices of α_i's in terms of total allocation I and the resources.

Theorem 3. *Let $\mathcal{K}$ be a collection of states having positive resources $r_1, ..., r_k$ with $k \ge 2$.*

(i) An equilibrium can be attained in $\mathcal{K}$ if and only if

$$r_i \le r(\mathcal{K}) - \frac{I}{2} \ \forall\, i \in \mathcal{K}. \tag{14}$$

(ii) If (14) holds, then there is a unique choice of α_i, $i \in \mathcal{K}$, satisfying (13) if and only if either $k = 2$, $I = 0$, $I = r(\mathcal{K})$, or equality is attained in (14) for some i.

If the total allocation is $I = r(\mathcal{K})$, then (14) reduces to $r_i \le r(\mathcal{K})/2 \ \forall\, i \in \mathcal{K}$ and hence Theorem 3.i reduces to Theorem 1.i. Note that the uniqueness of α_i's does not necessarily imply that there is a unique b-equilibrium in $\mathcal{K}$.[6] Also note that (14) is equivalent to $r_i \le (2r(\mathcal{K}) - I)/2 \ \forall\, i \in \mathcal{K}$. If equality is attained for some i, then state-i is a (almost) hegemon in $\mathcal{K}$ as $r(\mathcal{K}) \ge I$. Hence, the third possibility of unique α_i's in Theorem 3.ii can be realized only if there is a (almost) hegemon in $\mathcal{K}$.

[6]If $I = r(\mathcal{K})$, $k > 3$, and no state is an almost hegemon in $\mathcal{K}$, then Theorem 1.ii gives that there are infinitely many equilibria inside $\mathcal{K}$.

3 ALLIANCES

Let us now consider a system of n states partitioned into $l \leq n$ alliances $\mathcal{A}_i$, $i = 1, ..., l$ which are nonempty and disjoint subsets of $\mathcal{N} := \{1, ..., n\}$ such that $\mathcal{A}_1 \cup ... \cup \mathcal{A}_l = \mathcal{N}$. Internally, each alliance $\mathcal{A}_i$ is a system of n_i states. Externally, each alliance is perceived as a unit or as an effective-state by the other alliances. A fraction I_i of the total resource

$$r(\mathcal{A}_i) := \sum_{j \in \mathcal{A}_i} r_j$$

of $\mathcal{A}_i$ is consumed internally and the remaining fraction E_i is allocated externally to other alliances so that $E_i + I_i = r(\mathcal{A}_i)$. The *internal allocation* I_i and the *external allocation* E_i are formed by fractions of the resource of each state in the alliance i. Thus, for each $i = 1, ..., l$, we have

$$I_i = \sum_{j \in \mathcal{A}_i} \alpha_{ij} r_j, \; E_i = \sum_{j \in \mathcal{A}_i} (1 - \alpha_{ij}) r_j \tag{15}$$

for some $\alpha_{ij} \in [0, 1] \;\; \forall \, j \in \mathcal{A}_i$.

Given a fixed alliance configuration of l alliances, a system of n states will be said to be *in alliance equilibrium* if *(i)* each $\mathcal{A}_i$ as a system of n_i states is in equilibrium for $i = 1, ..., l$ and *(ii)* the system of l alliances, wieved as effective-states with resources E_i, $i = 1, ..., l$, is in equilibrium. In Figure 4, a system of five states in alliance equilibrium with respect to a configuration of two alliances is illustrated. Note that each alliance is (internally) in equilibrium as well as the system of two alliances. For the case $l = 2$, the external allocations E_1, E_2 are necessarily the allocations of $\mathcal{A}_1$ and $\mathcal{A}_2$ against each other. When $l \geq 3$, the allocations of any alliance $\mathcal{A}_i$ to other $l - 1$ alliances sum up to E_i.

It will be helpful to define a *local hegemon* to be a state in an alliance with resource exceeding the sum of resources of the other members of that alliance. Further, let $\mathcal{A}_k$ be called a *hegemonic-alliance* if $r(\mathcal{A}_k)$ is strictly more than the sum of the total resources of the other alliances.

We now ask the most basic of possible questions for an alliance configuration where all alliances are alignments (levels of internal or external allocations are not constrained).

Q3. Does there exist an alliance equilibrium, with respect to a given alliance configuration, in a given system of n states?

Q4. What are the internal and external equilibrium allocation profiles provided they exist?

It is not difficult to see that *if a system of n states is in equilibrium, then it is also in alliance equilibrium for any given alliance configuration.* Thus, the absence of a global hegemon implies that an alliance equilibrium is possible for any alliance configuration.

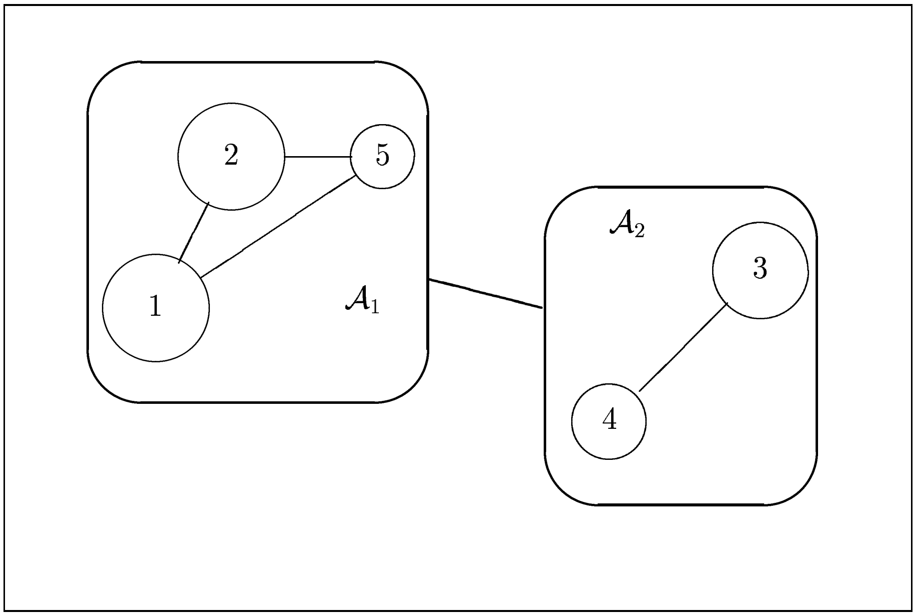

Figure 2: Alliance Equilibrium

An alliance equlibrium may hence be a less stringent requirement than equilibrium. The fact that alliance equilibrium poses no restriction as to how an external allocation of an alliance $\mathcal{A}_i$ to another alliance $\mathcal{A}_j$ should be distributed among the members of $\mathcal{A}_i$ also supports the expectation that an alliance equilibrium is easier to achieve than an equilibrium. It turns out, however, that the absence of a global hegemon is also necessary for the existence of an alliance equilibrium so that equilibrium and alliance equilibrium are equivalent as far as the existence conditions are concerned. The difference is in the equilibrium allocation profiles.

Theorem 4. *In a system of n states with positive resources $r_1, ..., r_n$, the following hold.*

(i) If there is a global hegemon, then no alliance equilibrium exists for any alliance configuration.

(ii) If there is no global hegemon, then there always exists an alliance equilibrium for any given alliance configuration. Any equilibrium allocation profile of Theorem 1 is also an alliance equilibrium allocation profile. The following particular external allocations also exist and allow an alliance equilibrium to be attained: For $i = 1, ..., l$

$$E_i = \begin{cases} 0 & \text{if } \mathcal{A}_i \text{ has no local hegemon,} \\ r(\mathcal{A}_i) & \text{if } \mathcal{A}_i \text{ is not a hegemonic-alliance and has a local hegemon,} \\ \sum_{j \neq i} r(\mathcal{A}_j) & \text{if } \mathcal{A}_i \text{ is a hegemonic-alliance and has a local hegemon.} \end{cases} \quad (16)$$

(iii) Suppose there is no global hegemon. There is a unique choice of alliance equilibrium allocation profiles (both internal and external equilibrium allocation profiles are unique) if and only if either $n \leq 3$ or an almost global hegemon exists.

4 A PERFECTLY BALANCED WORLD

The nonuniqueness of the equilibrium allocation scheme for $n > 3$ suggests considering a stronger notion of equilibrium than the one in the previous section. This is our objective in this section.

In an n-state-system with $n \geq 3$, let the equilibrium allocations further satisfy the following: For any given pair of states $\{k, l\}$,

$$a_{ik} - a_{il} = a_{jk} - a_{jl}, \tag{17}$$

for every $i, j \in \mathcal{N} - \{k, l\}$. By (17), $a_{ik} > a_{il}$ if and only if $a_{jk} > a_{jl}$ and $a_{ik} = a_{il}$ if and only if $a_{jk} = a_{jl}$. Thus, states i and j consistently emphasize one of any given two states k and l by their allocations. The condition (17) requires that the amount of emphases are equal in addition to being consistent. We call an allocation profile to be a *perfect equilibrium* if it is a b-equilibrium and the allocations of every pair of disjoint subsets $\{i, j\}$ and $\{k, l\}$ of $\mathcal{N}$ satisfy (17).

The following result shows that the world is perfectly balanced if and only if the sum of resources of every pair of states is not less than the average resource of the remaining states.

Theorem 3. *A world of n states with $n \geq 3$ and resources $r_1, ..., r_n$ satisfying (4) is perfectly balanced if and only if*

$$r_i + r_j \geq \frac{1}{n-2} \sum_{t \in \mathcal{N} - \{i,j\}} r_t \;\; \forall \;\; \{i, j\} \subset \mathcal{N}. \tag{18}$$

If (18) holds, then there is a unique set of allocations achieving the perfect equilibrium given by

$$a_{ij} = \frac{1}{(n-1)} \left(r_i + r_j - \frac{1}{n-2} \sum_{t \in \mathcal{N} - \{i,j\}} r_t \right) \;\; \forall \;\; \{i, j\} \subset \mathcal{N}. \tag{19}$$

A proof of Theorem 3 is given in the Appendix.

4.1 On Perfect Balance

Our definition of perfect balance is based on the equality of allocated resources between every pair of disjoint groups consisting of equal number of states. It might be wondered whether this definition can be strengthened by also requiring an equilibrium between every possible pairs of groups. This additional requirement however can never be satisfied since the equilibrium condition (3) implies that there are at least two groups of unequal number of states out of equilibrium.

The condition (18) reads: *the total resource of the weakest two states is not less than the average resource of the remaining states.* In fact, assuming (6), the condition (18) reduces to only one inequality for $i = n-1$, $j = n$.

For $n = 3$, the extra requirement (17) is inapplicable so that the condition (18) and the perfect equilibrium allocations are, respectively, the same as (5) and the equilibrium allocations. It is interesting to observe that even for $n > 3$, the condition (18) is equivalent to (5) with $n = 3$ provided that *each collection of $n-2$ states is considered as a single state with resource equal to the arithmetic average of their resources.* Also note that the perfect equilibrium allocations are all nonzero unless equality holds for the weakest two states in (18) (in which case only their allocations against each other are zero). This is in contrast with equilibrium allocations.

If $n \geq 4$, the set of resources satisfying (18) is a proper subset of the set of resources satisfying (5). Suppose (18) fails so that a perfect equilibrium is not possible. It is possible to obtain a perfect world by introducing new states into the world of n states, i.e., by an *expansion* of the world, as in the case of balance. Suppose with (4) and (6) that

$$r_{n-1} + r_n < \frac{1}{n-2}(r_1 + ... + r_{n-2}).$$

Let $l > 2n-1$ be the smallest integer satisfying $lr_n \geq r_1 + ... + r_{n-1}$ and let $r_j = r_n$; $j = n+1, ..., l-n+2$. In the new world of $l-n+2$ states a perfect balance is obtained since it can be verified that $(l-n)(r_{l-n+2} + r_{l-n+1}) \geq r_1 + ... + r_{l-n}$.

5 CONCLUSIONS

Starting with as few assumptions as possible concerning states and their resources, we have investigated some alternative concepts of balance among states. By sharply distinguishing between equilibrium, which is equality of allocations, and balance, which is capacity to achieve equilibrium, we examined the implications of three different concepts of static equilibrium on balances. Each balance is characterized by certain inequalities among resources.

The more fundamental of the three is the concept of balance, which is capacity to allocate resources to achieve equality in every one-against-one confrontation among states, and is characterized by the no hegemony condition. This condition is satisfied in a continuum of situations ranging from equality of resources among all states to the existence of an almost hegemon among states. For the case of three states, the equilibrium allocation is unique whenever it exists. For the case of more than three states, the equilibrium allocation is unique just in case there is an almost hegemon. The concept of perfect balance, which is capacity to allocate resources to achieve equality in every confrontation of two subsets of the same number of states, is characterized by the arithmetic average condition. The perfect equilibrium allocation is unique whenever it exists. The concept of threat-balance, which is the capacity to achieve equilibrium in a reduced world of three effective states, is characterized by every state being able to change the relative strengths of two coalitions formed by the other states.

The main conclusion one can draw from this analysis is that many crucial conditions on resource distribution that emerge in empirical and theoretical studies of balance of power correspond tightly with the capacity of states to achieve some kind of structural equilibrium. The foregoing analysis also clarifies the idea behind defining balance of power as the absence of a hegemon since, by Theorem 1, this is equivalent to the states being able to achieve an equilibrium in the sense of Section 2. Other plausible definitions perfect and threat-balance are consequences of stronger notions of equilibrium among states. Although they both imply balance, they are equally legitimate means of viewing balance of power within the realm of our model.

The model, resource allocation matrix, used in our analysis is intended to capture the common components present in a number of more sophisticated models and it would be superfluous to discuss how much of the reality it reflects. Nevertheless, it is necessary to discuss its limitations *if it were* viewed as a model with explanatory power. We start by pinpointing its mathematical limitations. The model has no dimension of time or sequentiality of decisions. It is therefore a static model and only allows questions concerning equilibrium. Since the dynamic aspects are lacking in the model, whether a given equilibrium is stable or not can not be answered. This means that wars, negotiations, threats, *formation* of coalitions, and the like can not be fully considered within the present model. Since the whole resource of a state is apportioned among the others and since all states are treated equally, the model can not accommodate "internal consumption vs. defense spending" (Powell, 1993), considerations and it disregards "geographic limitations or advantages" (Niou and Ordeshook, 1989). A more fundamental limitation of course is the fact that the model neglects behavioral features altogether. Some of the behavioral aspects incorporated in game theoretic models of Wagner (1986) and Niou, Ordeshook, and Rose (1989) are appropriate enrichments of the present model. As discussed by Wagner (1994), however, no single model may ever be rich enough to help us understand why states prefer one course of action to another in every given time and place. There are factors other than resources like beliefs in other states' motives or uncertainty about the outcome of a confrontation which are sometimes more determinative and which are difficult to justly incorporate into a model that claims sufficient generality. These cautionary advices while bringing forth the importance of structural analysis does not mean that the behavioral aspects can not be studied via formal models. It only means that those assumptions that characterize a specific situation should carefully be incorporated into the formal model developed for that particular situation. Some of the above mathematical limitations can be remedied remaining within the domain of structural analysis without imposing rules of behavior and strategy. The conclusions reached by such structural analysis would serve as useful inputs to game theoretic studies which may still be considered as *the* correct way of studying behavioral phenomena.

APPENDIX

The appendix contains proofs of Lemma 1 and Theorems 1-3 which are straightforward except those of the uniqueness claims of Theorems 1 and 3 that occupy most of the space.

Proof of Theorem 1. Let the states be numbered as in (6).

Necessity of (5): Necessity of (5) being clear for $n = 2$, suppose $n \geq 3$. Let a_{ij}'s satisfy (1)-(3). By (1), we can write

$$\sum_{i=1}^{n} r_i - 2r_1 = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij} - 2\sum_{j=2}^{n} a_{1j}. \tag{20}$$

Since by (3) $a_{ij} = a_{ji}$ for all i, j, we can write (20) as

$$\sum_{i=1}^{n} r_i - 2r_1 = 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} - 2\sum_{j=2}^{n} a_{1j} = 2\sum_{i=2}^{n-1} \sum_{j=i+1}^{n} a_{ij} \tag{21}$$

where the right hand side is nonnegative. Hence,

$$\sum_{i=1}^{n} r_i - 2r_1 \geq 0$$

which, in view of (6), implies (5).

Sufficiency of (5): Suppose $n = 2$. Then (5) gives $r_1 = r_2$ so that a balance is achieved by $a_{12} = a_{21} = r_1$. Suppose $n \geq 3$. Let a_{ij}'s be chosen as in (9). They satisfy (3) by definition. We now show that they also satisfy (2) and (1). First observe that $c \geq 0$ and $d \geq 0$ where the first inequality follows by (7) and the second by (8). Taking into account (6), it follows that all the other a_{ij}'s are also nonnegative. Hence, (2) holds. Consider (1) with $i = 1$. We have

$$\begin{aligned} \sum_{j=2}^{n} a_{1j} &= d + r_{2k} - r_{2k+1} + \sum_{l=1}^{k-1}(r_{2l} - r_{2l+1}) + \sum_{j=2k+2}^{n} r_j \\ &= r_1 - \sum_{i=1}^{k-1}(r_{2i} - r_{2i+1}) + \sum_{l=1}^{k-1}(r_{2l} - r_{2l+1}) \\ &= r_1, \end{aligned}$$

where the second equality follows by

$$\begin{aligned} d + r_{2k} - r_{2k+1} + \sum_{j=2k+2}^{n} r_j &= 2\sum_{i=1}^{k+1} r_{2i-1} - \sum_{j=1}^{2k+1} r_j + r_{2k} - r_{2k+1} \\ &= \sum_{i=1}^{k} r_{2i-1} - \sum_{j=1}^{k-1} r_{2j} \\ &= r_1 - \sum_{i=1}^{k-1}(r_{2i} - r_{2i+1}). \end{aligned}$$

For the sum (1) with $i = 2k+1$, we have

$$\begin{aligned} \sum_{j=1, j\neq 2k+1}^{n} a_{(2k+1)j} &= (c+d)/2 \\ &= r_{2k+1}. \end{aligned}$$

For the sum (1) with $i = 2k$, we have

$$\begin{aligned} \sum_{j=1, j\neq 2k}^{n} a_{(2k)j} &= (c+d)/2 + r_{2k} - r_{2k+1} \\ &= r_{2k}. \end{aligned}$$

Finally, for the sums (1) with $i = 2l$, $l = 1, ..., k-1$, we have

$$\sum_{j=1, j\neq 2l}^{n} a_{(2l)j} = r_{2l} - r_{2l+1} + r_{2l+1} = r_{2l}.$$

For any other value of i, each sum (1) contains only one nonzero term which is equal to r_i. Therefore, (1) holds for all $i \in \mathcal{N}$. The above choice of allocations also proves the last statement of the theorem.

Uniqueness: The statement on uniqueness of the equilibrium allocations for $n = 2, 3$ follows since (1) and (3) impose, respectively, $3, 6$ independent equality constraints for $2, 6$ variables.

Suppose $n > 3$. If in (5) an equality exists, then by (6) it must be that

$$\sum_{j=2}^{n} r_j = r_1. \tag{22}$$

Let a_{ij}'s be any choice of allocations achieving equilibrium. Then, by (1) and (22),

$$\sum_{j=2}^{n} a_{1j} = r_1 = \sum_{j=2}^{n} r_j \tag{23}$$

where, by (2) and (3), we have

$$0 \leq a_{1j} = a_{j1} \leq r_j; \quad j = 2, ..., n. \tag{24}$$

We claim that

$$a_{1j} = a_{j1} = r_j; \quad j = 2, ..., n. \tag{25}$$

In fact, if $a_{n1} < r_n$, then by (23),

$$\sum_{j=2}^{n-1} a_{1j} > \sum_{j=2}^{n-1} r_j$$

which contradicts (24). Hence, $a_{n1} = r_n$ and (23) holds with $n-1$ replacing n. By induction, it follows that (25) holds. Then, as (25) gives $a_{j1} = a_{j1} = r_j,\ j = 2, ..., n$, the equalities (1) give that $a_{jk} = 0$ for all $j, k = 2, ..., n$ and there is a unique set of values for equilibrium allocations. To prove the converse, suppose

$$\sum_{j=2}^{n} r_j > r_1. \tag{26}$$

Let k be the smallest integer such that $c > 0$ and $d \geq 0$ and alter the definitions of a_{ij}'s in (9) by ϵ as follows: If n is even, let

$$\begin{aligned}
&a_{1(2k+1)} = a_{(2k+1)1} = [d + (n-2k-2)\epsilon]/2,\\
&a_{1(2k)} = a_{(2k)1} = [d + (n-2k)\epsilon]/2 + r_{2k} - r_{2k+1},\\
&a_{(2k)(2k+1)} = a_{(2k+1)(2k)} = [c - (n-2k)\epsilon]/2,\\
&a_{(2l)1} = a_{1(2l)} = r_{2l} - r_{2l+1},\\
&a_{l(2l+1)} = a_{(2l+1)l} = r_{2l+1}, \text{ for } l = 1, ..., k-1 \text{ if } k \geq 1,\\
&a_{1j} = a_{j1} = r_j - \epsilon, \text{ for } j = 2k+2, ..., n \text{ if } 2k+2 \leq n,\\
&a_{(i+1)(n-i+1)} = a_{(n-i+1)(i+1)} = \epsilon,\\
&a_{(i+2k)(n-i+1)} = a_{(n-i+1)(i+2k)} = \epsilon;\ i \in \mathcal{N} - \{2k\}, \text{ and}\\
&a_{ij} = 0 \text{ if it does not appear above.}
\end{aligned}$$

If n is odd, let

$$\begin{aligned}
&a_{1(2k+1)} = a_{(2k+1)1} = [d + (n-2k-1)\epsilon]/2,\\
&a_{1(2k)} = a_{(2k)1} = [d + (n-2k-1)\epsilon]/2 + r_{2k} - r_{2k+1},\\
&a_{(2k)(2k+1)} = a_{(2k+1)(2k)} = [c - (n-2k+1)\epsilon]/2,\\
&a_{(2l)1} = a_{1(2l)} = r_{2l} - r_{2l+1},\\
&a_{l(2l+1)} = a_{(2l+1)l} = r_{2l+1} - \delta, \text{ for } l = 1, ..., k-1, \text{ if } k \geq 1,\\
&a_{1j} = a_{j1} = r_j - \epsilon, \text{ for } j = 2k+2, ..., n, \text{if } 2k+2 \leq n,\\
&a_{(i+1)(n-i+1)} = a_{(n-i+1)(i+1)} = \epsilon,\\
&a_{(i+1)(n-i+1)} = a_{(n-i+1)(i+1)} = \epsilon;\ i \in \mathcal{N} - \{1, 2k, 2k+1\}, \text{ and}\\
&a_{ij} = 0 \text{ if it does not appear above,}
\end{aligned}$$

where $\delta = 0$ if $4l = n + 1$ and $\delta = \epsilon$ otherwise. It is now straightforward to verify that (1)-(3) hold for every sufficiently small $\epsilon > 0$. □

Proof of Theorem 2.

Assuming the order (6), if (5) fails, then

$$r_1 > \sum_{i=2}^{n} r_i.$$

It follows that the allocations (11) satisfy (1) and (2). A simple computation shows that (11) achieves the value

$$m = \frac{1}{n-1}\left(r_1 - \sum_{t=2}^{n} r_t\right) = |a_{1j} - a_{j1}|;\ j = 2, ..., n.$$

Suppose the true minimum $\hat{m}$ is achieved by $\hat{a}_{ij}$ satisfying (1) and (2). Hence,

$$\hat{m} = \max_{\substack{i,j \\ i \neq j}} |\hat{a}_{ij} - \hat{a}_{ji}| \leq m.$$

If $\hat{a}_{ij} \neq a_{ij}$ for $i = 1$ and for some $j = 2, ..., n$, then by the fact that $\hat{a}_{1i}$'s satisfy (1), there exists $\epsilon > 0$ and $j = 2, ..., n$ such that $\hat{a}_{1j} = a_{1j} + \epsilon$. We must then have

$$|\hat{a}_{1j} - \hat{a}_{j1}| \leq \hat{m} \leq m = |a_{1j} - a_{j1}| = a_{1j} - a_{j1}. \tag{27}$$

Since $\hat{a}_{j1} \leq r_j$, it follows that $\hat{a}_{1j} - \hat{a}_{j1} \geq a_{1j} + \epsilon - r_j > 0$. Hence $|\hat{a}_{1j} - \hat{a}_{j1}| = \hat{a}_{1j} - \hat{a}_{j1}$ and (27) gives $r_j \leq \hat{a}_{j1} - \epsilon \leq r_j - \epsilon$ which contradicts $\epsilon > 0$. Hence, $\hat{a}_{1j} = a_{1j};\ j = 2, ..., n$. If, on the other hand, $\hat{a}_{kj} > 0$ for some $k, j = 2, ..., n$, then since $\hat{a}_{kj}$'s satisfy (1), we have $\hat{a}_{j1} = r_j - \epsilon$ for some $\epsilon \geq 0$ which gives $|\hat{a}_{1j} - \hat{a}_{j1}| = a_{1j} - r_j + \epsilon \geq m$. This implies $\hat{m} \geq m$ which is only possible if $\epsilon = 0$. Therefore, $\hat{a}_{ij} = a_{ij}$ for all $i \neq j$, $\hat{m} = m$, and the proof is complete. □

Proof of Theorem 3. (i) If (13) holds, then, for all $i \in \mathcal{K}$, we have $\alpha_i r_i \leq I/2$ so that $\sum_{j \in \mathcal{K}} \alpha_j r_j = I$ implies $\sum_{j \in \mathcal{K}-\{i\}} \alpha_j r_j \geq I/2$. The last inequality gives (14). Conversely, suppose (14) holds. Suppose the numbering is such that $r_1 \geq r_2 \geq ... \geq r_k$. Then, (14) is equivalent to $r(\mathcal{K}) - r_1 \geq I/2$. We first consider the case $r_1 \leq I/2$, or equivalently, $r_i \leq I/2$ for all $i \in \mathcal{K}$. Since $r(\mathcal{K}) \geq I$, there exist $\alpha_j \in [0, 1]$ such that $\sum_{j \in \mathcal{K}} \alpha_j r_j = I$. Note that $\alpha_j r_j \in [0, \min\{r_j, I/2]$ for all $j \in \mathcal{K}$ as required by (13). If, as the second case, $r_1 > I/2$, then let $\alpha_1 \in [0, 1)$ be such that $\alpha_1 r_1 = I/2$. By $r(\mathcal{K}) - r_1 \geq I/2$, there also exist $\alpha_j \in [0, 1]$ for $j = 2, ..., k$ satisfying $\sum_{j=2}^{k} \alpha_j r_j = I/2$ and thus (13) is satisfied. (ii) If $I = 0$, then (13) gives that $\alpha_i = 0,\ i \in \mathcal{K}$. If $I = r(\mathcal{K})$, then it gives $\alpha_i = 1,\ i \in \mathcal{K}$. If $k = 2$, then $\alpha_1 = I/(2r_1$ and $\alpha_2 = I/2r_2$ is the only possible choice for α_i's. Suppose, with the ordering $r_1 \geq ... \geq r_k$, that $r(\mathcal{K}) - r_1 = \sum_{i=2}^k r_i = I/2$. Then, $\sum_{i=2}^k \alpha_i r_i \leq I/2$ and by (13), $\alpha_1 r_1 = I/2$. But this implies in turn that $\sum_{i=2}^k \alpha_i r_i = I/2 = \sum_{i=2}^k r_i$ so that $\alpha_i = 1$ for all $i = 2, ..., k$. This proves the "if" part of the statement (ii). To see the "only if" part, let us suppose that $k > 2$, $0 < I < r(\mathcal{K})$, and $r(\mathcal{K}) - r_i > I/2$ for all $i \in \mathcal{K}$. We show that (13) is satisfied by infinitely many α_i's. Let α_i, $i \in \mathcal{K}$ be a set of choices satisfying (13). Since $r(\mathcal{K}) > I$, $\alpha_j < 1$ for some $j \in \mathcal{K}$. Suppose $\alpha_j r_j < I/2$. If $\alpha_t = 0$ for all $t \in \mathcal{K} - \{j\}$, then $\alpha_j > 0$ and for an arbitrary $t \in \mathcal{K} - \{j\}$, (13) implies that

$$(\alpha_j - \frac{\epsilon}{r_j})r_j + (\alpha_t + \frac{\epsilon}{r_t})r_t + \sum_{i \in \mathcal{K} - \{j,t\}} \alpha_i r_i = I \tag{28}$$

holds for all sufficiently small $\epsilon > 0$. If, on the other hand, $\alpha_t > 0$ for some $t \in \mathcal{K} - \{j\}$, then (13) implies that

$$(\alpha_j + \frac{\epsilon}{r_j})r_j + (\alpha_t - \frac{\epsilon}{r_t})r_t + \sum_{i \in \mathcal{K} - \{j,t\}} \alpha_i r_i = I$$

holds for all sufficiently small $\epsilon > 0$. This shows that if $\alpha_j r_j < I/2$, then (13) is satisfied by infinitely many choices of α_i's. Suppose now that $\alpha_j < 1$ but $\alpha_j r_j = I/2$. Then, $\sum_{t \in \mathcal{K} - \{j\}} \alpha_t r_t = I/2$ and $r(\mathcal{K}) - r_j > I/2$ implies that either $\alpha_t r_t < \min\{r_t, I/2\}$ for some $t \in \mathcal{K} - \{j\}$ in which case (28) again holds for infinitely many $\epsilon > 0$ or $\alpha_t < 1$ and $\alpha_t r_t = I/2$. In this case, for all $s \in \mathcal{K} - \{j, t\}$, we must have $\alpha_s = 0$ and (28) holds with t replaces by s. Note that such an s exists by $k > 2$. □

Proof of Theorem 4. We first note by Theorem 3.i that for any $i = 1, ..., l$, the internal allocation I_i of $\mathcal{A}_i$ should always satisfy

$$0 \leq I_i \leq \min\{r(\mathcal{A}_i), 2[r(\mathcal{A}_i) - \max_{j \in \mathcal{A}_i} r_j]\}. \tag{29}$$

Moreover, any I_i satisfying (29) is realizable as an internal equilibrium allocation, i.e., there exist equilibrium allocations inside $\mathcal{A}_i$ such that the total allocated resource is I_i. Also note that, by Proposition 1, an external equilibrium can be attained among $\mathcal{A}_i, i = 1, ..., l$ if and only if

$$2E_j \leq E_1 + ... + E_l, \; \forall \; j = 1, ..., l. \tag{30}$$

(i) Suppose state-1 is a global hegemon and is a member of $\mathcal{A}_1$. Then, $2r_1 > r(\mathcal{A}_1) + ... + r(\mathcal{A}_l)$. Moreover, state-1 is a local hegemon of $\mathcal{A}_1$ as well and by (29), $E_1 \geq 2r_1 - r(\mathcal{A}_1)$. For the other alliances, $E_j \leq r(\mathcal{A}_j)$, $j = 2, ..., l$. It follows that $E_1 > E_2 + ... + E_l$. With such external allocations an equilibrium can not be attained, by Theorem 1 applied to l states of resources $E_1, ..., E_l$.

(ii) Suppose there is no global hegemon. By Theorem 1, an equilibrium allocation exists. Any equilibrium allocation profile clearly gives an (internal) equilibrium for any alliance

$\mathcal{A}_i$, $i = 1, ..., l$. Moreover, the external allocation of $\mathcal{A}_i$ to another alliance $\mathcal{A}_k$ is given by $E_{ik} := \sum_{j \in \mathcal{A}_i} \sum_{t \in \mathcal{A}_k} a_{jt}$. Since by equilibrium $a_{jt} = a_{tj}$, we have E_{ik} equal to the allocation of $\mathcal{A}_k$ to $\mathcal{A}_i$, i.e., $E_{ki} = \sum_{t \in \mathcal{A}_k} \sum_{j \in \mathcal{A}_i} a_{tj}$. We now show that the external allocations given in (16) also yield an alliance equilibrium.

If there is an alliance $\mathcal{A}_k$ containing no local hegemon, by (16), $E_k = 0$ and it is thus enough to establish (30) for those alliances containing a local hegemon. That is, we can assume without loss of generality that all alliances contain a local hegemon. Suppose first that there is a hegemonic-alliance, say alliance 1. Then, all the other alliances are not hegemonic-alliances. By (16), $E_1 = E_2 + ... + E_l$ and $E_j = r(\mathcal{A}_j)$ for $j = 2, ..., l$. It follows that (30) holds with equality and that (29) is satified for $i = 2, ..., l$. Moreover, $2\max_{j \in \mathcal{A}_1}\{r_j\} - r(\mathcal{A}_1) \le E_1 < r(\mathcal{A}_1)$, where the first equality follows by the fact that there is no global hegemon and the second inequality by the fact that $\mathcal{A}_1$ is a hegemonic-alliance. It follows that (29) is satisfied by E_1 also. Suppose second that there is no hegemonic-alliance. Then, by (16), $E_j = r(\mathcal{A}_j), j = 1, ..., l$ and (30) holds by the fact that there is no hegemonic-alliance. These external allocations also satisfy the constraint (29).

(iii) By (29) and (30), for any I_i satisfying

$$\max\{0, r(\mathcal{A}_i) - \sum_{j=1, j \neq i} r(\mathcal{A}_i)\} \le I_i \le \min\{r(\mathcal{A}_i), 2[r(\mathcal{A}_i) - \max_{j \in \mathcal{A}_i} r_j]\} \tag{31}$$

an internal equilibrium allocation in $\mathcal{A}_i$ exists. If the internal equilibrium allocations are all unique, then (31) gives that for at least one i either $2[r(\mathcal{A}_i) - \max_{j \in \mathcal{A}_i} r_j] = r(\mathcal{A}_i) - \sum_{j=1, j \neq i} r(\mathcal{A}_i)$ or $r(\mathcal{A}_i) = \max_{j \in \mathcal{A}_i} r_j$, i.e., either $\mathcal{A}_i$ is a singleton or there is an almost global hegemon. If there is no almost global hegemon, then by the uniqueness of internal allocations, at least one alliance must be a singleton. If, in addition, the external equilibrium allocation is unique, then we must have $l \le 3$ or $E_i = \sum_{j=1, j \neq i}^{l} E_j$. Suppose $l > 3$. If $\mathcal{A}_i$ is a singleton, then it must be an almost global hegemon. If $\mathcal{A}_i$ is not a singleton, then for some $k \neq i$, $\mathcal{A}_k$ is a singleton, i.e., $E_i - r_k = \sum_{j=1, j \notin \{i,k\}}^{l} E_j$. In this case, both internal allocations I_i and $I_i - \epsilon$ for some small $\epsilon > 0$ allow an external equilibrium allocation profile and two different internal equilibrium allocation profiles in $\mathcal{A}_i$ by Theorem 3.ii. It follows that either $l \le 3$ or an almost global hegemon exists. If $l = 3$ and one of the three alliances, say $\mathcal{A}_1$ is not a singleton, then $E_1 \le E_2 + E_3$ or one of $\mathcal{A}_1, \mathcal{A}_2$ is a global almost hegemon. In the first possibililty, both internal allocations I_1 and $I_1 - \epsilon$ for some small $\epsilon > 0$ allow an external equilibrium allocation profile and two different internal equilibrium allocation profiles in $\mathcal{A}_1$. It follows that $n = l = 3$ or there is an almost global hegemon. Finally let $l = 2$. Then, external equilibrium gives $E_1 = E_2$. Suppose $\mathcal{A}_1$ is a singleton. If $\mathcal{A}_2$ has more than two members, then Proposition 4.ii and the fact that there is a unique choice of α_{2i}'s imply that there is an almost global hegemon. Consequently, $n \le 3$ or there is an almost global hegemon. This proves the "only if" part of the statement (iii).

Suppose now that state-1 is an almost global hegemon in an alliance $\mathcal{A}_j$. Then $2r_1 =$

$\sum_{t=1}^{l} \mathcal{A}_t$ and since an internal equilibrium in $\mathcal{A}_j$ exists, by Proposition 4, $E_j \geq 2r_1 - r(\mathcal{A}_j)$. Hence, $E_j \geq \sum_{t=1,t\neq j}^{l} \mathcal{A}_t \geq \sum_{t=1,t\neq j}^{l} E_t$. By the existence of an external equilibrium, we must have $E_j = \sum_{t=1,t\neq j}^{l} E_t = \sum_{t=1,t\neq j}^{l} r(\mathcal{A}_t)$. It follows that $E_t = r(\mathcal{A}_t)$ for all $t \neq j$, which implies by Proposition 4.ii that internal equilibrium allocations in all alliances are all zero and are thus unique except possibly in $\mathcal{A}_j$. However, $E_j = \sum_{t=1,t\neq j}^{l} r(\mathcal{A}_t)$ so that $I_j = 2r(r(\mathcal{A}_j) - \sum_{t=1}^{l} r(\mathcal{A}_t)$ and hence $2r_1 = \sum_{t=1}^{l} r(\mathcal{A}_t) = 2r(\mathcal{A}_j) - I_j$. By Proposition 4.ii, the internal equilibrium allocations in $\mathcal{A}_j$ is unique as well. □

Proof of Theorem 5. Let us order the states as in (6).

Necessity of (18): Suppose there exist a_{ij}'s satisfying (1)-(3) and (17) for all disjoint subsets $\{i,j\}$, $\{k,l\}$ of $\mathcal{N}$. By the fact that the world is balanced, we can write the following equalities similarly to (21) for $k \in \mathcal{N}$:

$$\sum_{i=1}^{n} r_i - 2r_k = \sum_{i=1,i\neq k}^{n} r_i - r_k = 2\sum_{i=1,i\neq k}^{n-1}\sum_{j=i+1,j\neq k}^{n} a_{ij}. \tag{32}$$

Employing (32) for $k = n$, we have

$$\begin{aligned}\sum_{i=1}^{n-2} r_i - (n-2)(r_{n-1}+r_n) &= \sum_{i=1}^{n-1} r_i - r_n - (n-1)r_{n-1} - (n-3)r_n \\ &= 2\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} a_{ij} - (n-1)r_{n-1} - (n-3)r_n.\end{aligned}$$

Since

$$r_{n-1} = \sum_{j=1}^{n-2} a_{j(n-1)} + a_{(n-1)n}, \quad r_n = \sum_{j=1}^{n-2} a_{jn} + a_{(n-1)n},$$

we also have

$$\begin{aligned}&\sum_{i=1}^{n-2} r_i - (n-2)(r_{n-1}+r_n) \\ &= 2\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} a_{ij} - (n-3)\sum_{j=1}^{n-2} a_{j(n-1)} - (n-3)\sum_{j=1}^{n-2} a_{jn} - 2(n-2)a_{(n-1)n},\end{aligned} \tag{33}$$

We now add and subtract

$$2\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} a_{jn} = 2\sum_{k=2}^{n-2}(k-1)a_{kn}$$

on the right hand side of (33) to obtain

$$\begin{aligned}&\sum_{i=1}^{n-2} r_i - (n-2)(r_{n-1}+r_n)\\&= 2\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}(a_{ij}-a_{jn}) + 2\sum_{k=2}^{n-2}(k-1)a_{kn} - (n-3)\sum_{j=1}^{n-2}(a_{j(n-1)}+a_{jn}) - 2(n-2)a_{(n-1)n}.\end{aligned} \tag{34}$$

Since, by (17), we have $a_{ij} - a_{jn} = a_{i(n-1)} - a_{(n-1)n}$, and also noting that

$$\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} a_{i(n-1)} = \sum_{k=1}^{n-3}(n-k-2)a_{k(n-1)}, \quad 2\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} a_{(n-1)n} = (n-2)(n-3)a_{(n-1)n},$$

the equality (34) gives

$$\begin{aligned}&\sum_{i=1}^{n-2} r_i - (n-2)(r_{n-1}+r_n)\\&= 2\sum_{k=1}^{n-3}(n-k-2)a_{k(n-1)} + 2\sum_{k=2}^{n-2}(k-1)a_{kn} - (n-3)(\sum_{j=1}^{n-2} a_{j(n-1)} + \sum_{j=1}^{n-2} a_{jn}) - (n-1)(n-2)a_{(n-1)n}\\&= \sum_{k=1}^{n-2}(n-2k-1)a_{k(n-1)} - \sum_{k=1}^{n-2}(n-2k-1)a_{kn} - (n-1)(n-2)a_{(n-1)n},\end{aligned}$$

where the second equality is obtained by combining the first sum with the third and the second sum with the fourth. We now observe that

$$\begin{aligned}&\sum_{k=1}^{n-2}(n-2k-1)a_{k(n-1)} - \sum_{k=1}^{n-2}(n-2k-1)a_{kn}\\&= \sum_{k=1}^{n-3}(n-2k-1)[a_{k(n-1)} - a_{kn}] - (n-3)[a_{(n-2)(n-1)} - a_{(n-2)n}]\\&= \sum_{k=1}^{n-3}(n-2k-1)[a_{(n-2)(n-1)} - a_{(n-2)n}] - (n-3)[a_{(n-2)(n-1)} - a_{(n-2)n}]\\&= \left(\sum_{k=1}^{n-3} n-2k-1\right)[a_{(n-2)(n-1)} - a_{(n-2)n}] - (n-3)[a_{(n-2)(n-1)} - a_{(n-2)n}]\\&= 0\end{aligned}$$

since $a_{k(n-1)} - a_{kn} = a_{(n-2)(n-1)} - a_{(n-2)n}$ by (17) and since $\sum_{k=1}^{n-3}(n-2k-1) = (n-3)$. Therefore,

$$\sum_{i=1}^{n-2} r_i - (n-2)(r_{n-1}+r_n) = -(n-1)(n-2)a_{(n-1)n} \le 0$$

which by (6) gives (18) as we wanted to show.

Sufficiency of (18): If a_{ij}'s are defined by (19), then $a_{ij} \geq 0$ by (18) and (2) is satisfied. To see that (1) is also satisfied, we write

$$
\begin{aligned}
(n-1)(n-2) \sum_{j=1, j \neq i}^{n} a_{ij} &= \sum_{j=1, j \neq i}^{n} \left((n-1)(r_i + r_j) - \sum_{t=1}^{n} r_t \right) \\
&= (n-1)^2 r_i + (n-1) \sum_{t=1, t \neq i}^{n} r_t - (n-1) \sum_{t=1}^{n} r_t \\
&= (n-1)^2 r_i - (n-1) r_i \\
&= (n-1)(n-2) r_i
\end{aligned}
$$

which gives (1). The fact that a_{ij}'s of (19) satisfy (3) is obvious. We finally show that they also satisfy (17). In fact by (19) we have

$$
\begin{aligned}
&(n-1)(n-2)(a_{ik} - a_{il}) = (n-1)(r_k - r_l), \\
&(n-1)(n-2)(a_{jk} - a_{jl}) = (n-1)(r_k - r_l),
\end{aligned}
$$

for every pair of disjoint subsets $\{i, j\}$, $\{k, l\}$ of $\mathcal{N}$.

Uniqueness: If the allocations achieve perfect equilibrium, then (1)-(3), (32) hold and, for every pair of disjoint subsets $\{i, j\}$, $\{k, l\}$ of $\mathcal{N}$

$$
a_{ik} - a_{il} = a_{jk} - a_{jl}, \tag{35}
$$

holds. Since there are $3\frac{n!}{(n-4)!4!}$ pairs of disjoint subsets of cardinality 2 in $\mathcal{N}$, (35) gives $3\frac{n!}{(n-4)!4!}$ equalities. We first show that some of these are redundant, i.e., *if (35) is satisfied for the following $\frac{n(n-3)}{2}$ pairs of disjoint subsets*

$$
\begin{aligned}
&\{1,2\} \leftrightarrow \{3,i\}, \quad i = 4, ..., n, \\
&\{1,3\} \leftrightarrow \{2,i\}, \quad i = 4, ..., n, \\
&\{1,4\} \leftrightarrow \{2,i\}, \quad i = 5, ..., n, \\
&\{1,5\} \leftrightarrow \{2,i\}, \quad i = 6, ..., n, \\
&\vdots \\
&\{1,n-2\} \leftrightarrow \{2,i\}, \quad i = n-1, n, \\
&\{1,n-1\} \leftrightarrow \{2,n\}
\end{aligned} \tag{36}
$$

of $\mathcal{N}$, then it is also satisfied for every pair of disjoint subsets $\{i, j\}$, $\{k, l\}$ of $\mathcal{N}$. In order to prove this claim, we first note the following fact. Let $\{i, j\} \leftrightarrow \{k, l\}$ denote the equality (35). Then,

$$
\{i,j\} \leftrightarrow \{k,l\} \;\&\; \{i,j\} \leftrightarrow \{k,m\} \quad \Rightarrow \quad \{i,j\} \leftrightarrow \{l,m\} \tag{37}
$$

for all distinct $i, j, l, m \in \mathcal{N}$. This fact is easily obtained by subtracting both sides of the antecedent equalities and employing the balance condition. Suppose now that the equalities (36) are satisfied. To prove the claim we need to show that (35) or $\{i, j\} \leftrightarrow \{k, l\}$ holds for every distinct $i < j$, $k < l$, where we can always assume that $i < k$. If $i = 1$, $j = 2$, then $k \geq 3$, $l \geq 4$. By (36), we have $\{1, 2\} \leftrightarrow \{3, m\}$ for $m = 4, ..., n$ which by (37) gives $\{1, 2\} \leftrightarrow \{k, l\}$. If $i = 1$, $j > 2$, then by (36), we have $\{1, j\} \leftrightarrow \{2, j + 1\}$ for $j = 3, ..., n - 1$ which contain $\{1, j\} \leftrightarrow \{2, k\}$ and $\{1, j\} \leftrightarrow \{2, l\}$. These two equalities again imply by (37) that $\{1, 2\} \leftrightarrow \{k, l\}$. If $i = 2$, then $j \geq 3$ and, by (36), we have $\{1, 2\} \leftrightarrow \{3, m\}$ for $m = 4, ..., n$ which by (37) give $\{1, 2\} \leftrightarrow \{k, l\}$. Again by (36), we also have $\{1, j\} \leftrightarrow \{2, j + 1\}$ for $j = 3, ..., n - 1$ which by (37) give $\{1, j\} \leftrightarrow \{k, l\}$. Applying (37) once more, $\{1, 2\} \leftrightarrow \{k, l\}$ and $\{1, j\} \leftrightarrow \{k, l\}$ imply $\{2, j\} \leftrightarrow \{k, l\}$. Finally, if $i > 2$, then by (36) we can write $\{1, i\} \leftrightarrow \{2, i+1\}$ for $i = 3, ..., n-1$ which by (37) yield $\{1, i\} \leftrightarrow \{k, l\}$. Similarly, by (36), $\{1, j\} \leftrightarrow \{2, j+1\}$ for $j = 3, ..., n-1$ which by (37) yield $\{1, j\} \leftrightarrow \{k, l\}$. Combining $\{1, i\} \leftrightarrow \{k, l\}$ and $\{1, j\} \leftrightarrow \{k, l\}$ by (37), we obtain $\{i, j\} \leftrightarrow \{k, l\}$. This proves the italicized claim above. We next observe that the set of equalities in (36) are linearly independent since each term $a_{24}, ..., a_{2n}$ and $a_{i(i+j)}$ for $i = 3, ..., n-1$, $j \in \mathcal{N} - i$ is contained in one and only one of these equalities. Let us now consider (32) which is equivalent to

$$\sum_{i=1, i\neq k}^{n-1} \sum_{j=i+1, j\neq k}^{n} a_{ij} = \frac{1}{2}\left(\sum_{i=1, i\neq k}^{n} r_j - r_k\right); \quad k \in \mathcal{N}. \tag{38}$$

This set of equations can be written in matrix form $Ax = b$, where the $\frac{n(n-1)}{2}$-vector x in transposed form is

$$x^T = [a_{12}, ..., a_{1n}; a_{23}, ..., a_{2n}; ...; a_{(n-2)(n-1)}, a_{(n-2)n}; a_{(n-1)n}],$$

and the n-vector b is

$$b^T = [\frac{1}{2}(\sum_{i=2}^{n} r_i - r_1), \frac{1}{2}(\sum_{i=1, i\neq 2}^{n} r_i - r_2), ..., \frac{1}{2}(\sum_{i=1}^{n-1} r_i - r_n)].$$

Each entry of $\frac{n(n-1)}{2} \times n$-matrix A is either 1 or 0. In order to describe A explicitly, let us index the columns by the subscript of the corresponding entry in x, i.e., the columns of A are indexed by pairs of numbers $12, 13, ..., 1n, 23, ..., 2n, ..., (n-2)(n-1), (n-2)n, (n-1)n$. Then, an entry in the i-th row of A, say the entry $A(i, jk)$, is 0 if and only if $i = j$ or $i = k$ (it is equal to 1 if and only if $i \neq j$ and $i \neq k$). Each column indexed by jk hence contains exactly two zeros at its j-th and k-th rows. The rows of A are linearly independent since it is easy to show that its $n \times n$ submatrix consisting of the columns $12, 13, ..., 1n, 23$ is nonsingular. Let us now write (36) also in matrix form $Kx = 0$, where x is as defined above and K is a $\frac{n(n-3)}{2} \times \frac{n(n-1)}{2}$ matrix the columns of which are indexed as the columns of A and the rows of which are indexed by $24, ..., 2n, 34, ..., 3n, 45, ..., 4n, ..., (n-2)(n-1), (n-2)n, (n-1)n$. The row ij of K represents the equality

$$\begin{aligned} &a_{13} - a_{1j} - a_{i3} + a_{ij} = 0, \quad \text{if } \ i = 2, \\ &a_{12} - a_{1j} - a_{i2} + a_{ij} = 0, \quad \text{if } \ i > 2, \end{aligned}$$

so that it contains two 1's, two -1's, and 0's; the columns 13,ij of the ij-th row are 1 and the columns $1j$,$i3$ are -1 for $i = 2$; the columns 12,ij of the ij-th row are 1 and the columns $1j$,$i2$ are -1 for $i > 2$. It holds that $AK^T = 0$. To see this consider the m-th row of A which has a 0 in its lk-th column if $m = l$ or $m = k$. When multiplied with the column $2j$ of K^T, we have

$$\begin{aligned} &A(m,13)K(2j,13) + A(m,1j)K(2j,1j) + A(m,23)K(2j,23) + A(m,2j)K(2j,2j) \\ &= A(m,13) - A(m,1j) - A(m,23) + A(m,2j) = 0, \end{aligned}$$

and when multiplied with column ij of K^T for $i > 2$, we have

$$\begin{aligned} &A(m,12)K(ij,12) + A(m,1j)K(ij,1j) + A(m,i2)K(ij,i2) + A(m,ij)K(ij,ij) \\ &= A(m,12) - A(m,1j) - A(m,i2) + A(m,ij) = 0, \end{aligned}$$

where the second equalities follow by substituting values for $A(m,ij)$. We have so far shown that in a perfectly balanced world a_{ij}'s must satisfy $Ax = b,\ Kx = 0$, where A and K are matrices with linearly independent rows satisfying $AK^T = 0$. *It follows that (36) together with the n equalities of (38) form a linearly independent set of equations*, i.e., the rows of $\begin{bmatrix} A \\ K \end{bmatrix}$ are linearly independent. In fact, if it is not, then for some nonzero vector $y = \begin{bmatrix} y_A \\ y_K \end{bmatrix}$, $y^T \begin{bmatrix} A \\ K \end{bmatrix} = 0$ which gives $y_A^T AK^T + y_K^T KK^T = y_K^T KK^T = 0$ which in turn gives $y_K = 0$ and $y_A = 0$ since both K and A have linearly independent rows. This contradiction proves the statement in italics above. Taking into account that $\begin{bmatrix} A \\ K \end{bmatrix}$ is a square matrix, we conclude that there is a unique solution to

$$\begin{bmatrix} A \\ K \end{bmatrix} x = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

giving a unique solution for a_{ij}'s and the proof is complete. □

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